Many-body localized phase: dynamics and efficient numerical simulation

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[PRL 117, 160601(2016)]

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Ergodicity and integrability

**Ergodic systems**
- Chaos $\rightarrow$ Ergodicity

**Integrable systems**
- Stable to weak perturbations
- [Kolmogorov-Arnold-Moser theorem]

**Classical**
- Ergodic systems
- Integrable systems

**Quantum**
- Thermalizing phases
- MBL phases

"Toric cow" of non-ergodic systems
MBL: generic non-ergodic phase

- MBL = localized phase with interactions
  [Anderson, Fleishman’80]

  Perturbative arguments:  [Basko, Aleiner, Altshuler’05]  [Gorniy, Polyakov, Mirlin’05]

  Numerical evidence:      [Oganesyan, Huse’08]  [Znidaric, Prosen’08]  [Pal, Huse’10]

- Revived interest in MBL:
  - Experiments in cold atoms, ion chains…
  - Emergent integrability
    → universal non-ergodic dynamics
  - Breakdown of statistical mechanics
    → symmetry breaking at $T=\infty$,…
Towards understanding of MBL phase

I. Dynamics in MBL phase
   - Local integrals of motion
   - Entanglement growth and dephasing
   - New probes of dephasing dynamics

II. Highly excited MBL eigenstates
   - Structure of entanglement spectrum
   - Efficient numerical simulation with MPS

III. Summary and Outlook

[MS, Michailidis, Abanin, Papic, PRL 117, 160601(2016)]
I. Dynamics in MBL phase
Constructing local integrals of motion

\[ H_0 = \sum_i h_i S^z_i + J_z S^z_i S^z_{i+1} \]

\[ H = H_0 + \sum_i J_\perp (S^+_i S^-_{i+1} + h.c.) \]

Sequence of local unitaries:
\[
U^\dagger H U = H_{\text{diag}}
\]

Local integrals of motion
\[
[\hat{\tau}^z_i, H] = 0
\]

Effective spins form complete set
Emergent integrability

[MS, Papic, Abanin PRL’13]
[Huse, Nandkishore, Oganesyan PRB’14]
[Imbrie’14, PRL’16]

strong disorder RG:[Vosk&Altman, PRL’13]
Universal Hamiltonian of MBL phase

- If model is in MBL phase, apply sequence of local unitaries
  \[ H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + h_i S_i^z \]

- Hamiltonian expressed via \( \tau_i = U^\dagger S_i U \)

\[ \hat{H} = \sum_i H_i \tau_i^z + \sum_{ij} H_{ij} \tau_i^z \tau_j^z + \sum_{ijk} H_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots \]

- Effective spins cannot relax \( \rightarrow \) no transport
  Interactions \( \rightarrow \) dephasing & relaxation

\( H_{ij} \propto \exp(-|i - j|/\xi) \)
Entanglement growth from dephasing

\[ \hat{H} = \sum_i H_i \tau_i^z + \sum_{ij} H_{ij} \tau_i^z \tau_j^z + \ldots \]

\[ H_{ij} \propto J e^{-|i-j|/\xi} \]

- Phases randomize on distance \( x(t) \):

\[ tH_{ij} = tJ \exp(-x/\xi) \sim 1 \]

\[ x(t) = \xi \log(Jt) \]

- Logarithmic growth of entanglement

[MS, Papic, Abanin, PRL’13]

[Znidaric, Prosen, Prelovsek, PRB’08] [Bardarson, Pollmann, Moore, PRL’12]

**Q:** How to probe in experiment?
Local observables in a quench

- $\langle \tau^x(t) \rangle = \rho_{\uparrow\downarrow}(t) = \sum [N(t) = 2^{x(t)}$ oscillating terms$]$

- Decay of oscillations of $\langle \tau^x(t) \rangle$:

$$|\langle \tau^x_k(t) \rangle| \propto \frac{1}{\sqrt{N(t)}} = \frac{1}{(tJ)^a}$$

$$|\langle \hat{O}(t) \rangle - \langle \hat{O}(\infty) \rangle| \sim \frac{1}{t^a}$$

memory of initial state

[MS, Papic, Abanin, PRB’14]
Other probes of dephasing

- Modified spin echo protocol
  Quantum revivals, ...

All protocols assume: \[ \sigma_1^{\tilde{z}} \approx \tau_1^{\tilde{z}} \]

- How to probe “operator expansion”?

\[ \sigma_1^{\tilde{z}} = \sum \alpha_i \tau_i^{\tilde{z}} + \sum_{ij} \beta_{ij} \tau_i^{+} \tau_j^{-} + \sum_{ijk} \alpha_{ijk} \tau_i^{\tilde{z}} \tau_j^{\tilde{z}} \tau_k^{\tilde{z}} + \ldots \]

- Next: orthogonality catastrophe

[MS,Knap, et al., PRL’14]
[Vasseur, Parameswaran, Moore, PRB’15]
[MS&Abanin, in preparation]
Ramsey interferometry & operator expansion

- Impurity spin coupling: \( H_{\text{int}} = g \sigma_{\text{imp}}^z \sigma_{1}^z \)

- Initialize along \( x \), measure

\[
\langle \sigma_{\text{imp}}^x(t) \rangle = \text{Re} \langle \psi_0 | e^{i(H+g\sigma_{1}^z)t} e^{-i(H-g\sigma_{1}^z)t} | \psi_0 \rangle
\]

- Non-trivial dynamics comes from expansion:

\[
\sigma_{1}^z = \sum \alpha_i \tau_i^z + \sum_{ij} \beta_{ij} \tau_i^+ \tau_j^- + \sum_{ijk} \alpha_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots
\]

\[
\langle \sigma_{\text{imp}}^x(t) \rangle \approx \text{Re} \langle \psi_0 | e^{2igt(\sum \alpha_i \tau_i^z + \sum_{ijk} \alpha_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots)} | \psi_0 \rangle
\]

- Dephasing \( \rightarrow \) power law decay of fluctuations: \( | \langle \sigma_{\text{imp}}^x(t) \rangle | \propto \frac{1}{t^b} \)
Average vs fluctuations

\[ \langle \sigma_{\text{imp}}^x(t) \rangle \approx \text{Re} \langle \psi_0 | e^{2itg(\sum_i \alpha_i \tau_i^z + \sum_{ijk} \alpha_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots)} | \psi_0 \rangle \]

- Average: \( \alpha_i = \text{matrix elements} \rightarrow \text{universal function} \)
- Fluctuations: dephasing \( \rightarrow \text{power law decay} \)

Figure 3. Absolute value of the averaged coherence does not depend on the system size and interaction strength, and has a weak dependence on the disorder strength (solid lines correspond to \( W = 6.5 \) and dashed line to \( W = 7.5 \)). The numerical data agrees reasonably well with the theory suggesting the log-normal distribution of the localization length.
Off-diagonal terms in operator expansion

- Spin echo protocol:

\[
\langle \sigma_{\text{imp}}^x(t) \rangle_{\text{echo}} = \text{Re} \langle \psi_0 | e^{i(H-g\sigma_z^z)t} e^{i(H+g\sigma_z^z)t} e^{-i(H-g\sigma_z^z)t} e^{-i(H+g\sigma_z^z)t} | \psi_0 \rangle
\]

\[
\sigma_1^z = \sum \alpha_i \tau_i^z + \sum_{ij} \beta_{ij} \tau_i^+ \tau_j^- + \sum_{ijk} \alpha_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots
\]

- Spin-flip terms are less important:

![Graph showing the evolution of the spin-echo protocol with different system sizes (L = 8, L = 10, L = 12). The graph plots the time evolution of the spin-echo signal with logarithmic time axis.]
Probes of dephasing dynamics

- **Global probes**: quench, modified spin echo,…:

  \[ |\langle \hat{O}(t) \rangle - \langle \hat{O}(\infty) \rangle| \sim \frac{1}{t^a} \quad a \neq 0 \leftrightarrow \text{presence of interactions} \]

- **Local probes**: orthogonality catastrophe,…:

  \[ |\langle \sigma_{imp}^x(t) \rangle| \propto \frac{1}{t^b} \]
  
  decay depends on interactions

**Experimental challenge**: access fluctuations
II. Highly excited MBL eigenstates
From entanglement entropy to spectrum

• “Quantumness” of the pure state:

\[ \rho_L = \text{Tr}_R |\psi\rangle\langle\psi| \]

• Entanglement entropy: 
  \[ S_{\text{ent}} = - \sum_i \lambda_i \log \lambda_i \]
  *
  ground states: probes topological order  
  [Levin&Wen], [Kitaev&Preskill]
  *
  excited states: probes ergodicity

• Beyond entanglement? More information in \{\lambda_i\}
  [Li & Haldane]
Organization of entanglement spectrum

- Quantum Hall wave function:
  \[ k_y \text{ to organize ES} \]
  \[ \text{[Li & Haldane]} \]

- MBL phase: conserved quantities label ES

\[
|\uparrow\uparrow\uparrow\uparrow\uparrow\rangle = c_0 \, |\uparrow\uparrow\rangle|\uparrow\uparrow\rangle + e^{-\kappa} |\uparrow\downarrow\rangle|\uparrow\uparrow\rangle + e^{-2\kappa} |\uparrow\downarrow\rangle|\downarrow\uparrow\rangle + \ldots
\]

\[
\begin{array}{c}
|\uparrow\uparrow\rangle|\uparrow\downarrow\rangle + e^{-4\kappa} |\downarrow\downarrow\rangle|\downarrow\downarrow\rangle + \ldots \\
\end{array}
\]

- Coefficients decay as

\[
|C^{\uparrow\ldots\uparrow\downarrow\ldots\downarrow\ldots\uparrow}\rangle \propto e^{-\kappa r}
\]
Power-law entanglement spectrum

- Hierarchical structure of
  \[ \rho_L = \sum_{r=0}^{L/2} |\psi(r)\rangle \langle \psi(r)| \]

\[ \langle \psi(r) | \psi(r) \rangle \propto e^{-2\kappa r} \]
but non-orthogonal

- Orthogonalize perturbatively

\[ \lambda^{(r)} \propto e^{-4\kappa r} \]

multiplicity is \(2^r\)

\[ \lambda_k \propto \frac{1}{k\gamma}, \quad \gamma \approx \frac{4\kappa}{\ln 2} \]
Numerics for XXZ spin chain

- Spin chain in random field: $J_\perp = J_z = 1$

\[ H = \sum_i (h_i S_i^z + J_\perp S_i^+ S_{i+1}^- + h.c.) + \sum_i J_z S_i^z S_{i+1}^z \]

Disorder $W = 5$

\[ \lambda_k \propto \frac{1}{k^\gamma} \]

[MS, Michailidis, Abanin, Papic, PRL 117, 160601(2016)]
Decay of entanglement spectrum

- $\gamma$ controls decay of entanglement spectrum $\lambda_k \propto \frac{1}{k^\gamma}$

$$\gamma \approx \frac{4\kappa}{\ln 2}$$

perturbation theory

$$\kappa = 2\kappa' + \ln 2$$

Thouless conductance for MBL

$$G(L) \propto e^{-\kappa'L}$$

[MS,Papic,Abanin,PRX’15]

- Large value of $\gamma \rightarrow$ MPS description!

$$\frac{1}{\chi^{\gamma-1}} \approx \frac{1}{400^3} \approx 10^{-7}$$
Implementation of MPS algorithm

- Goal: access highly excited states
- “Shift-invert”: $H \rightarrow \frac{1}{(H - E)^2}$
- 50 DMRG-type sweeps; solve $|\psi_i\rangle = (H - E)^2|\psi_{i+1}\rangle$
- Conjugate gradient $\rightarrow$ large bond dimensions $\chi=400$

more details: [PRL 117, 160601(2016)]

Entanglement spectrum as a test

- Large bond dimensions are necessary close to transition
- DMRG underestimates entanglement spectrum for

\[ \lambda_k \geq e^{-15} \approx 10^{-6} \]
Estimates for the bond dimension

- To converge $S_{ent}$ up to 1%:
  - $\chi = 400 \rightarrow$ eigenstates close to MBL transition

**Q:** What can we learn from this?
Future directions

- Phase transitions within MBL phase
- MBL with fermions, $S>1/2$, bosons, etc.
- Structure of many-body resonances that drive transition?

Phenomenological RG: [Vosk,Huse,Altman,PRX’15] [Potter,Vasseur,Parameswaran,PRX’15]
Exact diagonalization: [Khemani et al, arXiv:1607.05756]
“Hot region” inside MBL phase

Identify structure of generic resonance from ES?

\[
\langle \ln \lambda_k \rangle = 2^{\ell/2} \quad \text{Power-law}
\]
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Outline and perspectives

- Orthogonality catastrophe in MBL phase:
  → power-law decay
  → probe relation between $\hat{\tau}_i$ and $\hat{S}_i$

- Power-law entanglement spectrum in MBL:
  $\lambda_k \propto \frac{1}{k^\gamma}$
  → power $\gamma \leftrightarrow$ scaling of matrix elements

→ MPS algorithm close to transition

[MS, Abanin in preparation]

III. DECAY OF SPIN COHERENCE WITH TIME

A. Analytic considerations

B. Numerical simulations

IV. SUMMARY AND OUTLOOK

Appendix A: Understanding time-averaged coherence

Figure 1.

Figure 2.