Topological nematic states and non-Abelian lattice dislocations

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Outline

• Fractional quantum anomalous Hall (FQAH) states---fractional quantum Hall states in translation invariant lattice models
• 1D Wannier state description of FQAH states
• FQAH states with higher Chern number and the topological nematic states
• Topological degeneracy induced by lattice dislocations
• Edge state picture and topological field theory description

Ref: XLQ, Phys Rev Lett. 107, 126803 (2011)
Maissam Barkeshli & XLQ, in preparation
Integer quantum Hall (IQH) state

$$\sigma_{xy} = ne^2/h \quad \text{(K von Klitzing 1980)}$$

- Topological origin of the quantized Hall conductance:
- Bulk gap (Landau level gap)
- The first Chern number (TKNN number) (Laughlin PRB 1981, Thouless, et al, PRL 1982)

$$n = \frac{1}{2\pi} \int d^2k (\partial_x a_y - \partial_y a_x)$$

$$a_i(k) = -i \langle k | \partial_i | k \rangle$$

- Chiral edge states on the boundary
(Integer) Quantum Anomalous Hall States

- A lattice model with nonzero Chern number in the occupied band
- General lattice Hamiltonian with translation symmetry \( H = \sum_k c_k^+ h(k) c_k \)
- There are \( n \) bands \( |n, \mathbf{k}\rangle \)
  - Chern number \( C_1 = \frac{1}{2\pi} \int d^2k \nabla \times \mathbf{a}_n \) defined for each band, \( \mathbf{a}_n(\mathbf{k}) = -i \langle n\mathbf{k}|\partial_i|n\mathbf{k}\rangle \)
- Example: two-band models \( H = \sum_a d_a(\mathbf{k}) \sigma^a \) (Haldane 1988, Qi Wu Zhang 2005)
- Material proposals: Hg(Mn)Te/CdTe (Liu et al PRL 2008), Cr or Fe doped Bi2Se3 film (Yu et al Science 2010)
Fractional Quantum Hall (FQH) States

- In partially filled Landau levels, electron interaction can lead to FQH states with nontrivial topological order and fractionalized quasiparticles (Tsui et al 1982)

- FQH states can be described by many-body wavefunctions such as the Laughlin wavefunction (Laughlin 1983)

  \[ \Psi_{\frac{1}{m}}(\{z_i\}) = \prod_{i<j} (z_i - z_j)^m \exp(-\sum_i |z_i|^2/2l_B^2) \]

- Moore-Read wavefunction for a non-Abelian state

  \[ \Psi_{MR}(\{z_i\}) = \text{Pf}(\frac{1}{z_i - z_j}) \prod_{i<j} (z_i - z_j)^q \exp(-\sum_i |z_i|^2/2l_B^2) \]

- Wavefunctions can be constructed systematically to describe many FQH states (e.g., Bernevig&Haldane2008, Wen&Wang 2008)
Can the QAH state be generalized to fractional QH states?

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Fractional quantum anomalous Hall (FQAH) states


Fractional quantum anomalous Hall (FQAH)

Flat band for
\[ t = 1, t' = \frac{1}{2+\sqrt{2}}, t'' = \frac{1}{2+2\sqrt{2}}, \phi = \frac{\pi}{4} \]
(Sun et al 2011)

Wave-function description of FQAH states

• What are the many-body wavefunctions describing FQAH states?
• Related to many other questions about FQAH, e.g., what states can be realized on the lattice?
• Idea: Finding the single-particle basis corresponding to the Landau level wavefunctions in the ordinary QH states.
Wave-function description of FQAH states: 1D Wannier functions

• The proper basis can be found by using 1D Wannier functions

• Consider FQAH state on a cylinder

• The states for each fixed $k_y$ forms a 1D chain.

• 1D Wannier functions: a local basis for the 1D system. Fourier transform of Bloch states

• $|W_{nk_y}\rangle = \frac{1}{\sqrt{L_x}} \sum_{k_x} e^{ik_xn} e^{i\varphi(k)} |k_x, k_y\rangle$
1D Wannier functions

- \[ |W_{nk_y} \rangle = \frac{1}{\sqrt{L_x}} \sum_{k_x} e^{ik_x n} e^{i\varphi(k)} |k_x, k_y \rangle \]

- The ambiguity of \( \varphi(k) \) can be fixed by requiring the Wannier functions to be maximally localized, i.e., by minimizing
  \[ (\Delta x)^2 = \langle W_{nk_y} | x^2 | W_{nk_y} \rangle - \langle W_{nk_y} | x | W_{nk_y} \rangle^2 \]

- In 1D, the maximally localized Wannier function (MLWF) can be obtained by diagonalizing the projected \( x \) operator (Kivelson 1982):
  \[ \hat{x} = P_- x P_- \text{ with } P_- = \sum_k |k\rangle \langle k| \text{ projection to the occupied band.} \]

\[ \hat{x} | W_{nk_y} \rangle = x_{nk_y} | W_{nk_y} \rangle \]

\[ x_{nk_y} = n - \theta(k_y)/2\pi \]
1D Wannier functions

- Wannier functions are shifted by $\theta / 2\pi$ with respect to the lattice sites.
- Correspondingly, charge polarization $P = -\theta / 2\pi$.
- Since $x = i\partial / \partial k_x$, the projected position operator $\hat{x} = i \frac{\partial}{\partial k_x} - a_x$,
  $a_x(k) = -i \langle k | \partial_{k_x} | k \rangle$ is the Berry’s phase gauge field.
- The shift of eigenvalues of $\hat{x}$ is determined by the flux of $a_x$. 

\[ \theta = \oint a_x \, dk_x \]

Coh & Vanderbilt, 2009 PRL
1D Wannier functions and the Chern number

- Chern number on the Brillouin zone torus is the winding number of the flux \( \theta(k_y) \)

\[
\frac{\sigma_H}{\hbar/e^2} = c_1 = -\frac{1}{2\pi} \int_0^{2\pi} \partial_{k_y} \theta(k_y) \, dk_y
\]

\( \theta(k_y) \)

\[
\langle \hat{x} \rangle
\]

\[
k_y/2\pi
\]

\[
k_y
\]
1D Wannier functions in QAH states

• “Twisted” boundary condition for Wannier functions

• $k_y \to k_y + 2\pi, |W_{nk_y}\rangle \to |W_{n+1,k_y}\rangle$

• A extended momentum $K$ can be defined, only if the Chern number is nontrivial

• $|W_{nk_y}\rangle = |W_K\rangle$
Using 1D Wannier functions to describe FQAH states

- After the redefinition, Wannier functions $|W_K\rangle$ are analog of Landau level wavefunctions

- $\psi_K(x, y) = e^{iky} e^{-(x-Kl_B^2)^2/2l_B^2}$
Using 1D Wannier functions to describe FQAH states

- Using this mapping of basis, every FQH wavefunction is mapped to the lattice FQAH states

FQH:

$$|\Psi\rangle = \sum_{\{n_K\}} \Phi(\{n_K\}) \prod_{n_K=1}^{x_K} |\psi_K\rangle$$

FQAH:

$$|\Psi\rangle = \sum_{\{n_K\}} \Phi(\{n_K\}) \prod_{n_K=1}^{W_K} |W_K\rangle$$

$$n_K = 0,1$$
Using 1D Wannier functions to describe FQAH states

- The occupation number wavefunction $\Phi(\{n_K\})$ is known for many FQH states

- For Laughlin state (Rezayi&Haldane 1994 PRB)

$$|\Psi_{1/3}\rangle = \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ + \ldots$$

- A generic construction by Jack polynomials (Bernevig&Haldane 2008 PRL)

- All knowledge on FQH wavefunctions can now be used to construct lattice wavefunctions with the same topological properties.

- Pseudo-potential Hamiltonians can be constructed on the lattice
FQAH state with higher Chern number

- Are there new states in the FQAH system that are absent in the ordinary FQH?
- Similar approach can be generalized to bands with Chern number >1
- Higher winding number of the Wannier state position

![Graphs showing the behavior of FQAH states with higher Chern numbers.]
Realizing multi-layer FQH states in one band

- For Chern number $C_1 = 2$, the Wannier states form two groups $|W_n^1\rangle, |W_n^2\rangle$, with each group equivalent to a Landau level

- ➜ Double-layer FQH states can be realized in a single band
Nontrivial representation of lattice translation symmetry

- Lattice translations $T_x, T_y$ acts differently on this basis
  - $T_x |W_n^1\rangle = |W_n^2\rangle$, $T_x |W_n^1\rangle = |W_n^1\rangle$
  - $T_y |W_n^{1,2}\rangle = e^{in2\pi/L_y} |W_n^{1,2}\rangle$
Topological nematic states

• Consider the Halperin \((mnl)\) states (Halperin ’83)

\[
\Phi(z_i, w_j) = \\
\Pi_{i<j}(z_i - z_j)^m (w_i - w_j)^n \Pi_{i,j}(z_i - w_j)^l
\]

• Lattice translation \(T_x\) exchanges the two “layers”.

• For \(m = n\) the state is translation invariant. However, the 4-fold lattice rotation symmetry (for a square lattice) is broken.

• We name such a state as a topological nematic state

• Lattice dislocations in a topological nematic state carry nontrivial topological degeneracy
Dislocations in topological nematic states

• Dislocations are described by the Burgers vector $\vec{b} = (b_x, b_y)$

\[ x\text{-dislocation } \vec{b} = \hat{x} \quad y\text{-dislocation } \vec{b} = \hat{y} \]

• Across the “branch-cut” of the $x$-dislocation, the two layers are exchanged!
Dislocations in topological nematic states

• A pairs of $x$-dislocations is equivalent to a “worm-hole”
Dislocations in topological nematic states

• Consider a simple case of \((mm0)\) state, which is a direct product of two Laughlin states

\[ |\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \]

• Consider two pairs of dislocations on the torus

• Ground state degeneracy \(N = m^3\)

• Degeneracy \(d = \sqrt{m}\) per dislocation
Dislocations in topological nematic states

- For more general \((mml)\) states, the topological degeneracy with \(n\) pairs of dislocation is
  \[
  N = |m^2 - l^2| \cdot |m - l|^{n-1}
  \]
- The degeneracy per dislocation, i.e., quantum dimension is \(d = \sqrt{|m - l|}\)
- This degeneracy can be obtained by studying the Chern-Simons theory with branch-cuts
  \[
  \mathcal{L} = \frac{1}{4\pi} a^I_\mu K_{IJ} \partial_\nu a^J_\tau \quad \text{(Barkeshli&Wen 2010)}
  \]
- Alternatively, it can be understood from an edge state picture
Edge state picture of dislocation-induced degeneracy

- Consider the torus as a cylinder glued along the edge
- Inter-edge tunneling exchanges the two layers across the branch-cut
Edge state picture of dislocation-induced degeneracy

- The edge states are described by the chiral Luttinger liquid theory

\[ \mathcal{L} = \frac{1}{2\pi} \left( \partial_t \phi^I_L K_{IJ} \partial_x \phi^J_L - \partial_x \phi^I_L V_{IJ} \partial_x \phi^J_L \right) + \frac{1}{2\pi} \left( -\partial_t \phi^I_R K_{IJ} \partial_x \phi^J_R - \partial_x \phi^I_R V_{IJ} \partial_x \phi^J_R \right) + \mathcal{L}_{\text{int}} \]

- Electron operators \( c^I_{L,R} = e^{iK_{IJ} \phi^J_{L,R}} \)

- Interedge tunneling

\[ \mathcal{L}_{\text{int}} = \begin{cases} g(c^1_L c^1_R + c^2_L c^2_R + \text{h.c.}), & \text{A region} \\ g(c^1_L c^2_R + c^2_L c^1_R + \text{h.c.}), & \text{B region} \end{cases} \]
Without dislocation, the inter-edge coupling potential $g \sum_i \cos K_{IJ} (\phi_L^J - \phi_R^J)$ has $\det |K|$ number of minima in the “Brillouin zone” $\phi_L^{1,2} - \phi_R^{1,2} \in [0, 2\pi)$, leading to the degeneracy of $\det|K| = m^2 - l^2$ of the torus.

With $n$ pairs of dislocations, each A region contributes $m^2 - l^2$, each B region contributes $m + l$, because the two operators $\phi_L^1 - \phi_R^1$ in A and $\phi_L^1 - \phi_R^2$ in B do not commute.

Each dislocation has a constraint:

$Q_i = \int_{\alpha_i}^{\alpha_{i+1}} \partial_x (\phi_1 + \phi_2) dx = 0$

$\Rightarrow$ Degeneracy

$N = \frac{(m^2 - l^2)^n (m + l)^n}{(m + l)^{2n-1}} = (m^2 - l^2)(m + l)^{n-1}$
Classification of topological nematic states

• The topological nematic states discussed above are sensitive to $x$-dislocations but not $y$-dislocations.
• Apparently, different topological nematic states can be obtained.
• Generically, 1D Wannier functions can be defined along any reciprocal lattice direction $\vec{K} = 2\pi (n_x, n_y)$
• The corresponding topological nematic states is sensitive to dislocations with burgers vector $\vec{b} \cdot \frac{\vec{K}}{2\pi}$ odd.
• 3 types of topological nematic states $(0,1), (1,0), (1,1)$
• The ordinary Halperin state can be viewed as a trivial class $(0,0)$
A topological field theory description of topological nematic states

- Without dislocations, the effective theory is an Abelian $U(1) \times U(1)$ Chern-Simons theory
  \[ \mathcal{L} = \frac{1}{4\pi} a^I_\mu K_{IJ} \partial_\nu a^J_\tau \]
- Around a dislocation, $a^1_\mu$ and $a^2_\mu$ are exchanged
- To describe this effect we introduce a $U(2)$ gauge field $A_\mu$ and a Higgs field $H = \sigma \cdot \vec{n} e^{i\theta}$ which breaks $U(2) \to U(1) \times U(1)$. The manifold of the Higgs field is $S^2 \times U(1)/\mathbb{Z}_2$.
A topological field theory description of topological nematic states

- Consider the Chern-Simons-Higgs theory

\[ \mathcal{L} = \frac{m-l}{4\pi} \epsilon^{\mu\nu\tau} \text{tr} \left[ A_\mu \partial_\nu A_\tau + \frac{2}{3} A_\mu A_\nu A_\tau \right] + \frac{l}{4\pi} \epsilon^{\mu\nu\tau} \text{tr} \left[ A_\mu \right] \partial_\nu \text{tr} \left[ A_\tau \right] + J \text{tr} \left[ D_\mu H^\dagger D_\mu H \right] \]

- A constant \( H \) breaks \( U(2) \) to diagonal \( U(1) \times U(1) \), leading to the Abelian Chern-Simons theory.

- A dislocation corresponds to a half vortex of \( H = \sigma \cdot \vec{n} e^{i\theta} \).

- The two \( U(1) \) are exchanged around the dislocation

- Generically, \( \theta = \pi \vec{u} \cdot \vec{N} \) with \( \vec{u} \) displacement field, and \( \vec{N} = (n_x, n_y) \) the type of the topological nematic state.
Wannier functions provide the proper basis for characterizing fractional topological states in lattice systems with nontrivial band structure.

- FQH states can be mapped to lattice models.
- A band with higher Chern number is mapped to a multi-layer FQH state.
- New states named as topological nematic states can be realized, with non-Abelian dislocations even if the state itself is Abelian.
- Provide the possibility of experimentally observe the topological degeneracy, which otherwise requires a high genus surface.