Nonlocal Lubrication Forces and the Sedimentary Jamming Front

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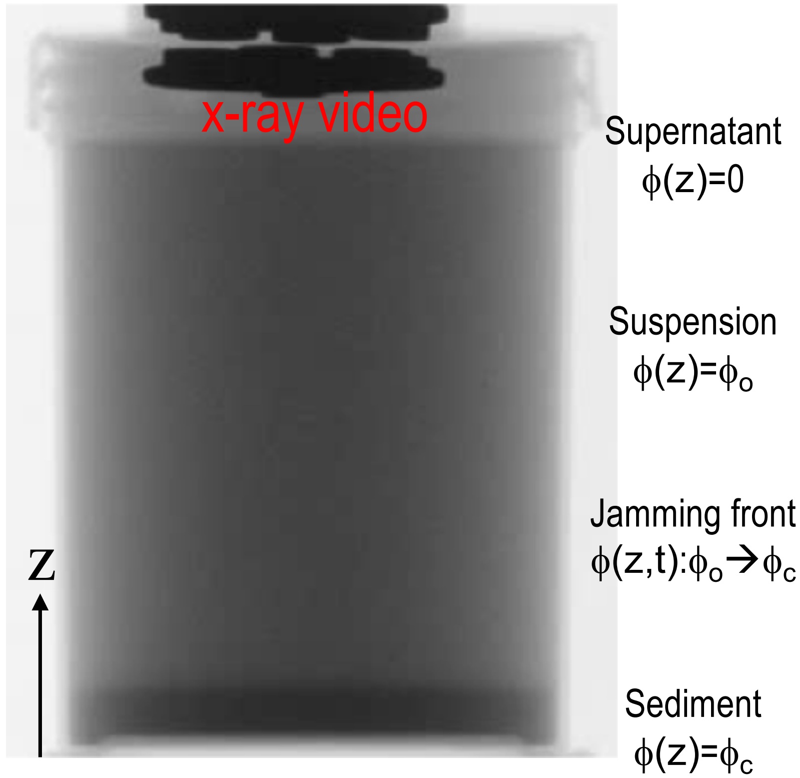
- Shape of the density profile across the jamming front?
  - Fluid is squeezed out between grains as they come into contact in jammed sediment
  - Coupled PDEs for $\phi(z,t)$ & $v(z,t)$
- Other $\phi(\mathbf{r},t)$ phenomena?
  - Sedimentation is a "simple" 1d/stationary case to study the kinetics of jamming



Physics beyond jamming

- Much is known about jamming & rheology of _uniform_ systems, represented by a single set of state variables (T, pressure, packing fraction, loading,…)
- But ***kinetics*** into / out of a jammed state usually involves flows and ***gradients*** in state variables
  - Boundary effects
  - Nonlocal effects
  - ***Jamming fronts***

- Examples from my lab
  - ***Sedimentation***, impact, clogging, creep, shear bands…
    - ★ spatially-varying density changes retarded by interstitial fluid

Plan of attack

- Background
  - Firm up the empirical hindered settling function
  - Review the lubrication force between two spheres
- Net lubrication force on sphere in suspension
  - Nonlocal: due to neighbors above and below
- Coupled PDEs for concentration & velocity fields
  - Linear response / dispersion relation in bulk
  - Asymptotic solutions for shape of jamming front
- Other nonlocal lubrication effects

Evolution of $\phi(z,t)$ under gravity?

[1] Continuity equation $\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial z}\left(-v + D_c \frac{\partial}{\partial z}\right)\phi$

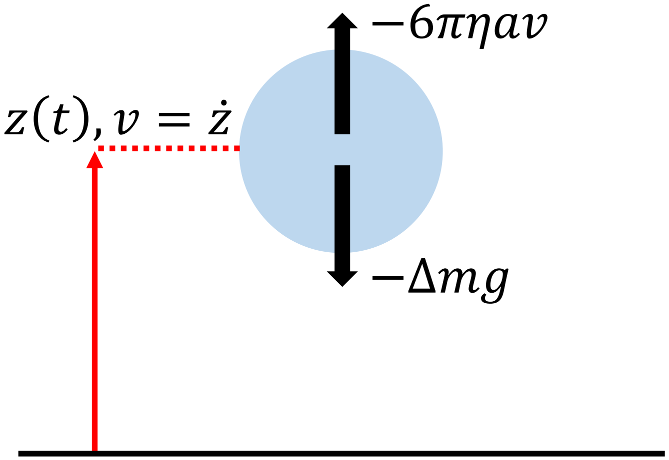
[2] Relation of particle velocity v & concentration ϕ:

- Kynch (1952): $v = -v_s(1 - \alpha\phi)$ where v_s = Stokes speed
- Hindered settling function H(ϕ) (eg reviewed in Guazzelli-Morris book (2012): $v = -v_s H(\phi) \approx -v_s[H_o + (\phi - \phi_o)H']$
- Burgers' equation (eg van Saarloos-Huse 1990): expand continuity equation to second order in $\varepsilon = \phi - \phi_o$
- Today: $v = -v_s H(\phi) +$ nonlocal lubrication term.

  Begin by considering the forces that act on grains…

Force balance in __uniform__ suspensions

- Dilute: grains of radius *a* in a fluid of viscosity η fall at the Stokes speed $v_s = \Delta mg/6\pi\eta a$



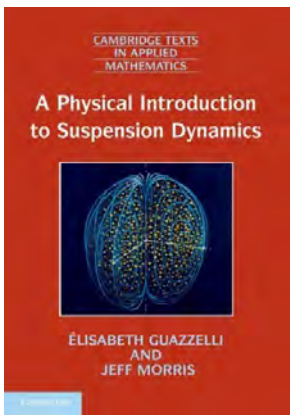
$$\Sigma F_{up} = -\Delta mg - 6\pi\eta av = 0$$

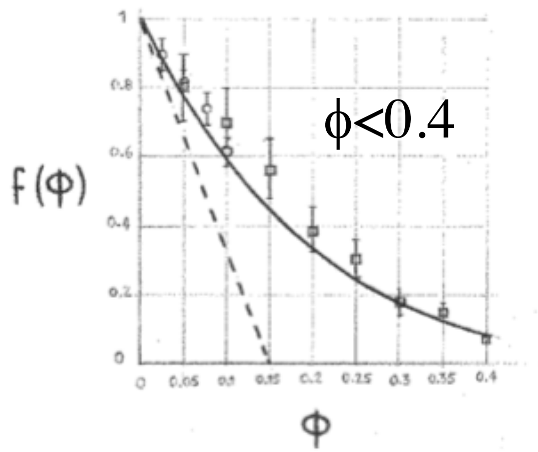
- Grains at volume fraction ϕ settle slower according to the "<u>hindered settling function,</u>" $H(\phi)<1$ *{form?}*

$$\begin{aligned} v &= v_s H(\phi) \\ \Sigma F_{up} &= -\Delta mg - 6\pi\eta av/H(\phi) = 0 \end{aligned}$$

Hindered settling function

- No expt'l/theoretical consensus for H(ϕ) versus ϕ
  - many empirical forms and contradictory statements[6-12]
    - eg Richarson-Zaki *$H(\phi) = (1-\phi)^n$* with $4<n<7$
    - Guazzelli-Morris' book (2012) recommends $n \approx 5$ for $\phi < 0.4$





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[8] R. H. Davis and A. Acrivos, Ann. Rev. Fluid Mech. **17**, 91 (1985).
[9] A. P. Philipse, Current Opinion in Colloid and Interface Science **2**, 200 (1997).
[10] E. Guazzelli and J. Hinch, Ann. Rev. Fluid Mech. **43**, 97 (2011).
[11] E. Guazzelli and J. F. Morris, *A Physical Introduction to Suspension Dynamics* (Cambridge Press, NY, 2012).
[12] R. Piazza, Rep. Prog. Phys. **77**, 056602 (2014).

There is a widely used empirical correlation[6] attributed to Richardson and Zaki (1954):

$$f(\phi) = (1-\phi)^n, \tag{6.12}$$

where a value of $n \approx 5$ most accurately represents the experimental data for small Reynolds numbers, as can be seen in Figure 6.7. Note that this correlation is likely to be inaccurate when approaching maximum packing, i.e. $\phi_{\max} \sim 0.60$.[7]

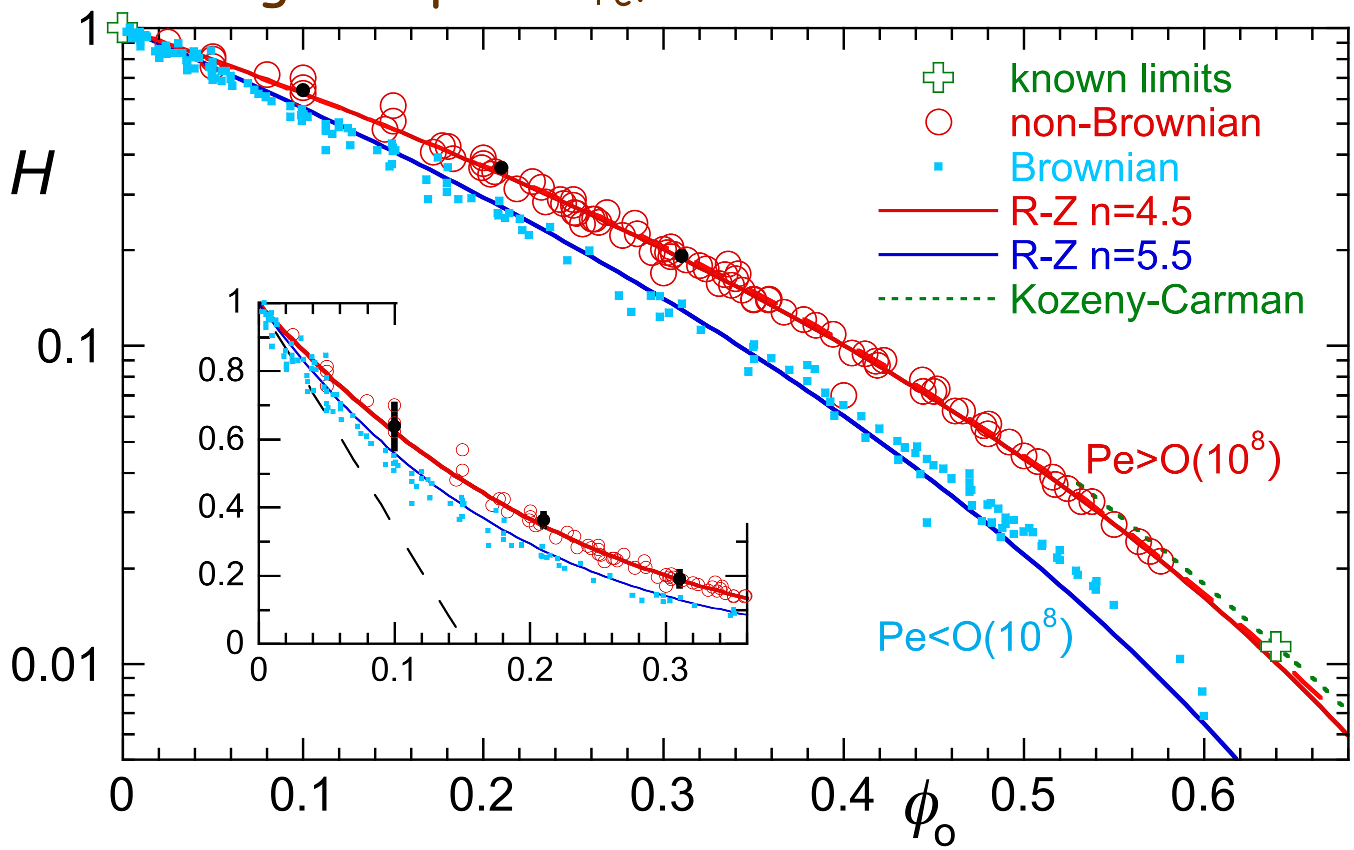
Digitize old data & take our own

- Monodisperse uncharged spheres, small Re
- Two very old and very standard methods
  - s uses speed of supernatant-suspension interface
  - f uses height of suspension vs fluidization speed

Source	Method	System	ρ_f (g/ml)	ρ_p (g/ml)	d (μm)	Type	Pe
[20] Kops82, Table IV	s	silica in cyclohexane	0.78	1.77	0.13	B	1.8E-04
[25] Buzzaccaro08, Fig. 8	s	polymer in water	1.00	2.15	0.15	B	4.1E-04
[24] Benes07, Fig. 1	s	polystyrene in water	1.00	1.05	0.72	B	4.8E-04
[23] Paulin90, Fig. 3a	s	pmma in decalin/tetralin	0.93	1.19	0.99	B	0.56
[19] Buscall82, Fig. 4	s	polystyrene in water	1.00	1.05	3.05	B	2.73
[27] Xue92, Fig. 1	f	polystyrene in water	1.00	1.05	31	B	2.9E+04
[21] Bacri86, Fig. 2	s	glass in water	1.00	2.50	40	B	2.4E+06
[28] Martin95, spreadsheet	f′	glass in water/glycerin	1.00	2.50	69	B	2.1E+07
[9] Richardson54, Fig.14a	s & f	divinylbenzene in water	1.00	1.06	217	n-B	7.9E+07
[18] Oliver61, Table 3	s	pmma in water/glycerin	1.00	1.19	161	n-B	8.1E+07
[22] Davis88, Fig. 1	s	glass in solution	1.02	2.49	130	B	2.6E+08
[26] Ham90, Fig. 3a	f	glass in solution	1.06	2.47	410	n-B	2.5E+10
[2] Ham88, Fig. 4	s	glass in solution	1.08	2.42	535	n-B	6.9E+10
[3] Nicolai95, Table 1	s	glass in solution	1.09	2.53	788	n-B	3.5E+11
[this work] Brzinski15	s	glass in water/glycerin	1.24	2.53	180 – 1000	n-B	8.4E+08 – 8.0E+11

Hindered Settling Compilation

- The data all sort onto two branches according to the Péclet number, Pe = v_s a / D ~ $\Delta\rho g a^4/kT$
  - Large Pe is non-Brownian, good fit to $H(\phi)=(1-\phi)^{4.48\pm0.04}$
  - "Small" Pe is Brownian, decent fit to $H(\phi)=(1-\phi)^{5.6\pm0.1}$
    - range is up to ~ϕ_c, where non-Brownian branch merges with K-C



Aside: why is crossover at such huge Péclet number ~O(10^8) ??

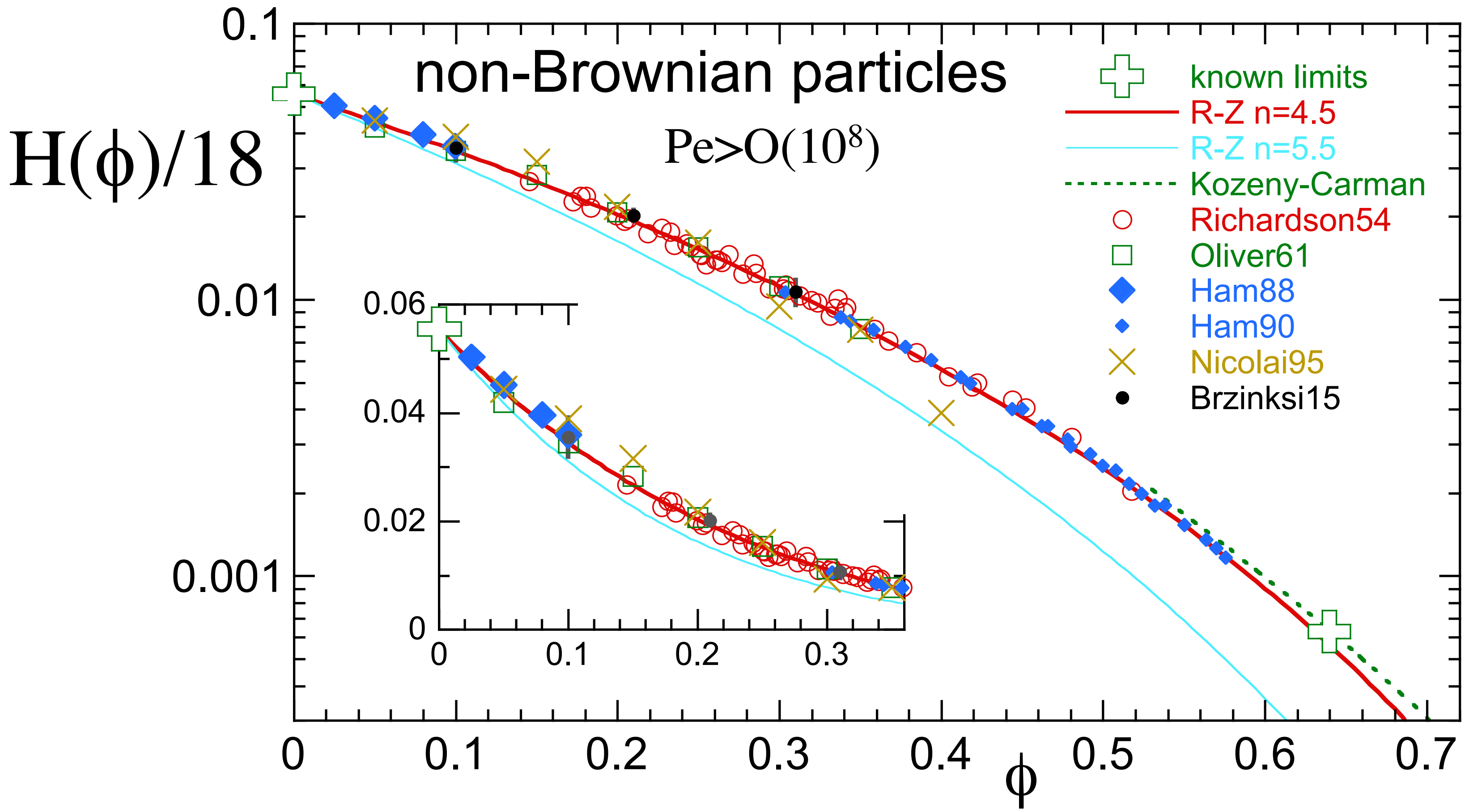
i.e. why so sensitive to very small thermal motion ?

Translate Pe=10^8 to particle radius:

$a_c \approx 70(k_B T/\Delta\rho g)^{1/4}$

Focus today: non-Brownian branch

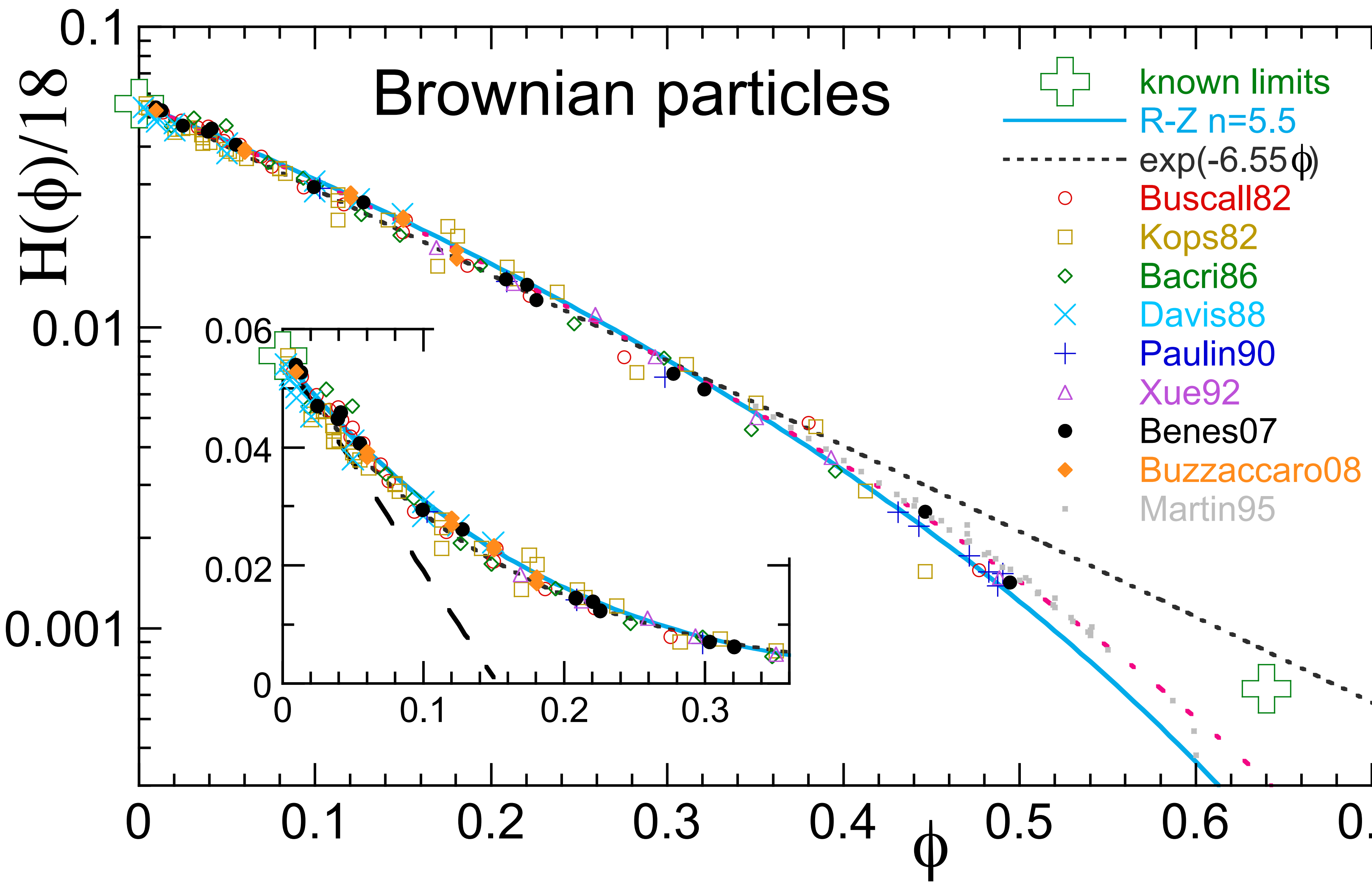
- Take $H(\phi)=(1-\phi)^{4.5}$ and hence $F_{drag} = 6\pi\eta av/H(\phi)$ with high confidence, for $\phi{:}0\rightarrow\phi_c$



{ask later to see (a) Brownian datasets and (b) theoretical predictions}

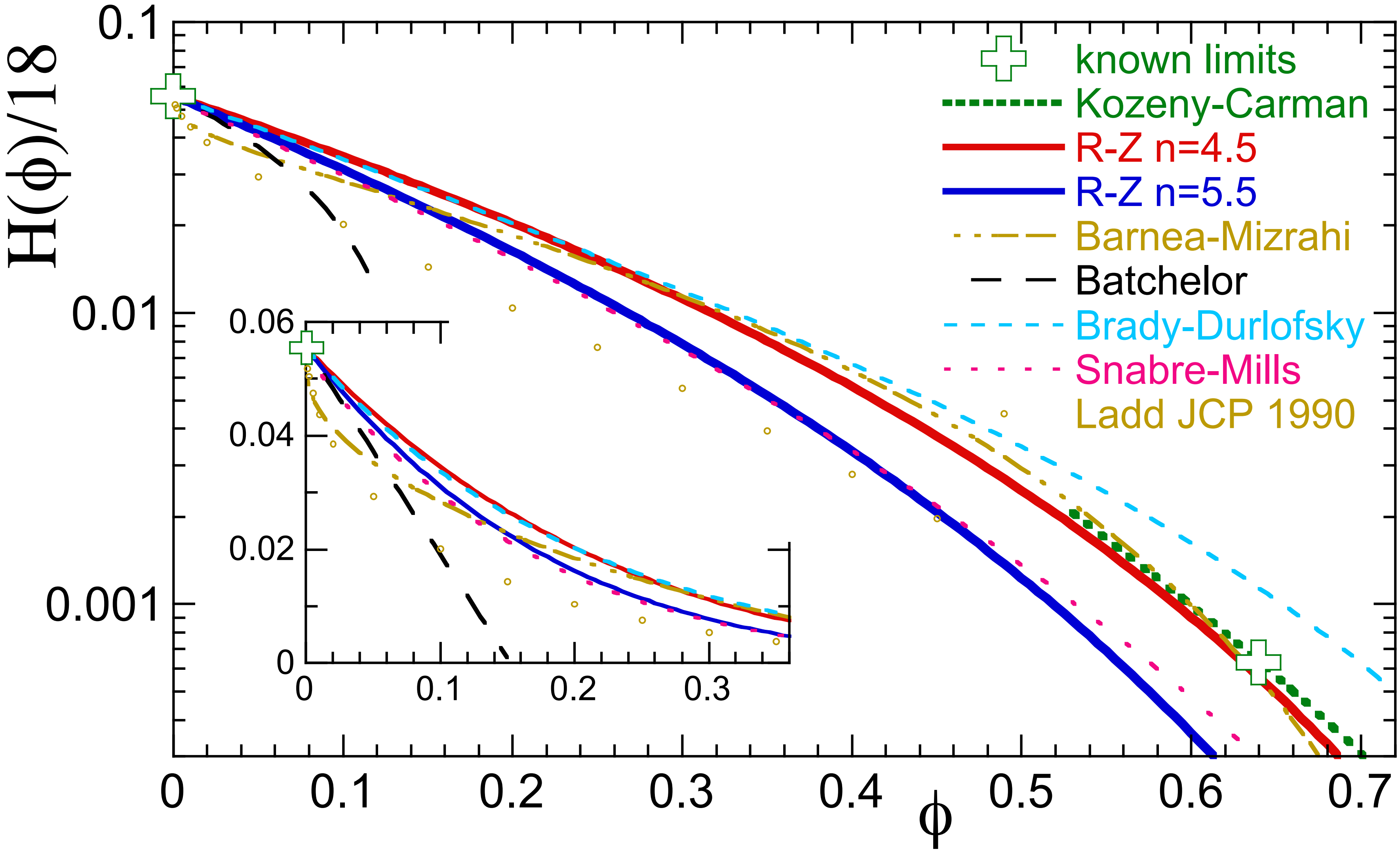
Brownian branch

- Datasets that fall on $H(\phi)=(1-\phi)^{5.5}$ {Pe<O(10^8)}



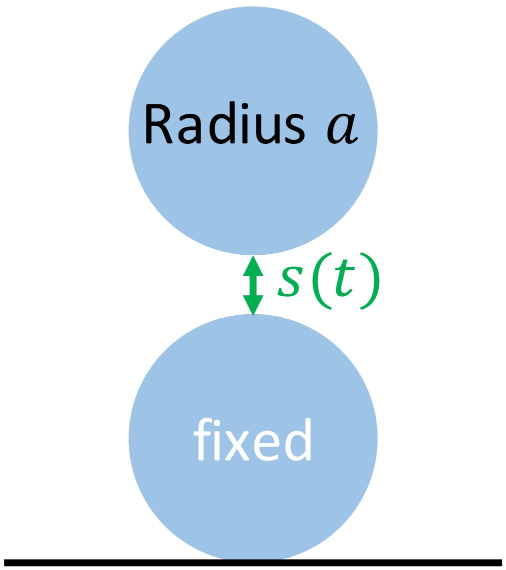
Predicted forms for $H(\phi)$

- Brady-Durlofsky (1988) matches nonBrownian data for $\phi<0.4$
- Snabre-Mills (2000) nonBrownian theory matches Brownian data



Lubrication Force

- For both ball-wall and ball-ball, the total viscous force for vertical motion of top ball at any surface separation s, is given to <7% by



$$F \approx \underbrace{6\pi\eta a\dot{s}}_{\text{Stokes drag}} + \underbrace{6\pi\eta a^2\dot{s}/s}_{\text{Lubrication force}}$$

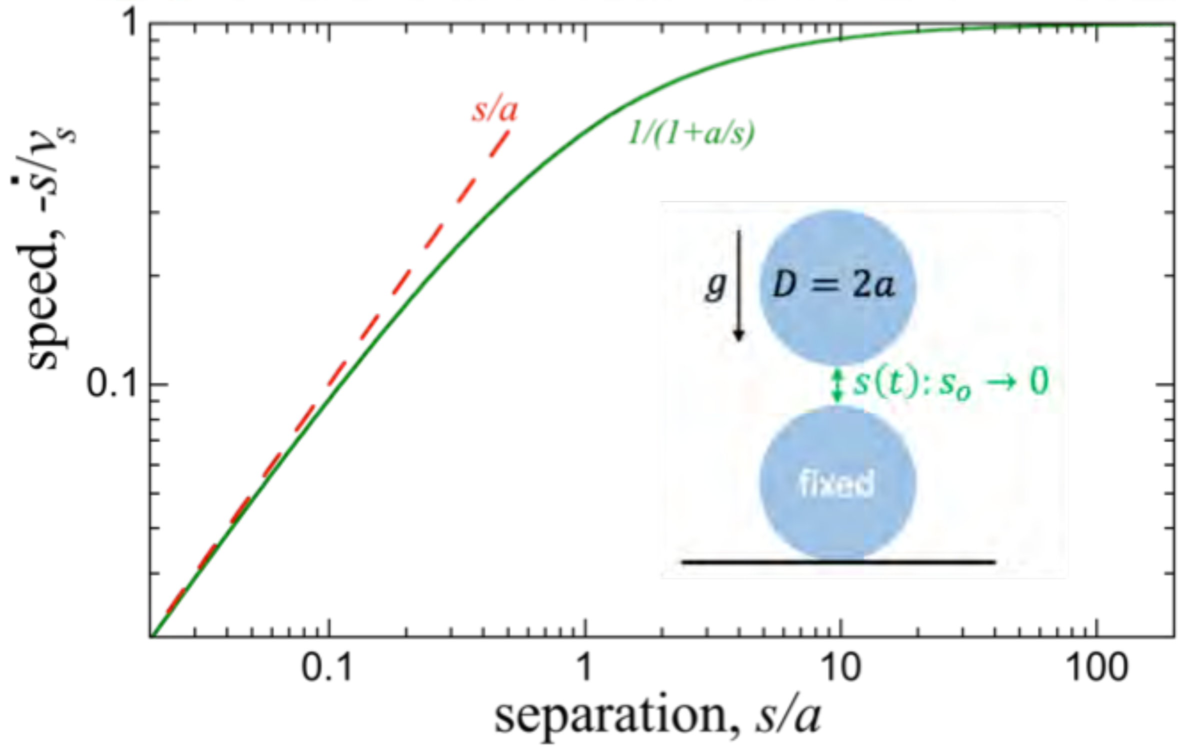
  - So it's usual to consider lubrication as a separate force that acts independently of the drag force.

[H. Brenner (1961) and e.g. Guazzelli & Morris (2012)]

Toy sedimentation problem

- One falling ball: comes to an effective rest by the balance of gravity & lubrication forces:

$$\sum F_{down} = \Delta m g + 6\pi\eta a\dot{s} + 6\pi\eta a^2\dot{s}/s = 0$$

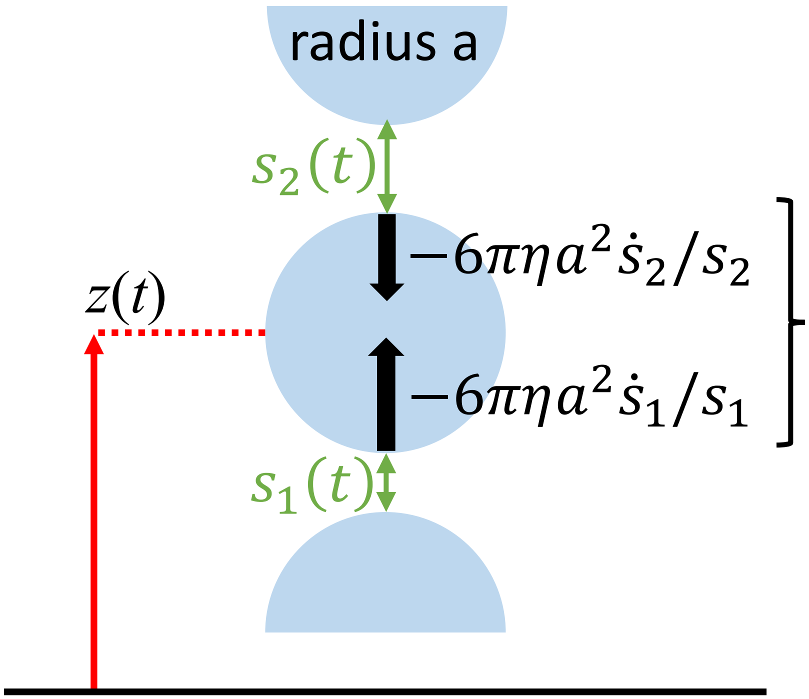


- NB: breakage of lubrication film is harmless

Collective lubrication effects can similarly bring grains to an effective rest in the sediment…

Lubrication in suspensions

- Middle sphere experiences a ***net*** lubrication force if there is a spatial gradient in the strain rate $\dot{s}/s$

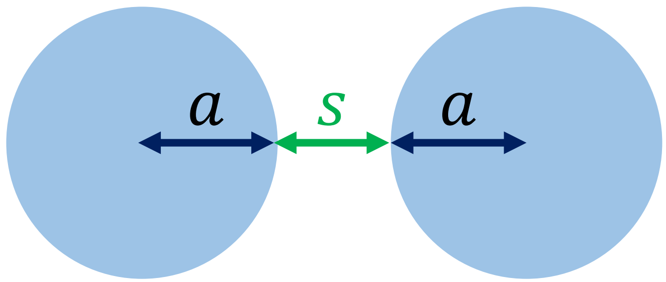


The net lubrication force on the middle sphere, due to neighbors above and below, is proportional to the gradient of $\dot{s}/s$. The rate of change of $s(t)$ depends on the material derivative of the volume fraction, $D\phi/Dt = \partial\phi/\partial t + \dot{z}\,\partial\phi/\partial z$.

(next slide)

Sphere-sphere separation

- For crystals, the volume fraction is $\phi = \phi_c/(1+s/2a)^d$



$$\phi \propto \frac{a^d}{(2a+s)^d}$$

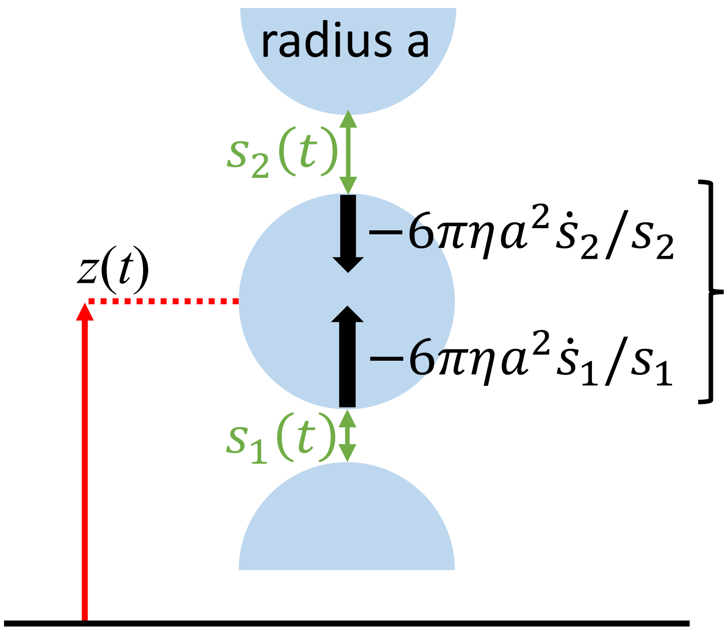
- So, the gap vanishes near ϕ_c as $s \approx (\phi_c-\phi)\cdot 2a/(d\phi_c)$

- Similarly, expect $s \propto (\phi_c-\phi)$ to hold for dense suspensions; therefore, the gap strain rate is

$$\boxed{\frac{\dot{s}}{s} = -\frac{D\phi/Dt}{\phi_c-\phi}}$$

Nonlocal Lubrication Force

- Acts on particles in suspensions when there is a gradient in the rate of change of volume fraction:



$$
\begin{aligned}
F_{lub} &= 6\pi\eta a^2 \frac{\dot{s}_2}{s_2} - 6\pi\eta a^2 \frac{\dot{s}_1}{s_1} \\
&= -6\pi\beta\eta a^3 \frac{\partial}{\partial z}\left(\frac{\mathrm{D}\phi/\mathrm{D}t}{\phi_c - \phi}\right)
\end{aligned}
$$

- Introduce β as a dimensionless parameter of to account for geometrical factors related to the three-dimensional constellation of neighbors. {Set β=0 to turn off lubrication}

Coupled PDEs for $\phi(z,t)$ & $v(z,t)$

- Force balance in upward (+z) direction:

$$\begin{aligned}\sum F_{up} &= -\Delta m g - \frac{6\pi\eta a}{H} v - 6\pi\beta\eta a^3 \frac{\partial}{\partial z}\left(\frac{\mathrm{D}\phi/\mathrm{D}t}{\phi_c - \phi}\right) \\ &= 0 \text{ for small Re} \\ &\downarrow\end{aligned}$$

$$v = -v_s H - \beta H a^2 \frac{\partial}{\partial z}\left(\frac{\frac{\partial\phi}{\partial t} + v\frac{\partial\phi}{\partial z}}{\phi_c - \phi}\right)$$

- Continuity (convection-diffusion equation):

$$\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial z}\left(-v + D_c\frac{\partial}{\partial z}\right)\phi$$

Linear Response

- Small-amplitude damped wave solution:

$$\phi(z,t) = \phi_o + \delta\phi e^{ik(z+v_p t)-\Gamma t} \quad \& \quad v(z,t) = -v_s H_o + \nu\delta\phi e^{ik(z+v_p t)-\Gamma t}$$

$$\begin{aligned} \beta(\phi) &= \beta_o + (\phi - \phi_o)\beta' \\ H(\phi) &= H_o + (\phi - \phi_o)H' \\ D_c(\phi) &= D_o + (\phi - \phi_o)D_c' \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} v_p(k) &= v_s H_o \frac{\left(1 + \frac{\phi_o}{H_o}H'\right) + \alpha}{1+\alpha} \quad \text{phase speed} \\ \Gamma(k) &= \frac{D_o k^2}{1+\alpha} \quad \text{damping rate} \\ \text{where } \alpha &= \frac{\beta_o H_o \phi_o a^2 k^2}{\phi_c - \phi_o} \text{ is lubrication correction} \end{aligned}$$

- Recover usual results for $\beta \rightarrow 0$ and also for $k \rightarrow 0$
- The non-local lubrication force alters the phase speed and reduces the diffusive damping at small wavelengths

Shape of the jamming front?

- Asymptotic solution of coupled PDEs for small-ε perturbation above jamming front, into suspension:

$$\phi(z,t) = \phi_o + \varepsilon e^{-\kappa_1 (z - v_c t)} \quad \& \quad v(z,t) = -v_s H_o (1 - \nu\varepsilon) e^{-\kappa_1 (z - v_c t)}$$

$$\begin{aligned} v_c &= \frac{v_s H_o \phi_o}{\phi_c - \phi_o} \quad \text{jamming front speed} \\ \beta(\phi) &= \beta_o + (\phi - \phi_o)\beta' \\ H(\phi) &= H_o + (\phi - \phi_o) H' \\ D_c &= 0 \text{ nonBrownian} \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \nu &= \frac{\phi_c}{(\phi_c - \phi_o)\phi_o} \\ \kappa_1 &= \sqrt{\frac{\phi_c - \phi_o}{a^2 \beta_o H_o \phi_o}} \end{aligned}$$

Shape of the jamming front?

- Asymptotic solution of coupled PDEs for small-ε perturbation below jamming front, into sediment:

$$\phi(z,t) = \phi_c - \varepsilon e^{-\frac{1}{2}\kappa_2^2(z-v_c t)^2} \quad \& \quad v(z,t) = -v_s h_o(0+\nu\varepsilon) e^{-\frac{1}{2}\kappa_2^2(z-v_c t)^2}$$

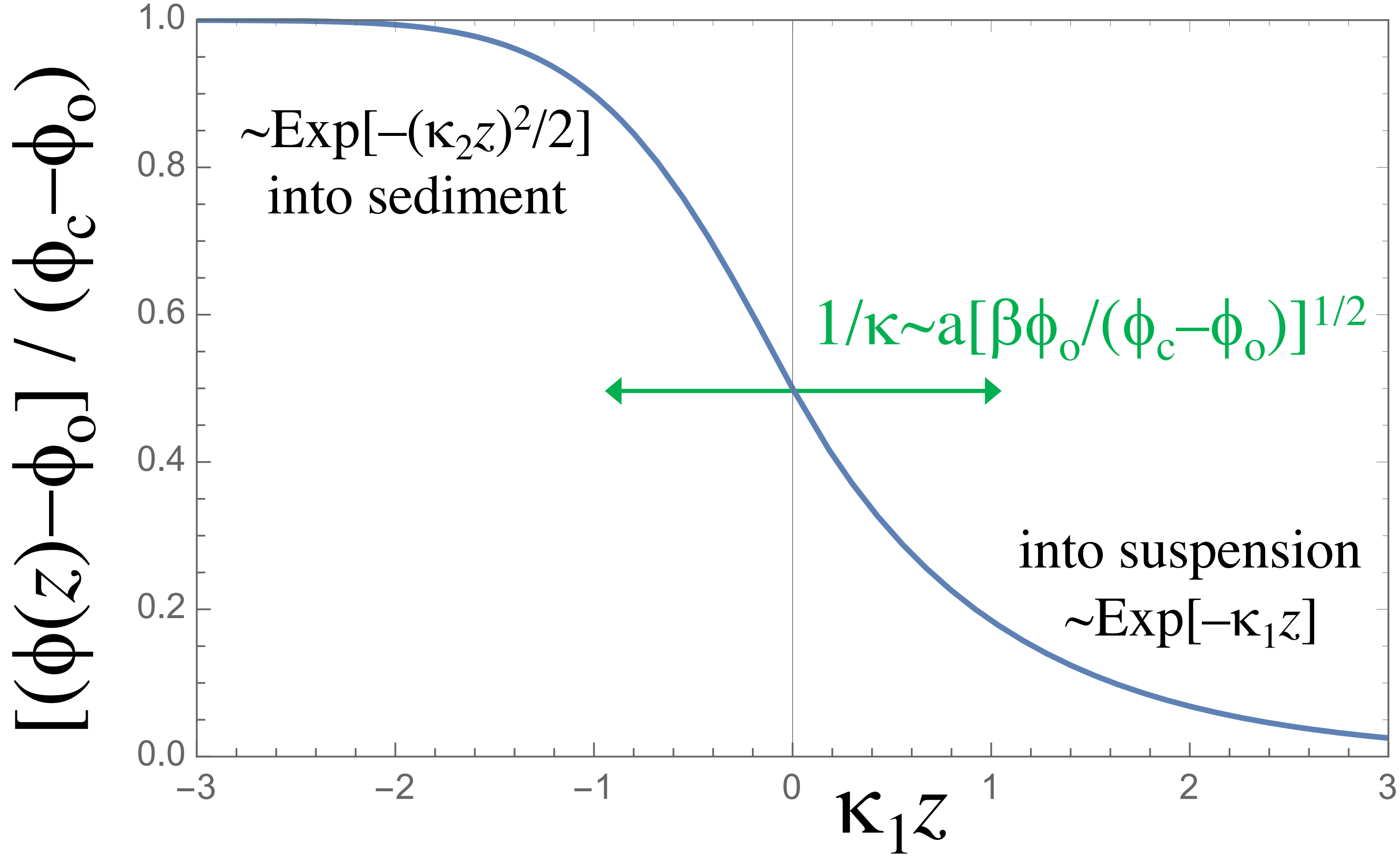
$$\begin{aligned} v_c &= \frac{v_s H_o \phi_o}{\phi_c - \phi_o} \quad \text{jamming front speed} \\ \beta(\phi) &= \beta_c + (\phi - \phi_c)\beta' \\ H(\phi) &= H_c + (\phi - \phi_c)H' \\ D_c &= 0 \text{ nonBrownian} \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \nu &= \frac{\phi_o}{(\phi_c - \phi_o)\phi_c} \\ \kappa_2 &= \sqrt{\frac{\phi_c - \phi_o}{a^2 \beta_c H_c \phi_o}} \end{aligned}$$

Shape of the jamming front

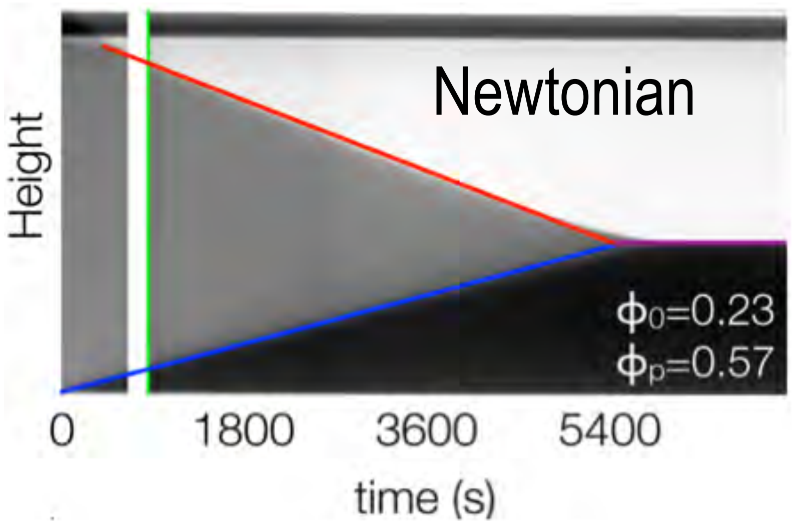
- Stationary concentration profile:

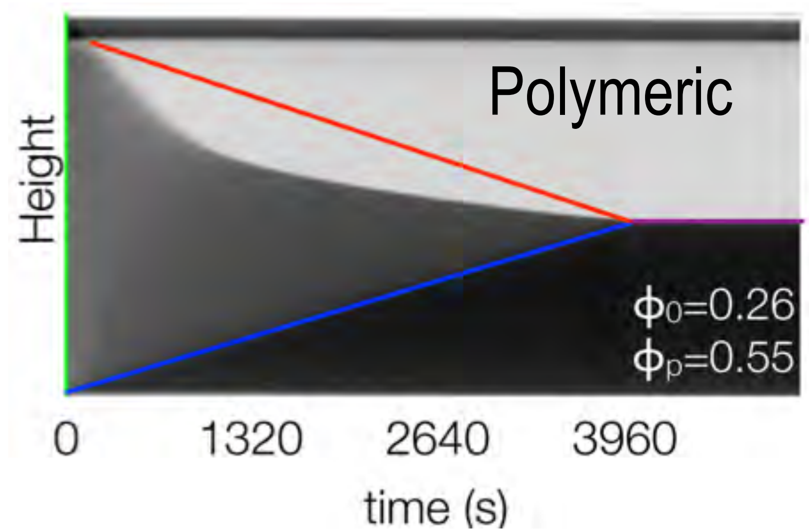


- Step-function for $\beta=0$ (no lubrication) and for $\phi_o \rightarrow 0$ (dilute)
- Width increases with ϕ_o and diverges for $\phi_o \rightarrow \phi_c$
- Velocity profile has the same asymptotics (easier to measure?)

Measure the front shape (I)

- First attempt for 300 μm diameter grains
  - spacetime plots of x-ray imaging videos:





Ted Brzinski

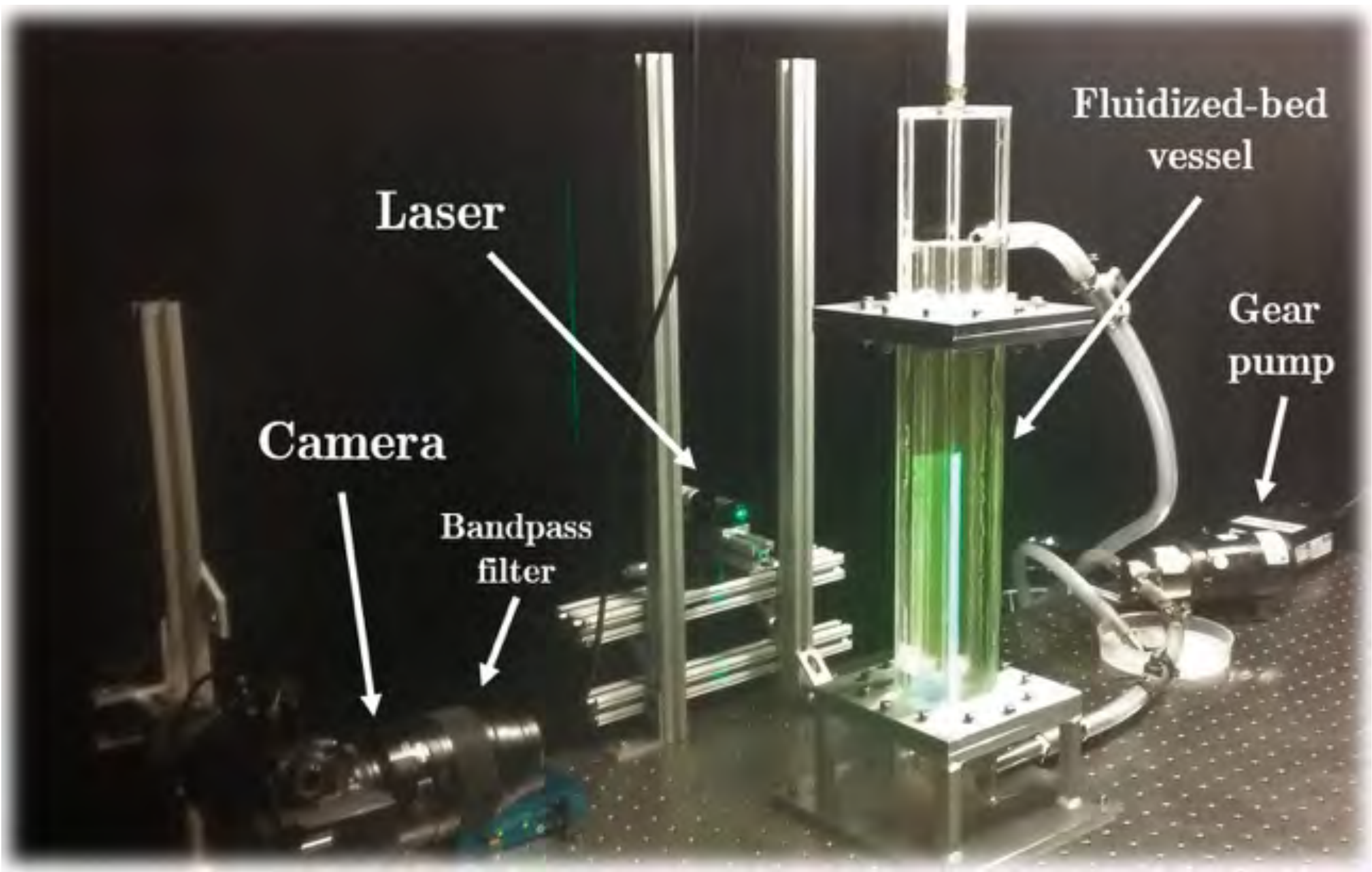
- Newtonian case: front is too sharp to be resolved with our collaborators' medical x-ray imaging device

- Polymeric (Boger fluid) case: perturbation extends far ahead of front, but isn't stationary. Modify PDEs using strain-rate dependent extensional viscosity in non-local lubrication force?

$$F_{lub} = -6\pi\beta\eta_e(\dot{\gamma})a^3\frac{\partial\dot{\gamma}}{\partial z} \text{ where } \dot{\gamma} = \frac{\dot{s}}{s} = \frac{\mathrm{D}\phi/\mathrm{D}t}{\phi_c-\phi}$$

Measure the front shape (II)

- Newly-commissioned apparatus to track particles in index-matched suspension that can be fluidized:



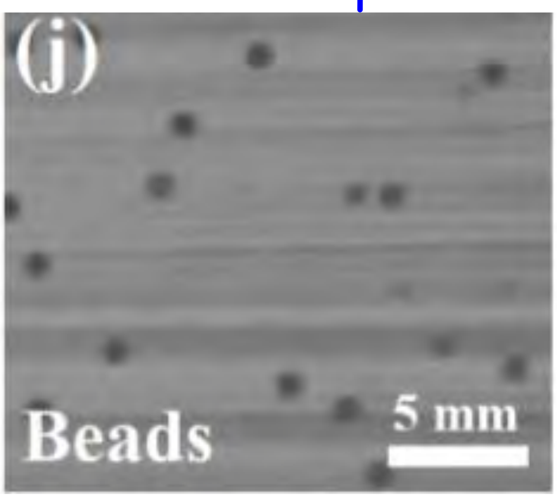
Seyyed Salili

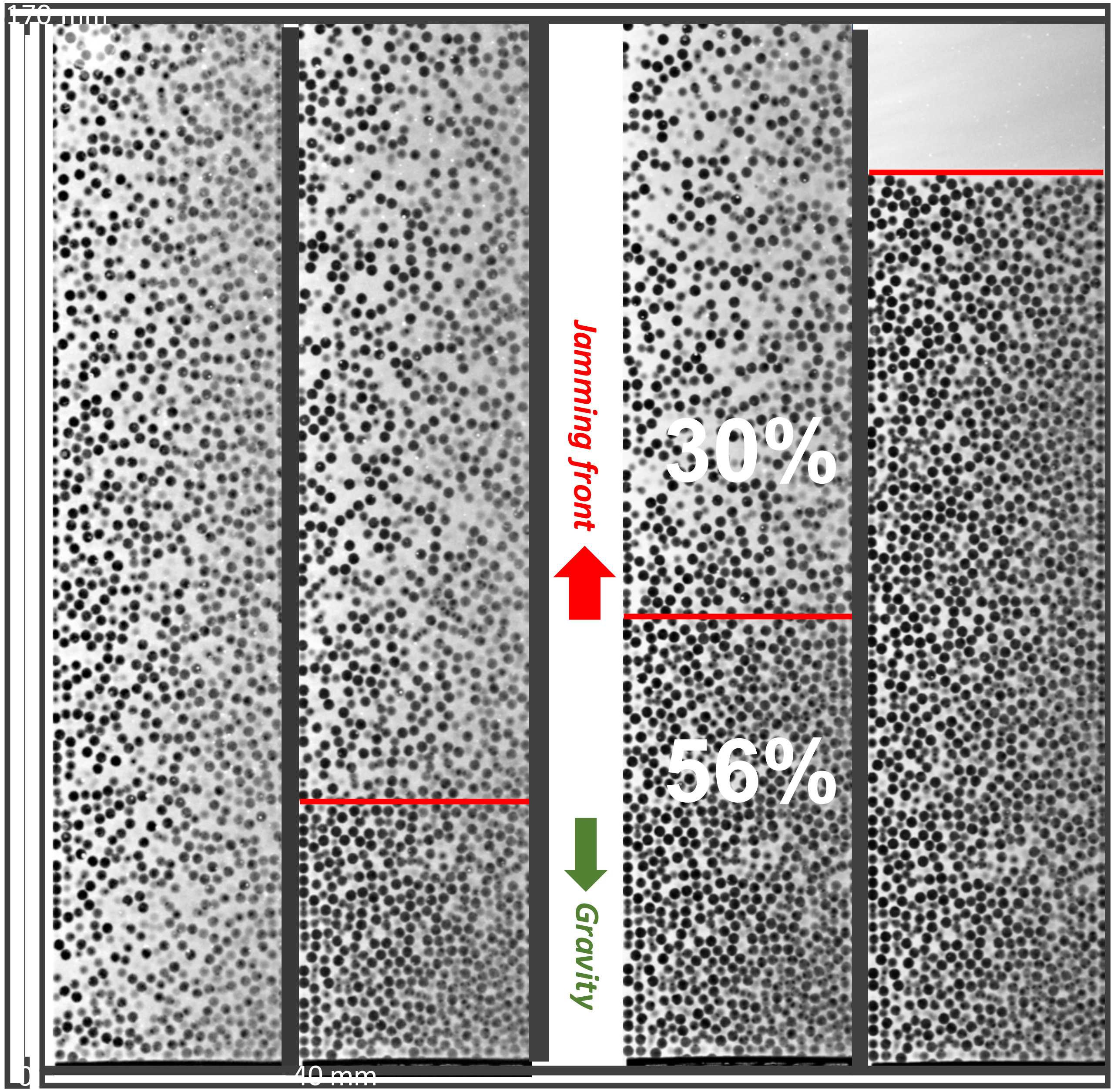
Preliminary data

- ***Raw*** images for non-Brownian PMMA beads:

  (Δn=0.002)

- NB: no stripes!

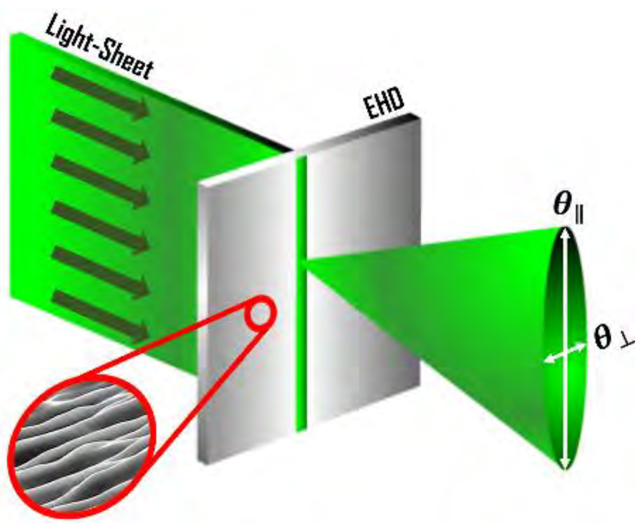


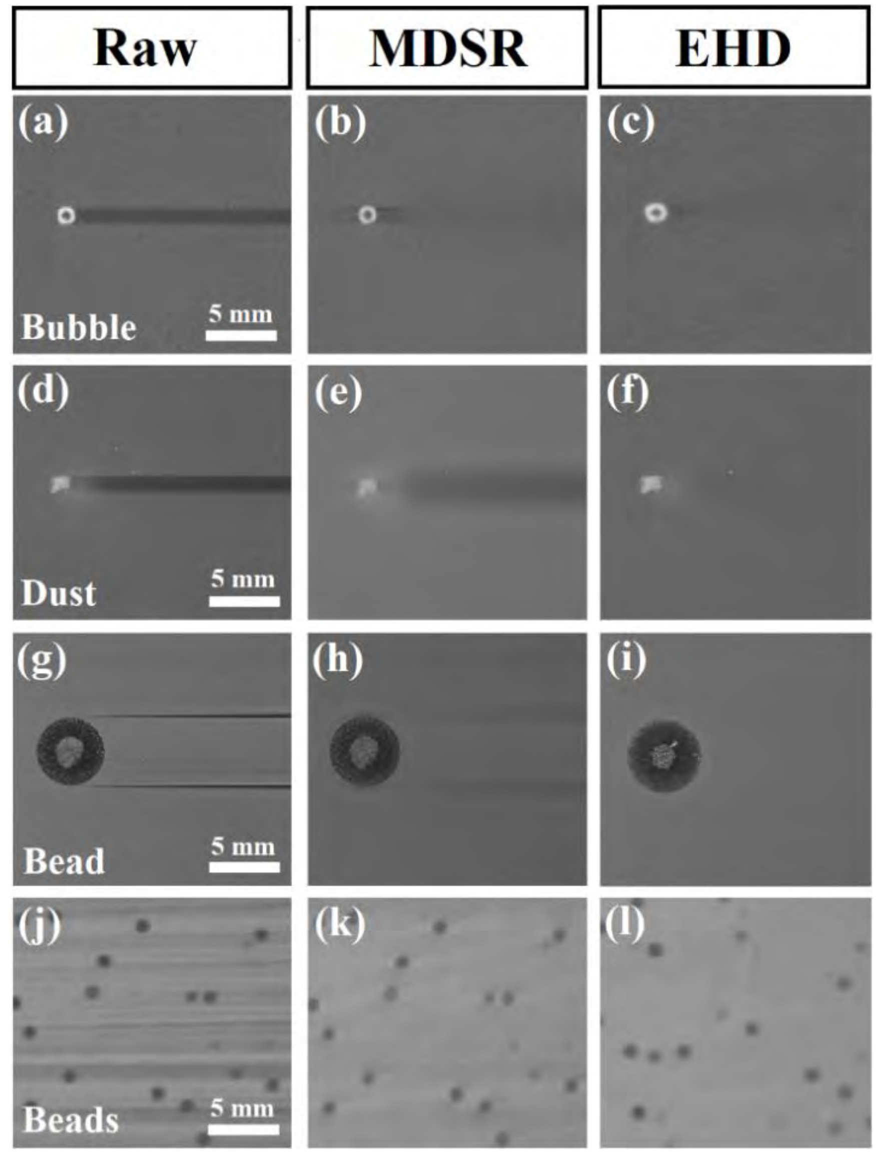


Elimination of stripe artifacts

- Our two methods:
  - MDSR algorithm
    - Filtering software
  - EHD
    - Multidirectional illumination

[Salili, Harrington, Durian arXiv:1711.07393]

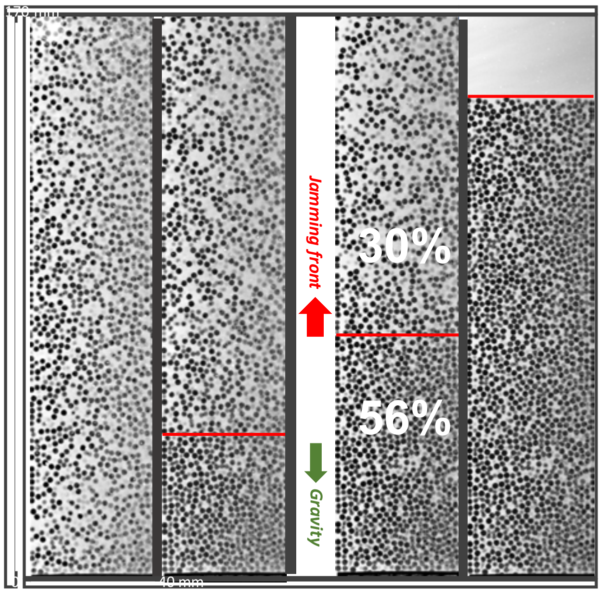


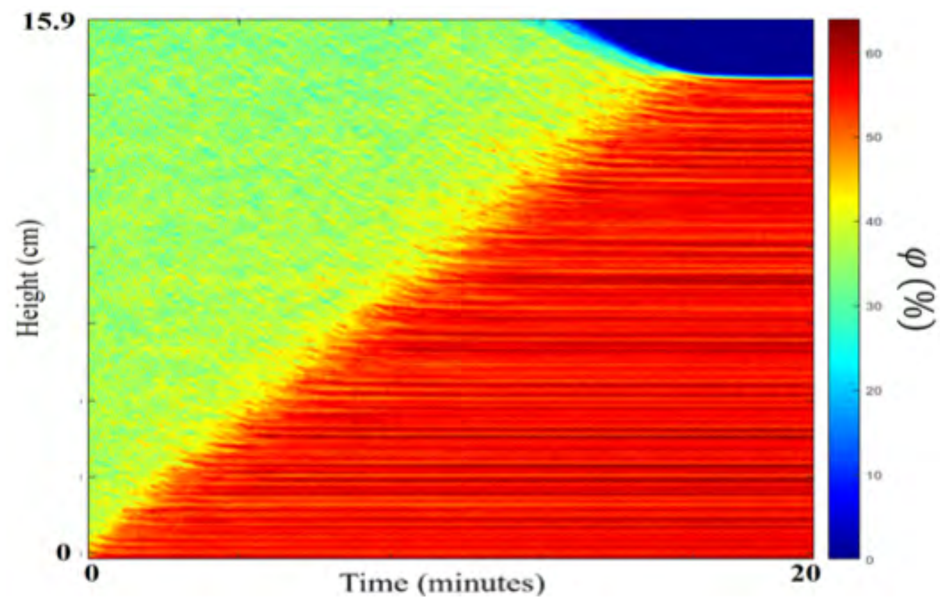


(Δn=0.002)

In progress…

- Deduce $\phi(z,t)$ and $v(z,t)$ fields, isolate asymptotics, compare with predictions for κ_1, κ_2 and with numerical solution for full profiles.

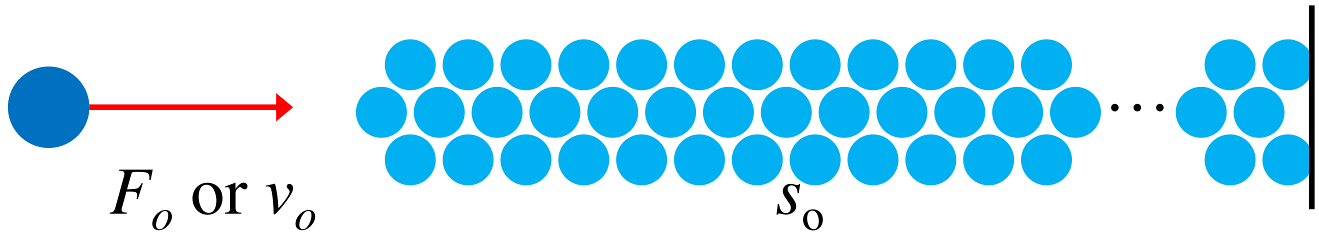




  - Repeat for other nonBrownian systems:
    - Different initial volume fractions
    - Polymeric (Boger) fluids

Other "fronts"

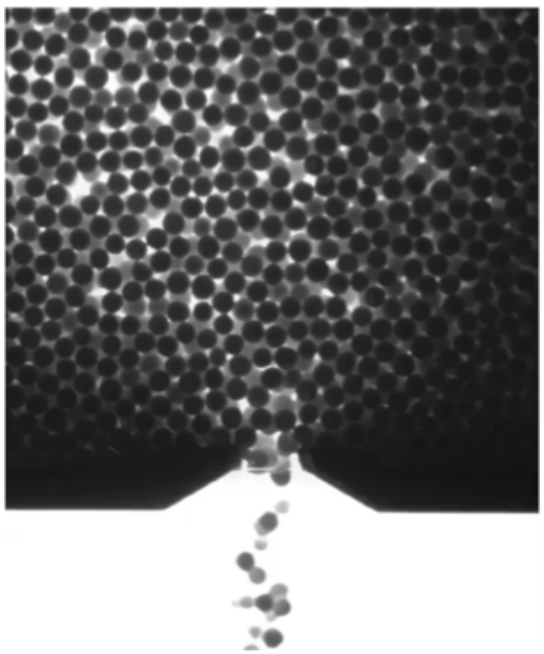
- Look elsewhere for non-local lubrication effects
  - Sedimentation: dispersion relation; densification front after fluidization speed is reduced; velocity & concentration fluctuations and their coupling
  - Impact:

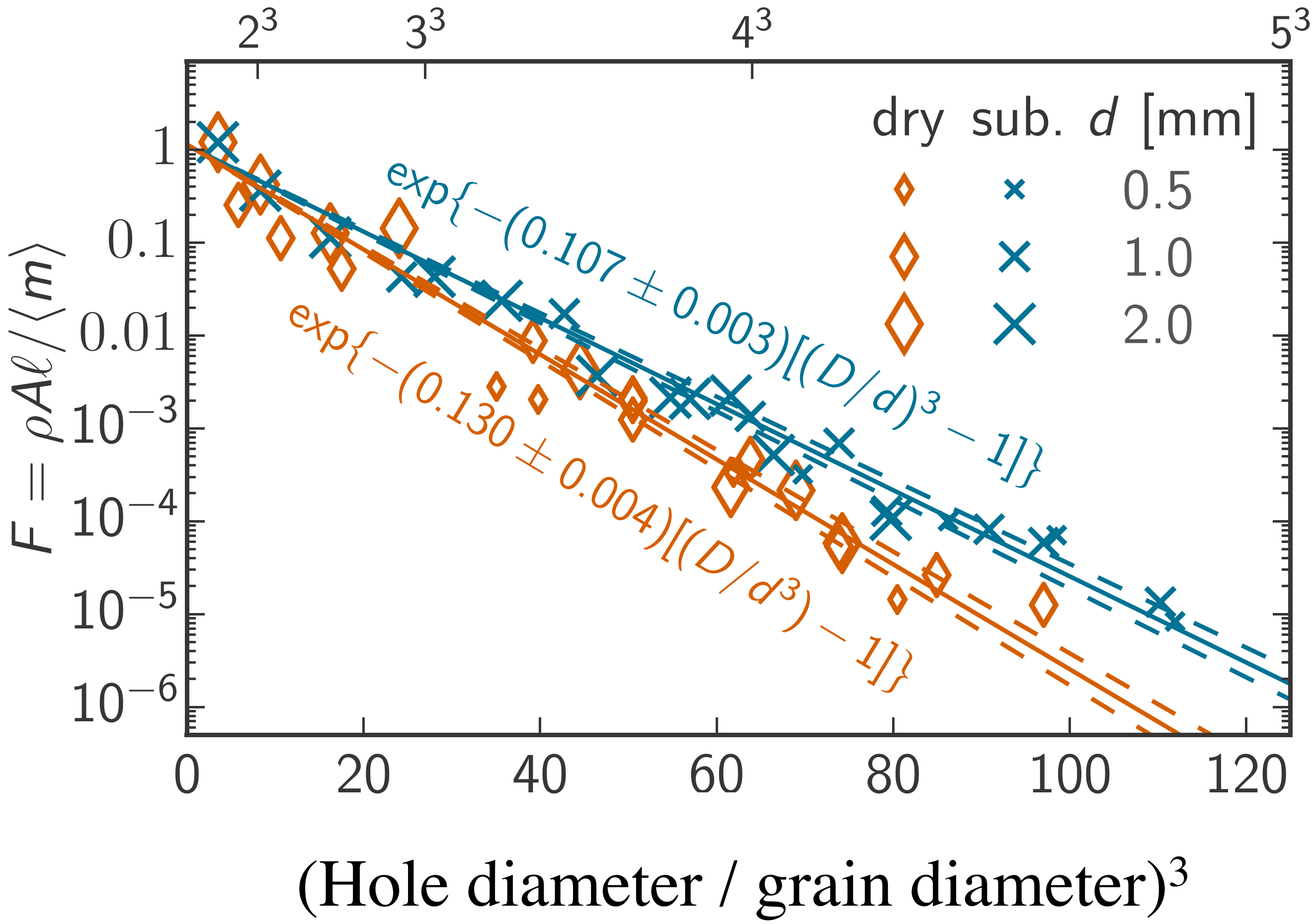


  - Evolution of $\phi(\mathbf{r},t)$ in flows with nonuniform shear (i.e. kinetics of particle migration in a pipe)
  - Clogging…

Granular clogging

- Fraction of flow microstates that cause a clog:





[Thomas-Durian PRL 2015; Koivisto-Durian Nat. Comm. 2017 & PRE 2017]

The END.

- New confidence in *two* hindered settling functions
- New expression for nonlocal lubrication force
  - Coupled PDEs for particle velocity & concentration fields:

$$\begin{aligned} 0 &= -\Delta m g \hat{\boldsymbol{z}} - \frac{6\pi\eta a \boldsymbol{v}}{H} \underbrace{- 6\pi\beta\eta_e a^3 \boldsymbol{\nabla}\left(\frac{\frac{\partial\phi}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\phi}{\phi_c - \phi}\right)}_{F_{lub}} \\ \frac{\partial\phi}{\partial t} &= \boldsymbol{\nabla}\cdot\left(-\boldsymbol{v} + D_c\boldsymbol{\nabla}\right)\phi \end{aligned}$$

  - Predicted width of jamming front: $1/\kappa \sim a[\beta\phi_o/(\phi_c-\phi_o)]^{1/2}$

- Comparing with data from new apparatus/technique…
- Looking for other nonlocal lubrication effects…

Ted Brzinski

Carlos Ortiz

Seyyed Salili

GSOFT

Topical Group on Soft Matter

www.aps.org/units/gsoft/

A new community of nearly 2000 soft matter scientists

What does GSOFT do?

- Soft matter sessions at March Meeting
- Travel/speaking/poster awards
- Early Career Award for Soft Matter Research
- APS Fellows nominations
- Other soft matter-related meetings

Why join as a student/post-doc?

- Travel grants for student speakers (up to $500)
- Student/post-doc poster prizes
- Short courses prior to March Meetings
  - 2018: “Short Course: Machine Learning and Data Science in Soft Matter”
- Get announcements about soft matter events
- Vote in GSOFT leadership elections
- Help GSOFT grow into an APS Division, so we can get more resources for soft matter!

How much does it cost?

- ***First year is free****
- Annual dues: $10.
- *Graduate students join 2 APS units for free.*
- APS membership: $37/yr for grad students

*send Email to djdurian@physics.upenn.edu

Background image by Qiong Tang, courtesy of Joe Zasadzinski