# Anderson localization and delocalization in 2D electron systems with strong spin-orbit interaction

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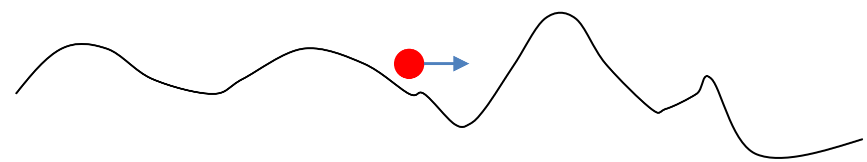
Ilya Gruzberg (Chicago)
Andreas Ludwig (Santa Barbara)

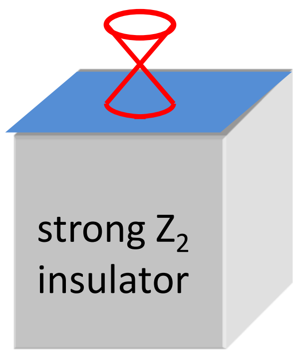
# Anderson localization

electron in random potential



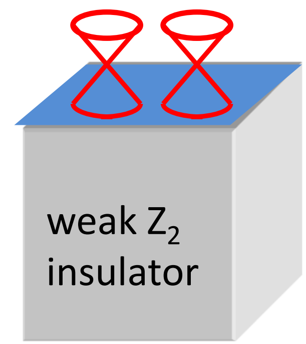
P.W. Anderson (1958)

(1) strong topological insulator

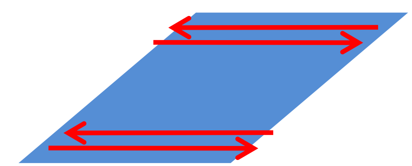



Never localized!
(topological metal)

(2) weak topological insulator




or



2D Z2 top. insulator

Localization-delocalization transition (in the standard symplectic class)

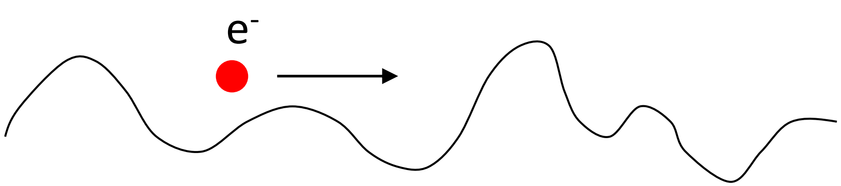
# Outline

- Anderson localization: short review
  - Scaling theory
  - before Quantum Spin Hall Effect (QSHE)
  - QSHE (& surface of weak topological insulators)
- Anderson delocalization
  - Nonlinear sigma model with Z2 topological term
  - Surface Dirac fermions of strong topological insulators

# Anderson localization

P.W. Anderson (1958)

a non-interacting electron moving in a random potential



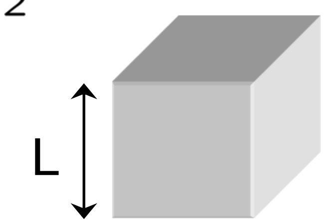

Scaling theory AALR (1979)

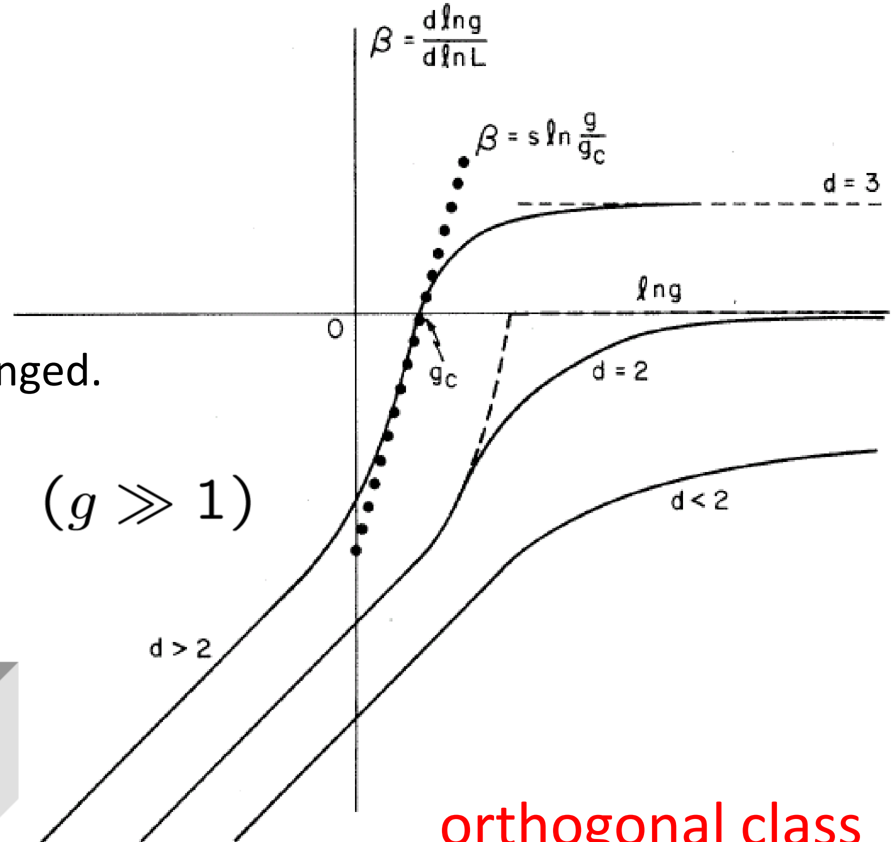
Conductance ***g*** changes as system size ***L*** is changed.

$$\beta(g) = \frac{d \ln g}{d \ln L} = d - 2 - \mathcal{O}(g^{-1}) \quad (g \gg 1)$$

Metal: $g \propto \frac{\text{area}}{\text{length}} = L^{d-2}$

Insulator: $g \propto e^{-L/\xi}$




orthogonal class

The metal-insulator transition at $g=g_c$ is continuous.

# Universality classes of disordered electron systems

- Dimensionality of space $d = 2$
- Symmetry of Hamiltonian
  time-reversal symmetry
  SU(2) rotation symmetry in spin space

3 standard classes (Wigner-Dyson random matrix theory)

| | time reversal symmetry | spin rotation | |
|---|---|---|---|
| orthogonal | ○ | ○ | $T^2 = +1$ |
| unitary | × | ○ / × | |
| symplectic | ○ | × | $T^2 = -1$ |

Prog. Theor. Phys. Vol. 63, No. 2, February 1980, Progress Letters

**Spin-Orbit Interaction and Magnetoresistance in the Two Dimensional Random System**

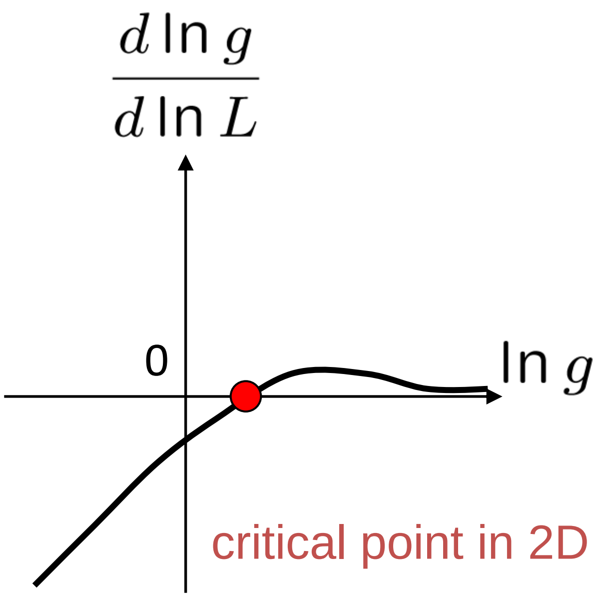
Shinobu HIKAMI, Anatoly I. LARKIN*) and Yosuke NAGAOKA
*Research Institute for Fundamental Physics*
*Kyoto University, Kyoto 606*

(Received November 5, 1979)

symplectic class: ○ time-reversal, × spin-rotation
spin-orbit interaction

anti-localization correction

$$\frac{d \ln g}{d \ln L} = d - 2 + \frac{c}{g} \qquad c > 0$$




Metal-insulator transition in 2D

Always localized in 1D

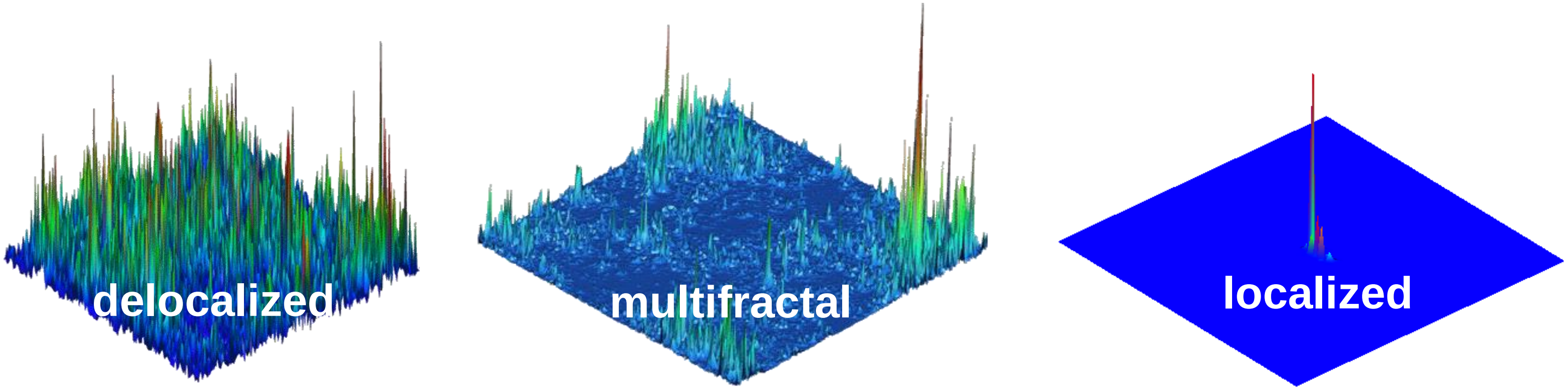
# Anderson transition (metal-insulator transition)

$d = 2$

Continuous phase transition induced by disorder

localization length $\xi \to \infty$ $\xi \sim |E - E_c|^{-\nu}$

Numerical studies (finite-size scaling) $\nu \approx 2.7$ (Asada, Ohtsuki & Slevin, 2002)




metallic phase critical point insulating phase

Conformal invariance has some consequence in multifractal spectra:
Obuse, Subramaniam, AF, Gruzberg & Ludwig, PRL 2007

# Some developments before QSHE (1)

$N \gg 1,\ L/l \gg 1\ \ (Nl/L \text{ fixed})$

(1-a) Nonperturbative calculculation for NLSM in Quasi 1D (thick wire limit)
Zirnbauer PRL (1992); Mirlin, Muller-Groeling & Zirnbauer, Ann. Phys. (1994)

$$\langle g \rangle = \frac{1}{s} + \frac{1}{3} + \cdots, \quad s \ll 1 \qquad \left( s \equiv \frac{2L}{Nl} \right)$$

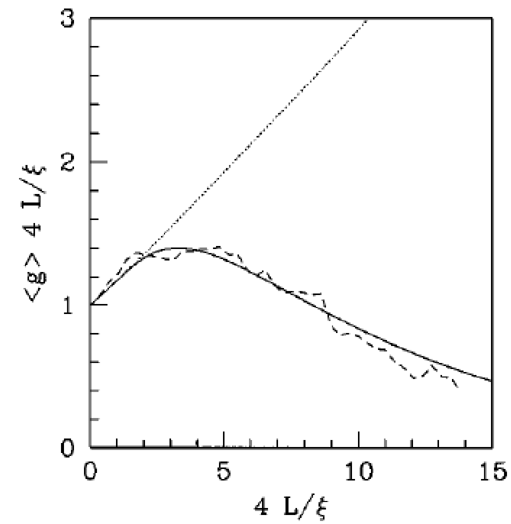
weak anti-localization

$\langle g^2 \rangle - \langle g \rangle^2$ was obtained as well

$$\langle g \rangle = \frac{1}{2} + C s^{-3/2} e^{-s/4} + \cdots, \quad s \gg 1$$

implies existence of a conducting channel
(in contradiction to the scaling theory
and the DMPK approach)

(1-b) exponential decay of ***g*** recovered by discarding zero-mode contributions
(Brouwer & Frahm, PRB 1996)



# Some developments before QSHE (2)

(2-a) (metallic) carbon nanotubes: Ando & Suzuura, JPSJ 2002

There is a perfectly conducting channel when disorder potential is smooth.

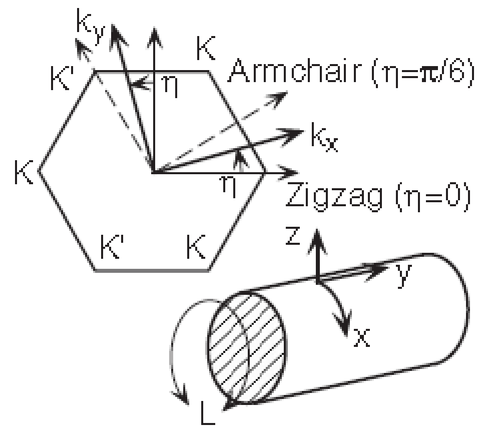
(scattering matrix elements between two valleys can be ignored.)

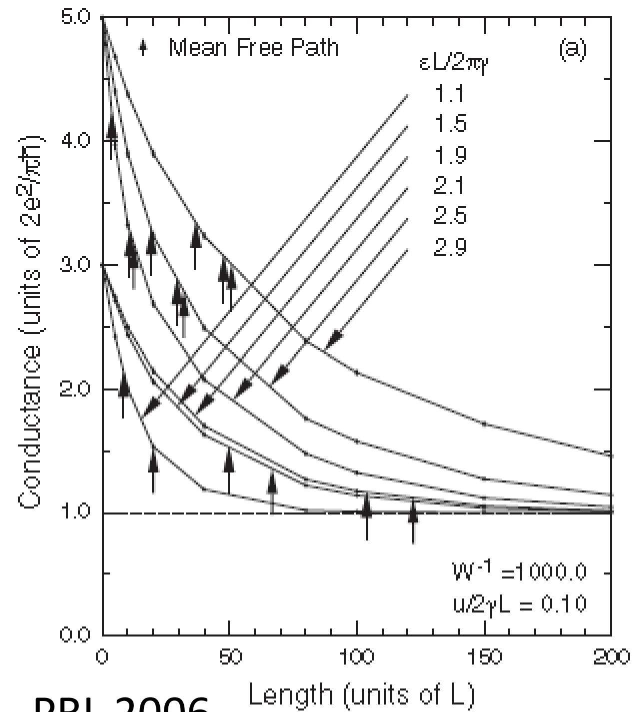
spin up & down = sublattice A & B

"time-reversal symmetry" $T^2 = -1$

Symplectic class

Berry phase $\pi$




(2-b) graphene: Suzuura & Ando, PRL 2002; McCann et al., PRL 2006

weak anti-localization for smooth potential

$$\Delta\sigma_{xx} = \frac{2e^2}{\pi^2\hbar}\ln\left(\frac{\ell_\phi}{\ell}\right)$$

inter-valley scattering ⟹ weak localization (orthogonal class)

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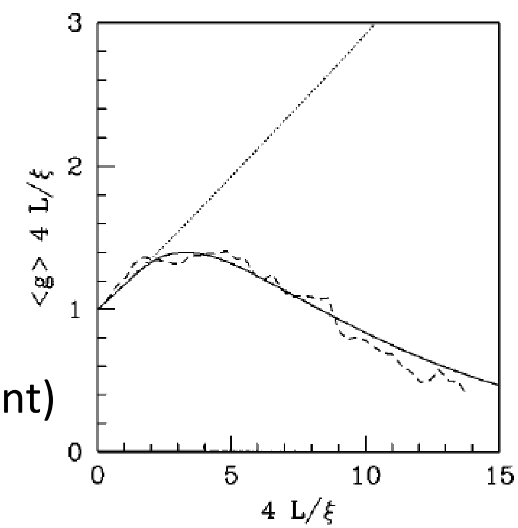
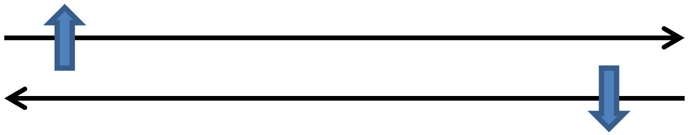



(1-c) significance of the parity of the number N of Kramers' pairs of conducting channels (Takane, JPSJ 2004, ..)

Odd N: a perfectly conducting channel (zero mode is important)
Even N: Brouwer-Frahm's result

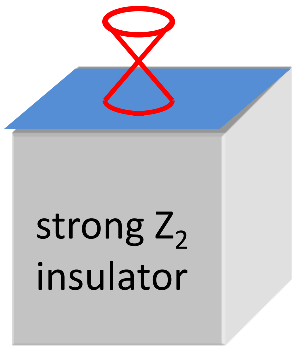
Metallic carbon nanotubes (2 Dirac points & up, down spins)

= 4 × edge states (a Kramers' pair) of 2D $Z_2$ topological insulator (QSHE)



Graphene (2 Dirac points & up, down spins)

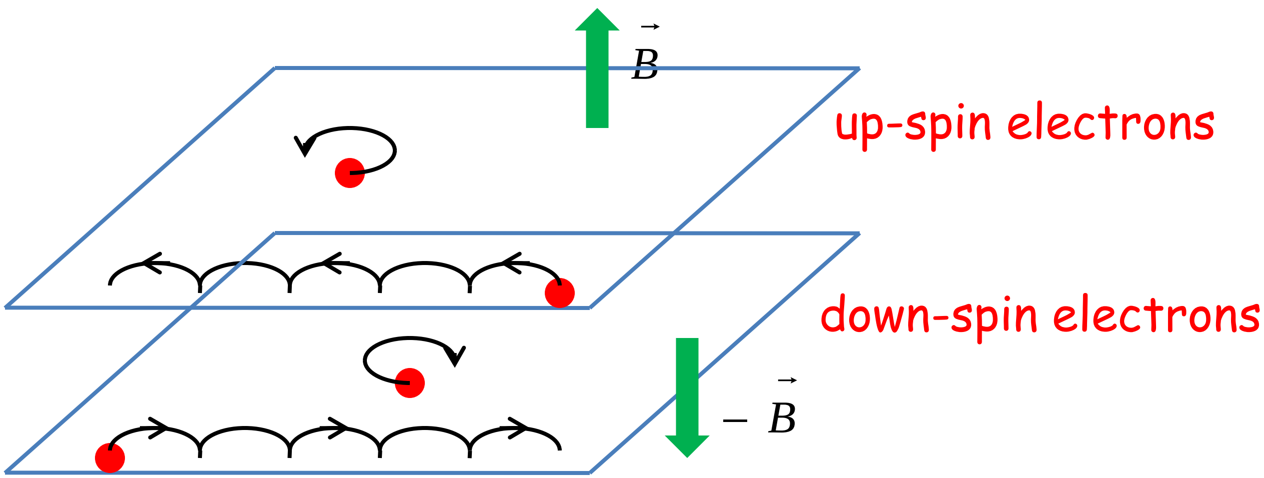
= 4 × surface Dirac (Weyl) fermions of 3D $Z_2$ topological insulator



# Quantum spin Hall effect ($Z_2$ top. Insulator)

Kane & Mele (2005, 2006); Bernevig & Zhang (2006)

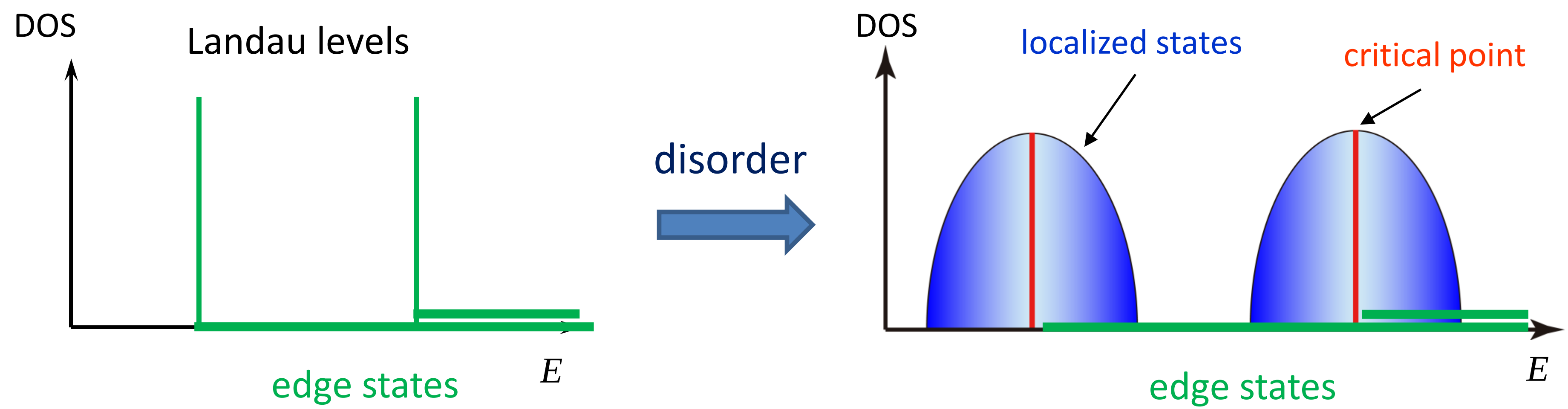
- Time-reversal invariant band insulator
- Strong spin-orbit interaction $\lambda \vec{L} \cdot \vec{S} \Rightarrow \left(\vec{p} \times \vec{E}\right) \cdot \vec{S}$
- Gapless helical edge mode (Kramers pair)



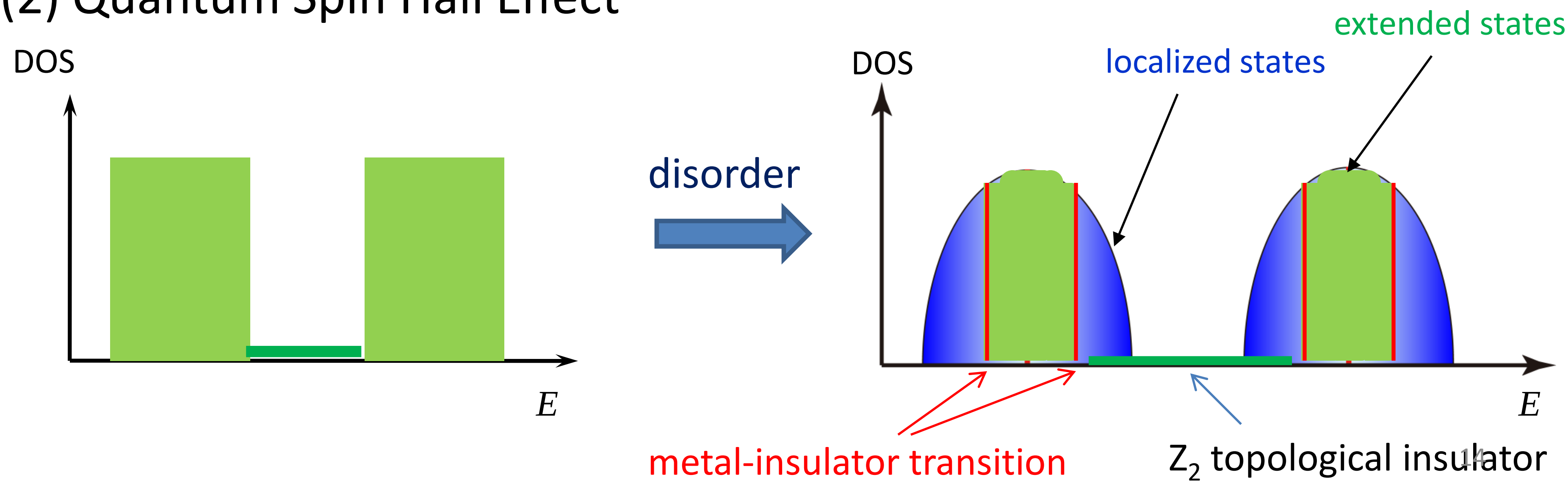

If $S_z$ is NOT conserved, Chern # (Z) ➡ $Z_2$ symplectic class

# Disorder effects

## (1) Integer Quantum Hall Effect



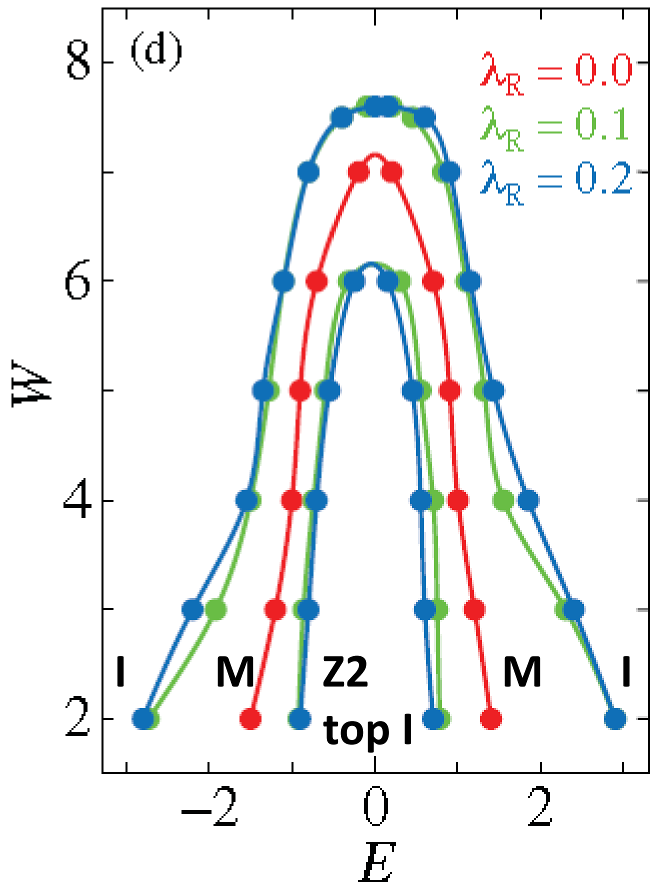

## (2) Quantum Spin Hall Effect



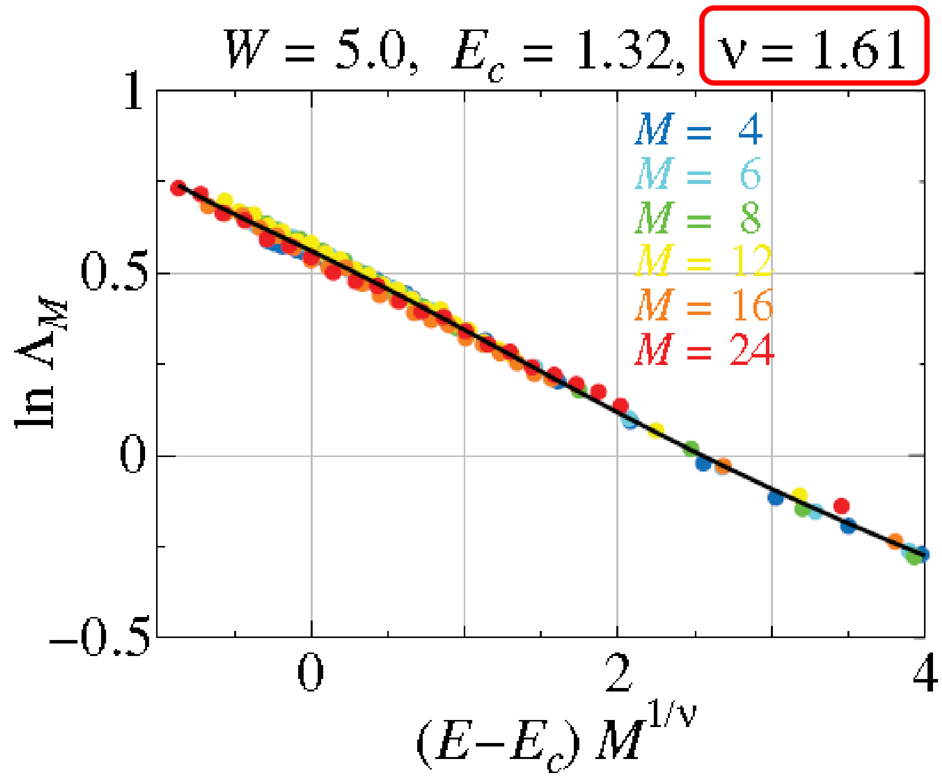
# Kane-Mele model with on-site disorder

Numerical simulations     Onoda, Avishai & Nagaosa, PRL (2007)

Phase diagram



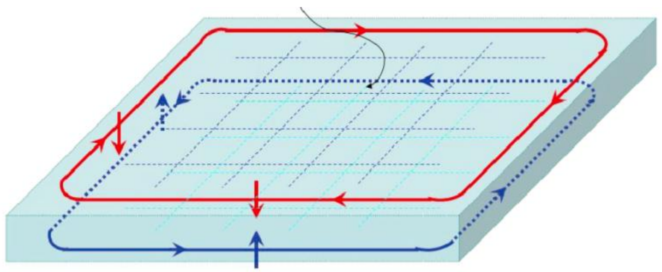

Finite-size scaling




small system size
poor statistics

# Network model for quantum spin Hall effect

Obuse, AF, Ryu & Mudry, PRB 76, 075301 (2007)



Chalker-Coddington network model for up-spin electrons

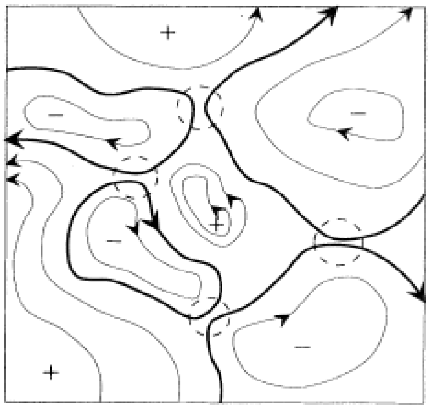
+ opposite chirality

Chalker-Coddington network model for down-spin electrons

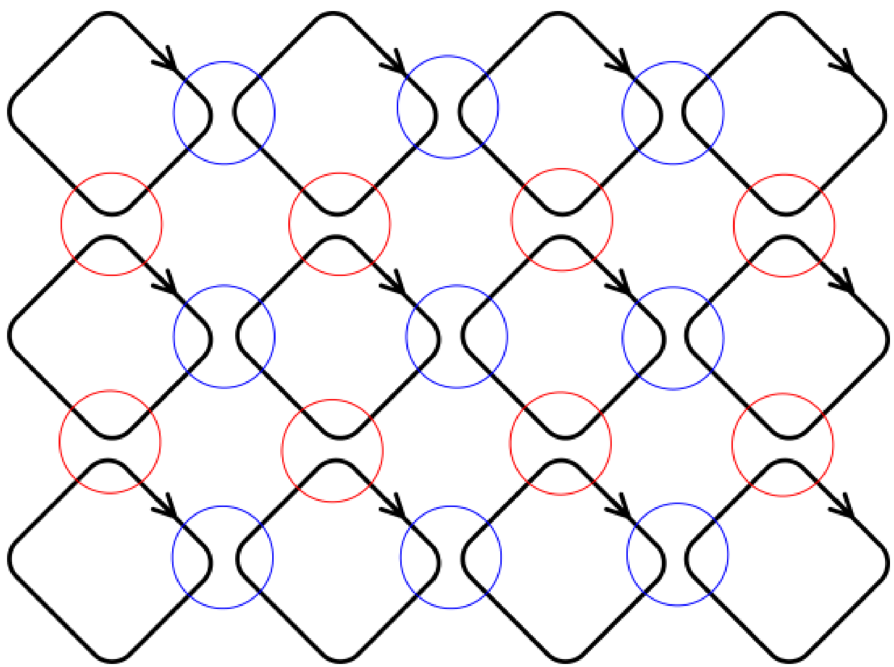
Coupled by general spin-dependent scattering vertices
that respect time-reversal symmetry

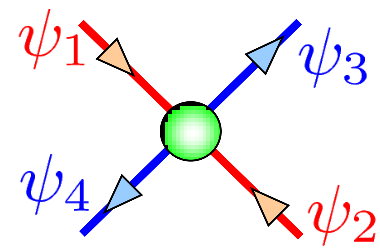
# Chalker-Coddington Network model

Chalker, Coddington (1988)



electron moving along equipotential lines

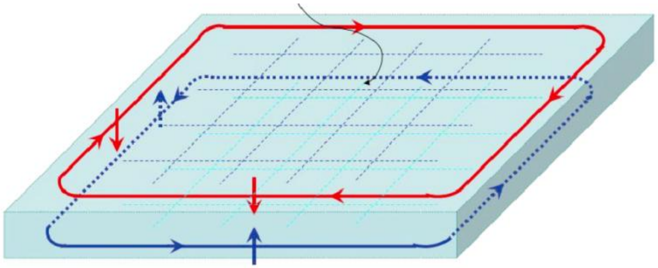




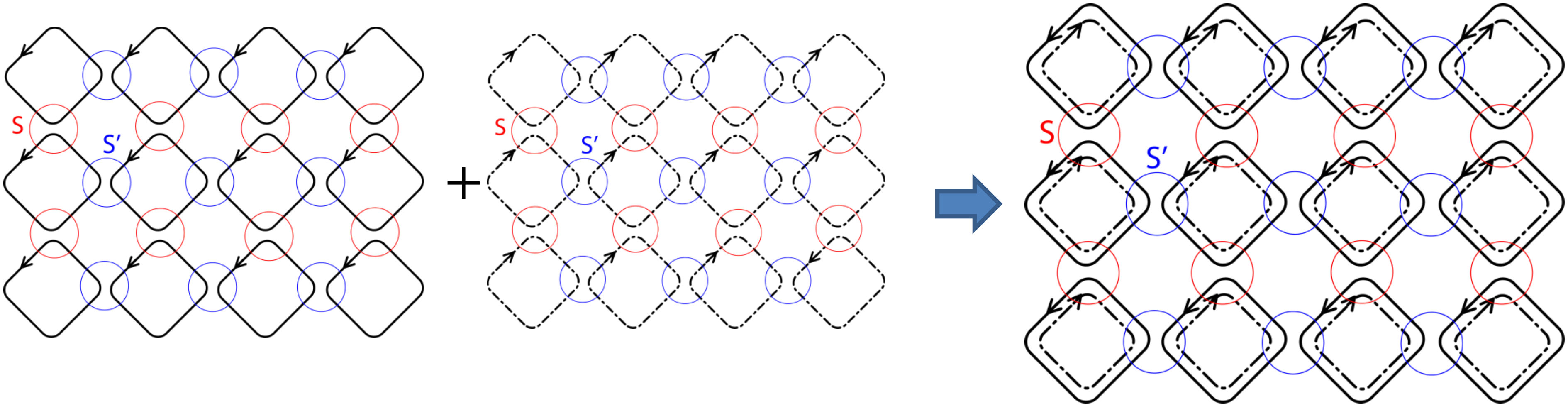

$$\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = S \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$S$ : unitary matrix

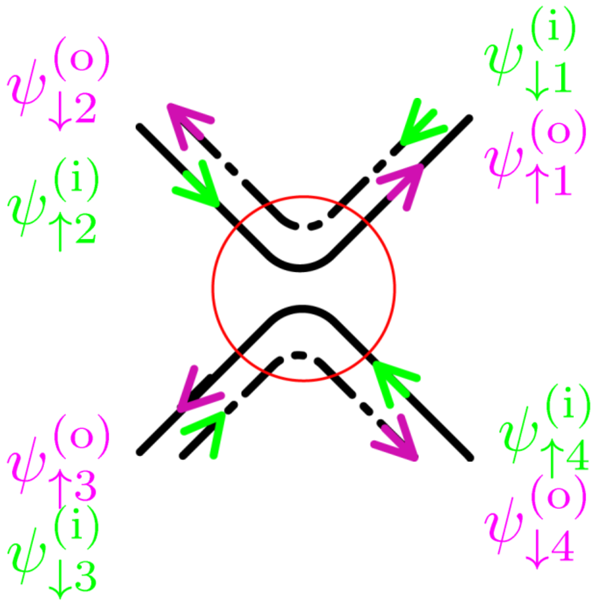
# Network model for QSHE (symplectic class)



2 coupled Chalker-Coddington networks of opposite chiralities



# S matrix




$$S = \begin{pmatrix} r\sigma_0 & tQ \\ -tQ^+ & r\sigma_0 \end{pmatrix} \qquad r^2 + t^2 = 1$$

spin flip

$$Q = \begin{pmatrix} e^{i\varphi_1}\cos\theta & e^{i\varphi_2}\sin\theta \\ e^{-i\varphi_2}\sin\theta & -e^{-i\varphi_1}\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \psi^{(o)}_{\uparrow 1} \\ \psi^{(o)}_{\downarrow 2} \\ \psi^{(o)}_{\uparrow 3} \\ \psi^{(o)}_{\downarrow 4} \end{pmatrix} = S \begin{pmatrix} \psi^{(i)}_{\uparrow 2} \\ \psi^{(i)}_{\downarrow 1} \\ \psi^{(i)}_{\uparrow 4} \\ \psi^{(i)}_{\downarrow 3} \end{pmatrix}$$

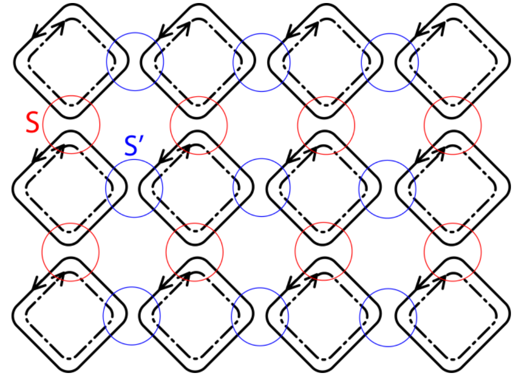
Time reversal symmetry:

$$S = \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix} S^T \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}$$

$\phi_{1,2}$ : random

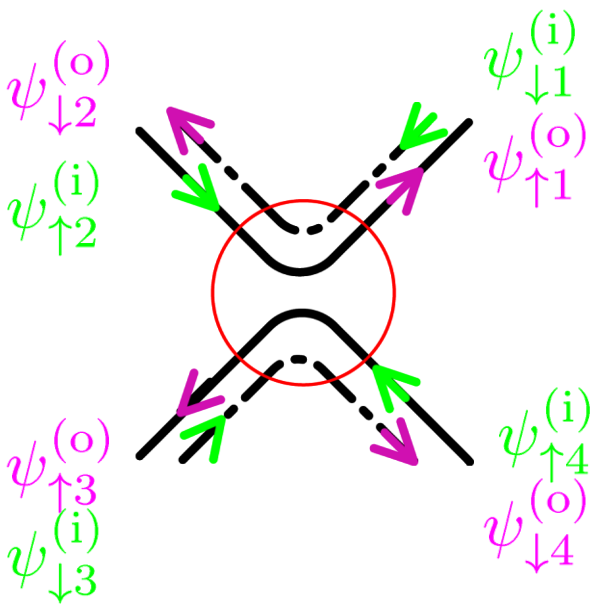
$\theta$: (i) fixed (ii) random

$r = \tanh x$ : fixed



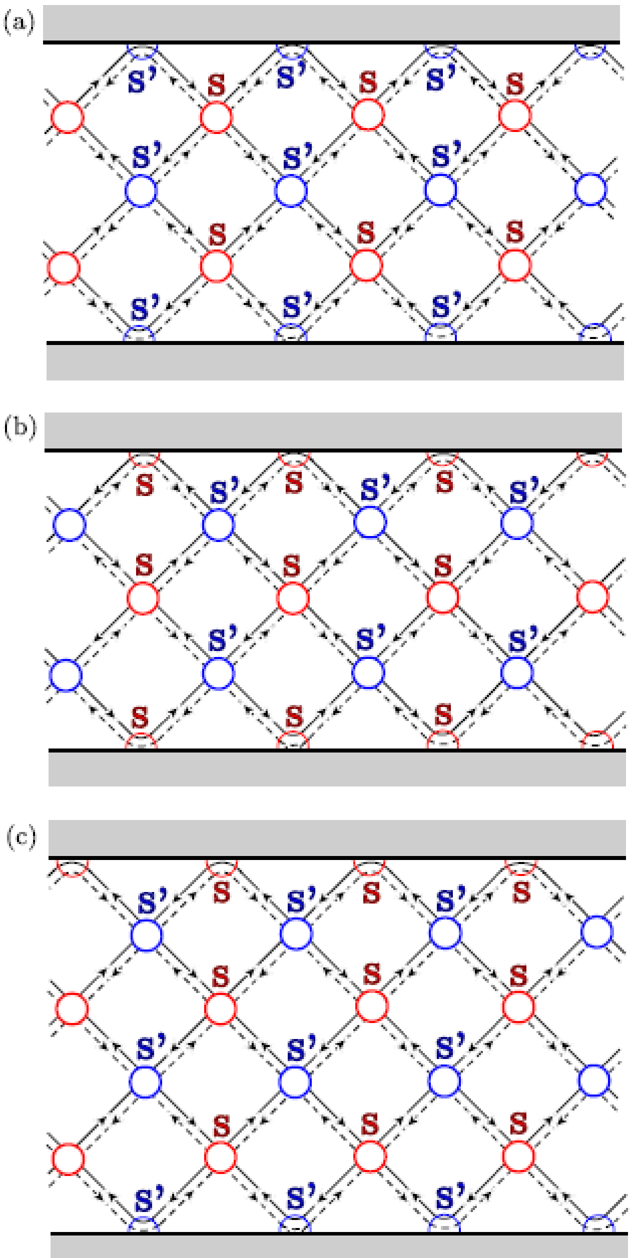

S

S'

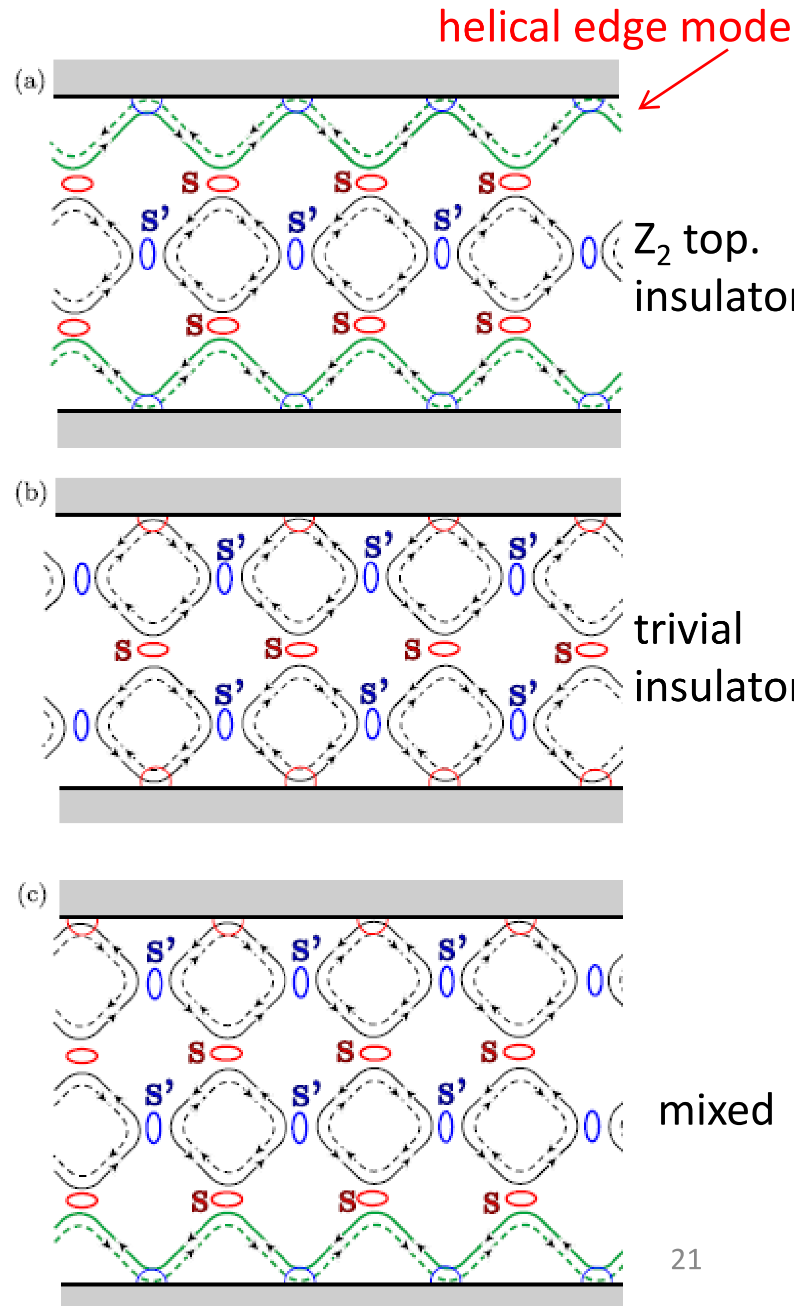



$90^{o}$ rotation

# Trivial (Insulating) limit: boundary conditions




$$S \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



The critical exponent $\nu$ is a bulk property. $\xi \approx \left| X - X_c \right|^{-\nu}$

Bulk properties should not depend on boundary conditions.

$\nu$ should be the same for

metal-to-(trivial insulator) transition

and

metal-to-($Z_2$ top. Insulator) transition.

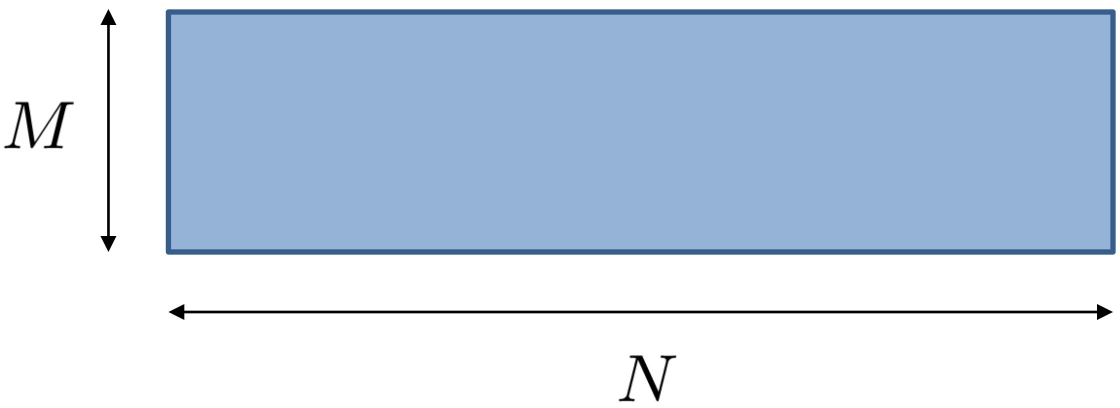
# Numerical simulation in quasi 1D geometry: localization length

Transfer matrix method for quasi-1d geometry $\quad y < 0$

$$\Lambda = \xi_M / M = F(\chi M^{1/\nu}, \zeta M^{y}, \ldots)$$

Finite-size scaling

$$M = 4, 8, 16, 32, 64 \qquad N = 5 \times 10^5 \sim 8 \times 10^6$$




$$\Lambda \sim \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} F_{p,q} \zeta^p \chi^q M^{py+q/\nu}$$

$$\Lambda' := \Lambda - \sum_{q=0}^{2} f_{1,q}^{(\theta)} \left( x - x_c^{(\theta)} \right)^q M^{y+q/\nu}$$

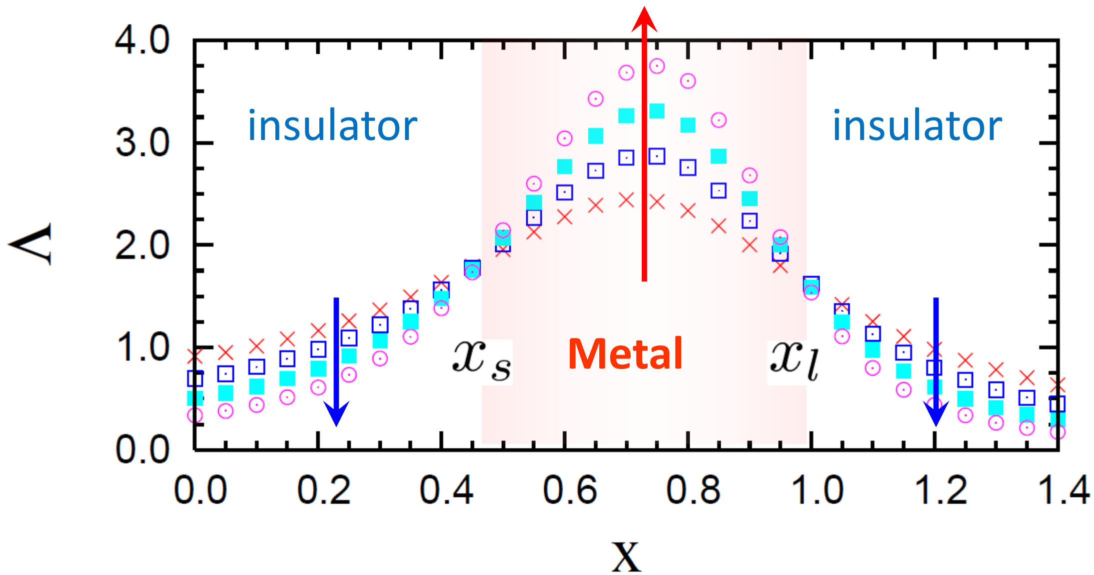
$$= \sum_{q=0}^{3} f_{0,q}^{(\theta)} \left( x - x_c^{(\theta)} \right)^q M^{q/\nu}.$$

Boundary conditions in the transverse direction:

(a) Periodic (cylinder)

(b) reflecting (strip)

# Numerical simulation in cylinder geometry (periodic b.c.)

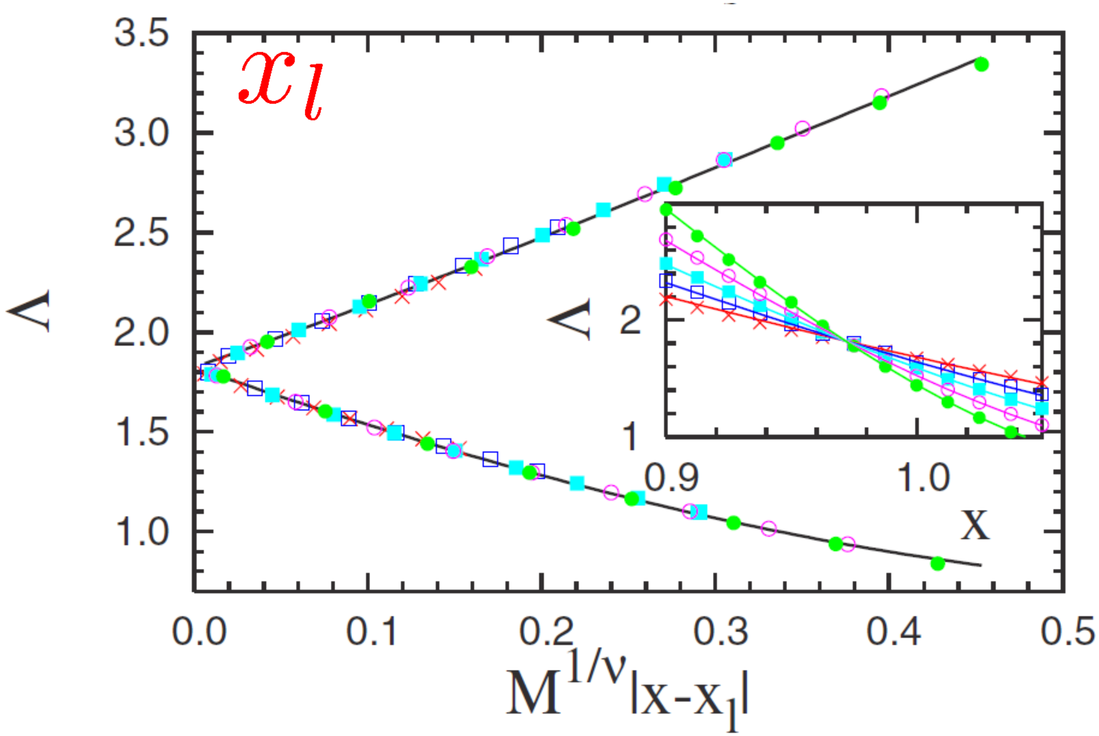



$$\theta = \frac{3\pi}{16}$$

$$r = \tanh x, \quad t = \frac{1}{\cosh x}$$

$\Lambda$ : Localization length divided by the transverse width in quasi 1D

2 critical points: $x_s, x_l$



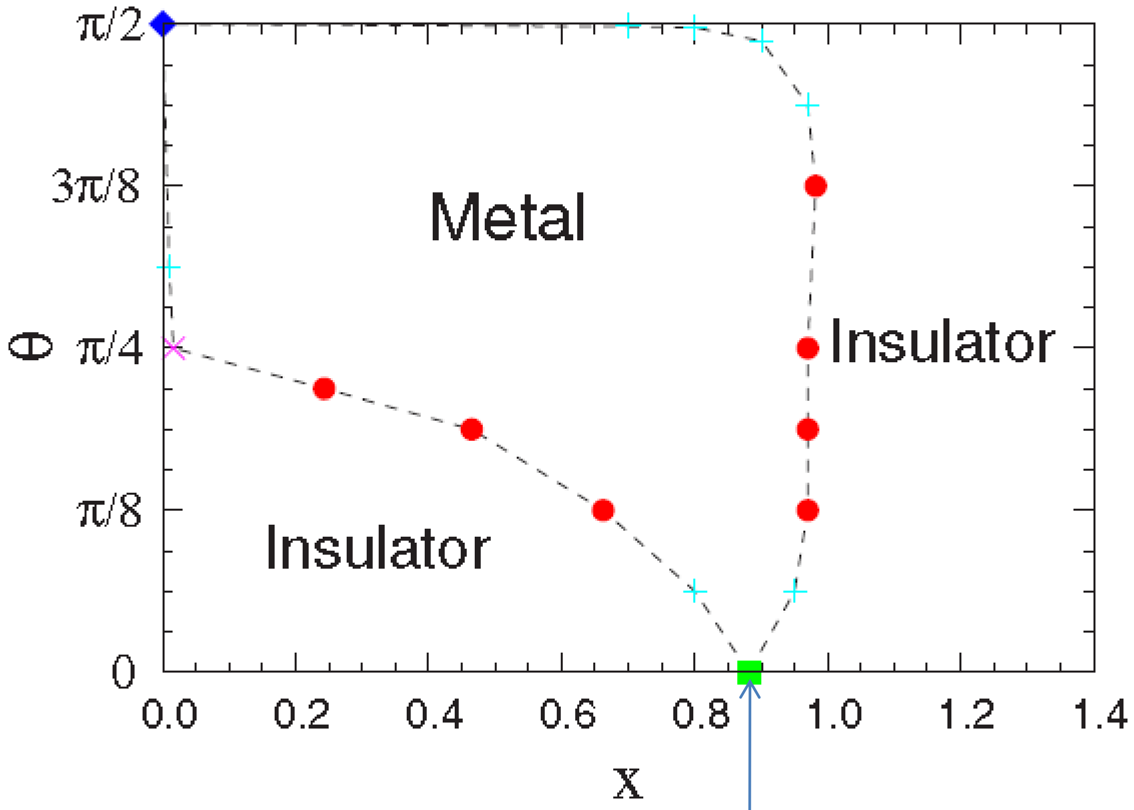

Finite-size scaling

$$\nu \approx 2.7$$

In agreement with the known value for the symplectic class (Asada, Slevin & Ohtsuki)

Obuse, AF, Ryu, Mudry, PRB ('07)

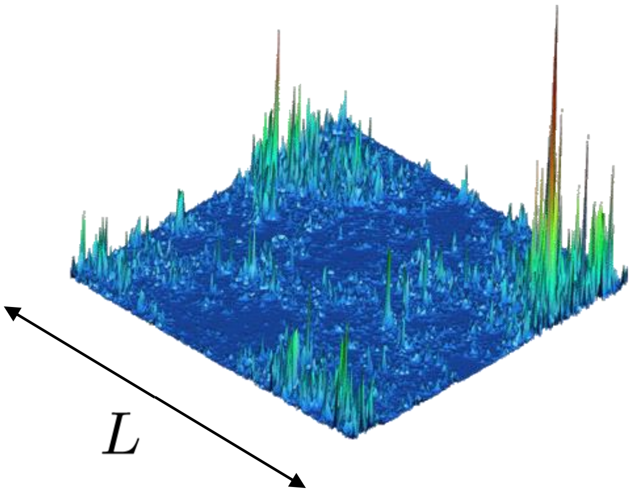
# Phase diagram

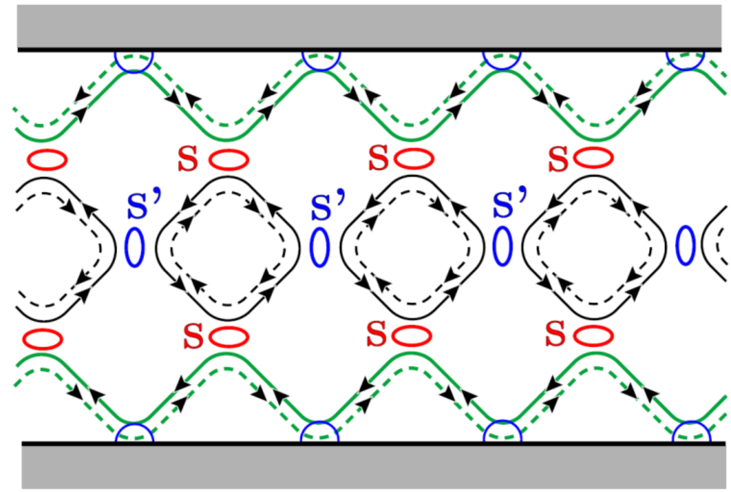



IQH plateau transition (unitary class)

# Multifractality: scaling behavior of moments of critical wave functions

network model (Obuse, AF, Ryu & Mudry, PRB 2008)



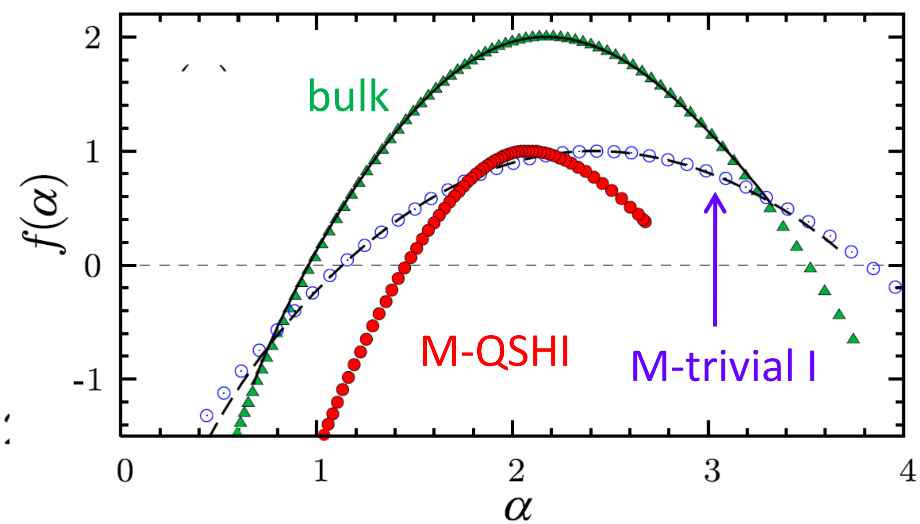




multifractal exponents $\tau_q$

In a metal
$|\psi(r)|^2 \sim L^{-d}$

$$L^d \, \overline{|\psi(r)|^{2q}} \sim L^{-\tau_q}$$

bulk & boundary multifractality




singularity spectrum

$$f(\alpha) = q\alpha - \tau_q \qquad \alpha = \frac{d\tau_q}{dq}$$

$N_\alpha \sim L^{f(\alpha)}$ : measure of $r$ where $|\psi(r)|^{\text{:}}$

# “Hamiltonian” formalism

$$S_{tot} = \exp\left(-iH\Delta t\right) \Rightarrow H = i\log S_{tot}$$

Expansion around the limit of two decoupled CC network models at criticality yields

$$H_4 = \begin{pmatrix} H_+ & \alpha\sigma_0 \\ \alpha\sigma_0 & H_- \end{pmatrix} \qquad 4\times 4$$

$$H_\pm = \sigma_x\left(-i\partial_x \pm A_x\right) + \sigma_y\left(-i\partial_y \pm A_y\right) \pm \sigma_z m + \sigma_0 A_0$$

$A_x,\ A_y$ random vector potential

$A_0$ random scalar potential

$m$ random mass

$$\mathrm{TRS}\ :\ -i\sigma_y\tau_x H^* i\sigma_y\tau_x = H$$

Note 1. CC network model $\Rightarrow$ $2\times 2$ Hamiltonian $H_+$ (Ho & Chalker PRB 1996)

Note 2. surface of 3D $Z_2$ top. Insulator: $2\times 2$ Hamiltonian $H_2 = -i\partial_x\sigma_x - i\partial_y\sigma_y + \sigma_0 A_0$

# Lattice Dirac fermions with on-site disorder

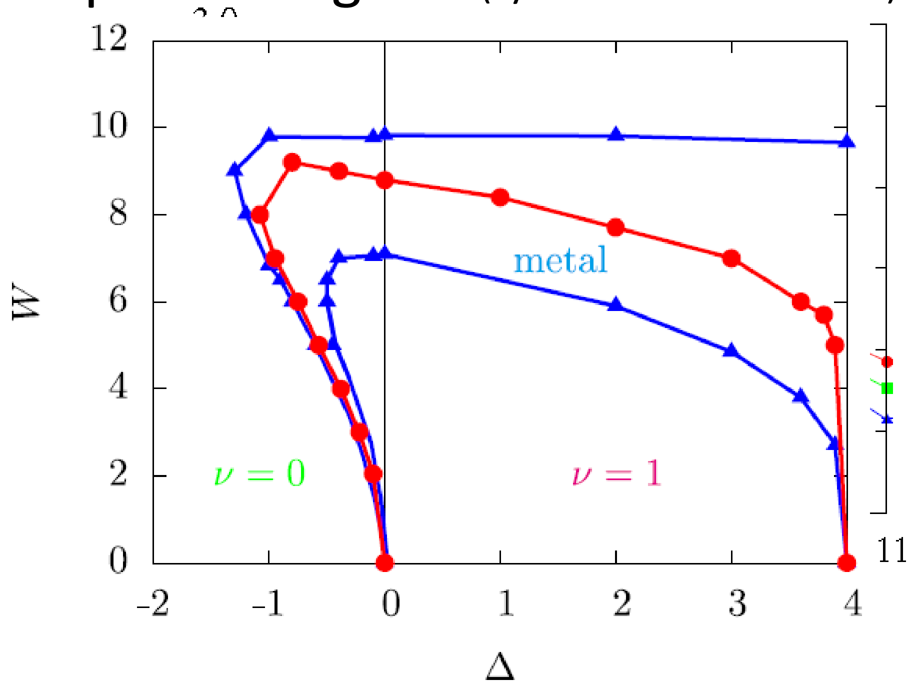
Yamakage, Nomura, Imura & Kuramoto, JPSJ 2011

$$H_{\vec{k}} = \begin{pmatrix} h_{\vec{k}} & \Gamma_{\vec{k}} \\ \Gamma^{\dagger}_{\vec{k}} & h^{*}_{-\vec{k}} \end{pmatrix} \qquad \Gamma_{\vec{k}} = i\alpha \begin{pmatrix} \sin k_x - i \sin k_y & 0 \\ 0 & \sin k_x + i \sin k_y \end{pmatrix}$$

$$h_{\vec{k}} = \vec{d}_{\vec{k}} \cdot \vec{\sigma} \qquad \vec{d}_{\vec{k}} = \left(\sin k_x, \sin k_y, \Delta - 2\left(2 - \cos k_x - \cos k_y\right)\right)$$

on-site disorder $\varepsilon_{\vec{r}} = \mathrm{diag}\left(W^{(1)}_{\vec{r}}, W^{(2)}_{\vec{r}}, W^{(1)}_{\vec{r}}, W^{(2)}_{\vec{r}}\right)$ BHZ+Rashba+disorder

phase diagram (symmetric about $\Delta = 4$)




Red: $\alpha = 0$ (2 × IQHE, unitary class)
Blue: $\alpha = 0.5$ (symplectic class)

# Outline

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  - 1980s
  - before Quantum Spin Hall Effect (QSHE)
  - QSHE (& surface of weak topological insulators)
- Anderson delocalization
  - Nonlinear sigma model with Z2 topological term
  - Surface Dirac fermions of strong topological insulators

# Nonlinear sigma model: (Wegner, Efetov, Altshuler-Kravtsov-Lerner, …)

low-energy effective field theory for Nambu-Goldstone bosons

$$E = \int (\partial \vec{n})^2 \, dx \, d\tau + i 2\pi S N \qquad\qquad E = \int \mathrm{tr}(\partial Q)^2 \, d^2 r + i\theta N$$

| | Antiferromagnets | Integer Quantum Hall effect |
|---|---|---|
| N-G bosons | magnons | Diffuson |
| Ordered phase | antiferromagnetic | metallic |
| Disordered phase | paramagnetic | insulating |
| Order parameter | $\vec{n} \in R^3 \quad \vec{n} \cdot \vec{n} = 1$ | $Q \in \mathrm{U}(2N) \quad Q \approx \mathrm{diag}(1_N, -1_N)$ |
| Target space | $G/H = \mathrm{O(3)/O(2)}$ | $G/H = U(2N)/U(N) \times U(N)$ |
| | $\pi_2(G/H) = Z$ | $\pi_2(G/H) = Z$ |
| | Haldane | Pruisken |

Topological terms lead to nonperturbative effects.

# IQHE (and 1d Antiferromagnet)

$$\pi_2(G/H) = \pi_2(U(2N)/U(N) \times U(N)) = Z$$ (Pruisken, 1983)

topological sectors labeled by an integer

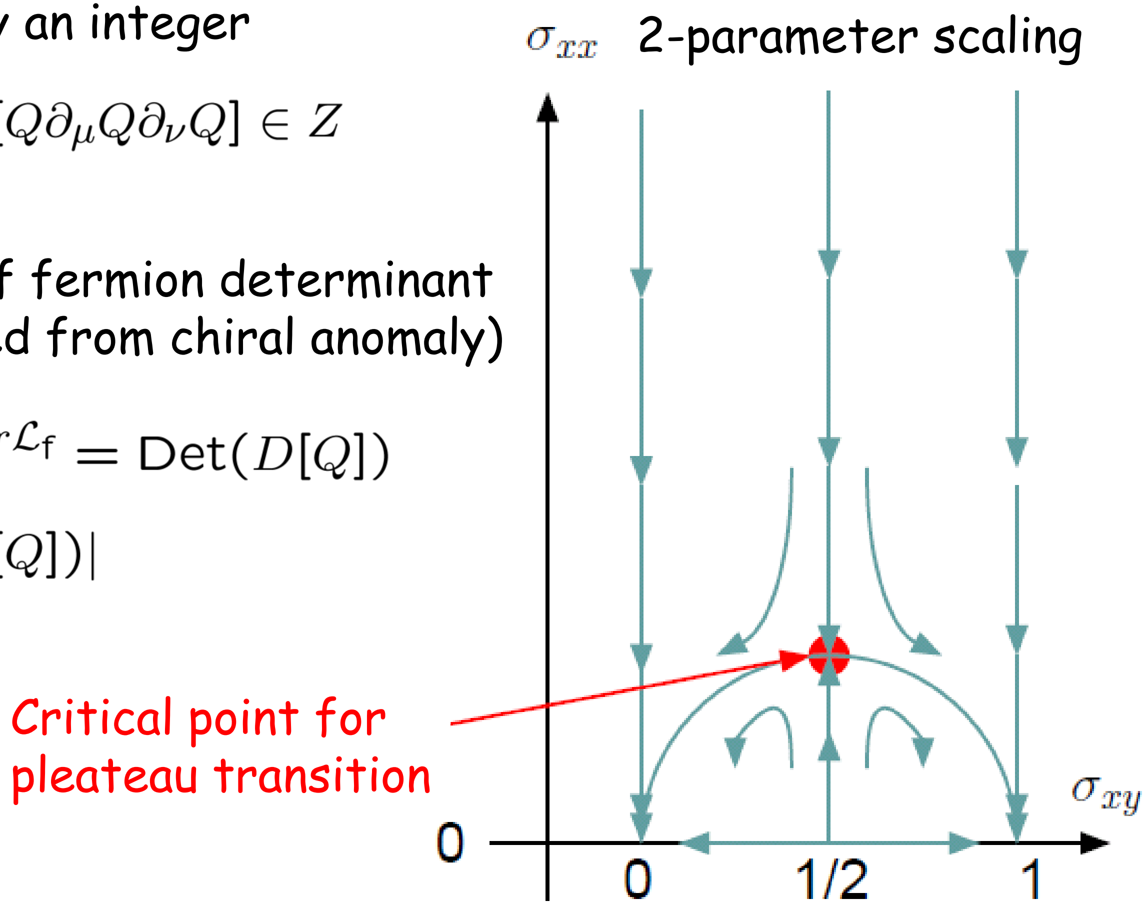
$$\text{Ch}[Q] := \frac{1}{16\pi i} \int d^2 r \epsilon_{\mu\nu} \text{tr}[Q \partial_\mu Q \partial_\nu Q] \in Z$$

topological term as a phase of fermion determinant
(can be obtained from chiral anomaly)

$$e^{-S_{\text{eff}}[Q]} = \int \mathcal{D}[\bar{\psi}, \psi] e^{-\int d^2 r \mathcal{L}_{\text{f}}} = \text{Det}(D[Q])$$
$$= e^{iS_{\text{top}}[Q]} |\text{Det}(D[Q])|$$

theta angle can be tuned

$$\theta = \sigma_{xy}/(e^2/h)$$



# Nonlinear sigma model for the symplectic class

N-G bosons — Diffuson & Cooperon

Ordered phase — metallic

Disordered phase — insulating

Matrix fields $\quad Q^2 = 1_{4N}, \quad Q^T = Q, \quad \mathrm{Tr}\, Q = 0$

Target space $\quad G/H = \mathrm{O(4N)/O(2N)} \times \mathrm{O(2N)}$

$$\pi_2(G/H) = Z_2$$

Fendley, PRB (2001)

2 distinct sectors in the space of field configurations


$$e^{-S_1} + e^{-S_2} \quad \text{or} \quad e^{-S_1} - e^{-S_2}$$

(no top. term) (with $Z_2$ top. term)

# A single Dirac fermion with random scalar potential

$$H = -iv\left(\sigma_x \partial_x + \sigma_y \partial_y\right) + V(x, y)\sigma_0$$

$$\overline{V(\vec{r})} = 0, \quad \overline{V(\vec{r})V(\vec{r}\,')} = g\delta(\vec{r} - \vec{r}\,')$$

time-reversal symmetry $i\sigma_y H^* \left(-i\sigma_y\right) = H$

(fermionic) replica, disorder averaging, H-S decoupling with a matrix field,
Integrating out fermions, gradient expansion around a saddle point

Nonlinear sigma model with Z2 topological term

$$S = \frac{1}{t}\int d^2 r \ \mathrm{tr}\left(\partial Q\right)^2 + i\pi n[Q]$$

Ryu, Mudry, Obuse, & AF, PRL 2007

Ostrovsky, Gornyi, & Mirlin, PRL 2007

$$\mathrm{e}^{i\pi n[Q]} = \pm 1$$

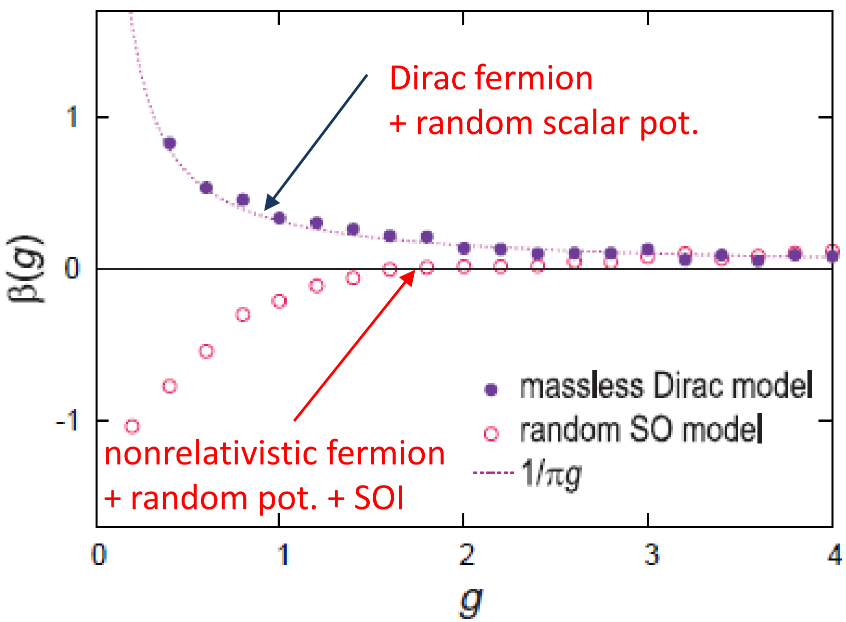
Similar to the NLSM at the IQH plateau transition $\theta = \pi$

→ gapless spectra (no localization)

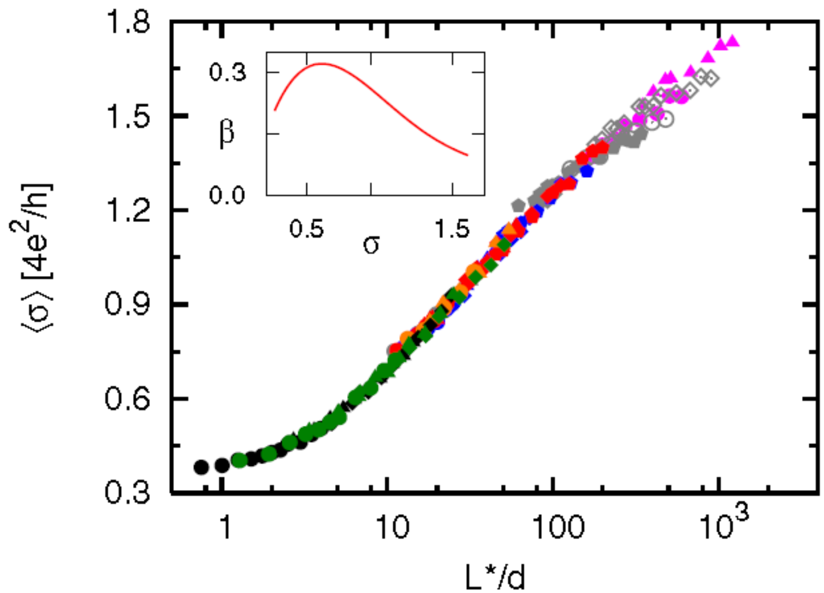
$$H = -iv\left(\sigma_x \partial_x + \sigma_y \partial_y\right) + V(x, y)\,\sigma_0$$

## Direct numerical calculations of the beta function $\beta(g) = \dfrac{d \ln g}{d \ln L}$

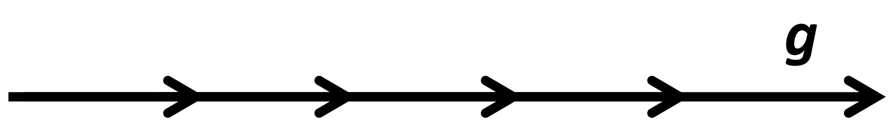
Nomura, Koshino, & Ryu, PRL (2007)




Bardarson, Tworzydlo, Brouwer & Beenakker PRL (2007)




The beta function is always positive!



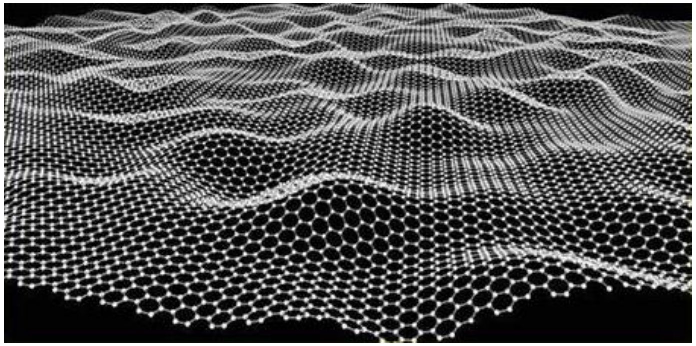

No localization at all

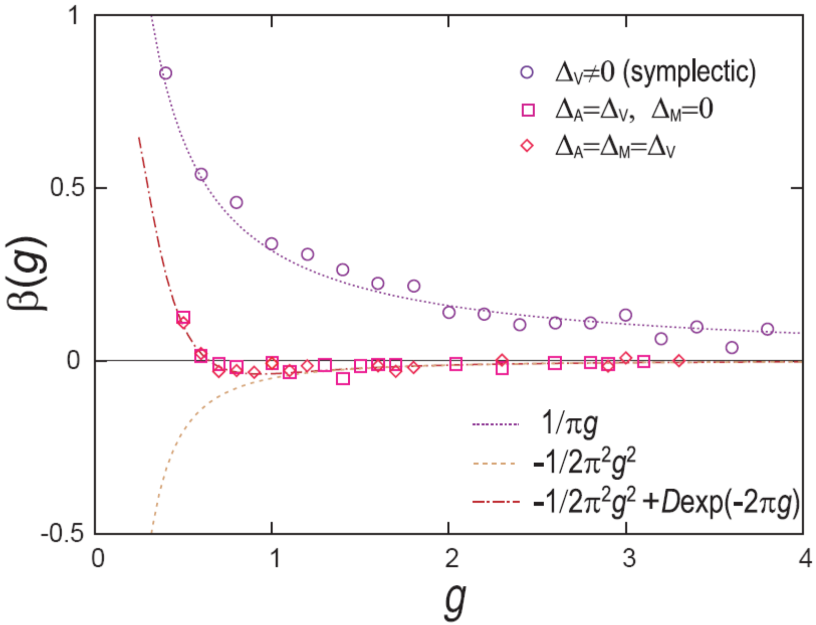
perfect metal

# Including other disorder terms:

$$H = -i\hbar v_F \boldsymbol{\sigma}\cdot\nabla + V(\mathbf{x}) + \boldsymbol{\sigma}\cdot\mathbf{a}(\mathbf{x}) + \sigma_z m(\mathbf{x})$$

Ripples in corrugated graphene = a *random vector potential*



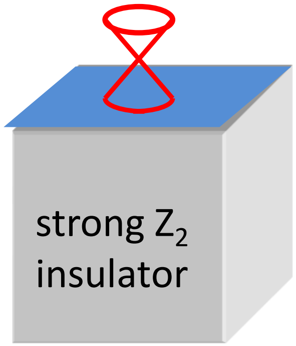



Nomura, Ryu, Koshino, Mudry, AF, PRL 100, 246806 (2008)

(i) Suppressed (anti)localization effect
(ii) Exactly at the IQH plateau transition point between $\sigma_{xy} = \pm 1/2$

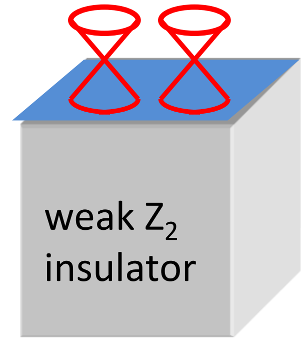
# Conclusions

(1) strong topological insulator

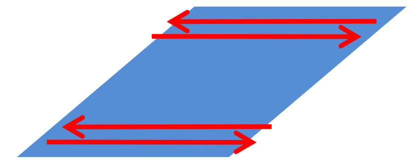



No localization at all!
(topological metal)

(2) weak topological insulator




or



2D Z2 top. insulator

In general, there is localization-delocalization transition