Topological Quantum Phenomena in Condensed Matter with Broken Symmetries

Non-Abelian topological orders in superconducting states

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Outline

1. Non-Abelian anyons in topological superconductors

    - chiral p-wave superconductor
    - non-Abelian anyon in s-wave SC

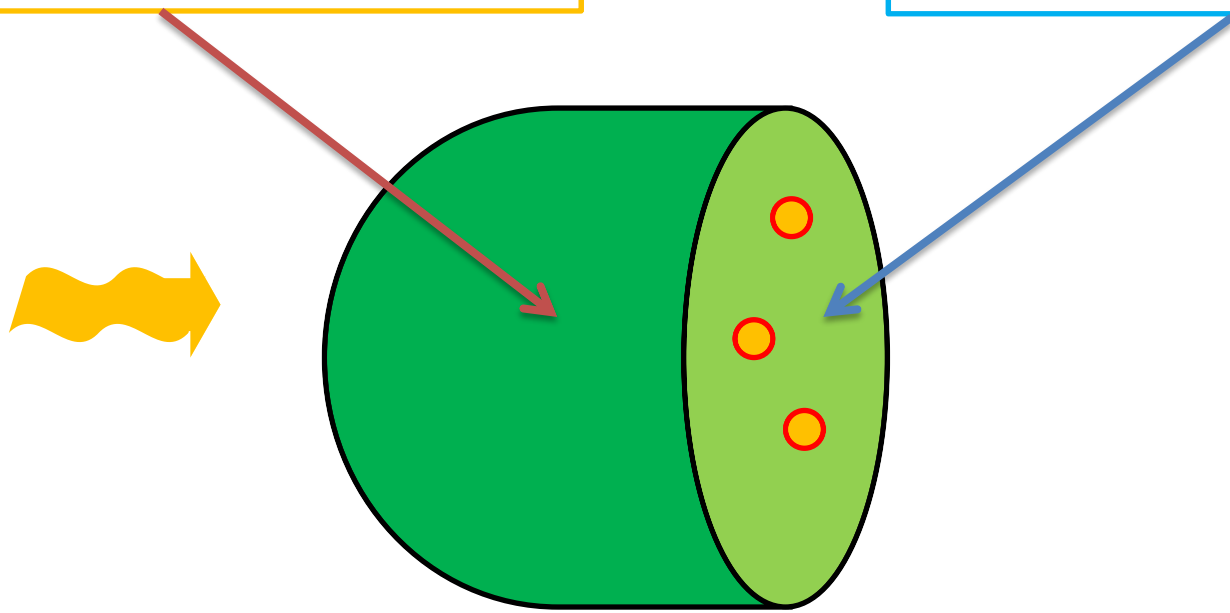
2. Gapless Topological Superconductors

What is topological superconductor ?

Topological superconductors

Bulk:
gapped state with
non-zero **topological #**

Boundary:
gapless state with
Majorana condition



The gapless boundary state = Majorana fermion

Majorana Fermion

Dirac fermion with Majorana condition

1. Dirac Hamiltonian

$$\mathcal{H}(\boldsymbol{k}) = \boldsymbol{\sigma} \cdot \boldsymbol{k}, \text{ or } \quad \mathcal{H}(k_x) = ck_x$$

2. Majorana condition

$$\Psi = C\Psi^*$$

particle = antiparticle

For the gapless boundary states, their Hamiltonians are naturally given by the Dirac Hamiltonians

How about the Majorana condition ?

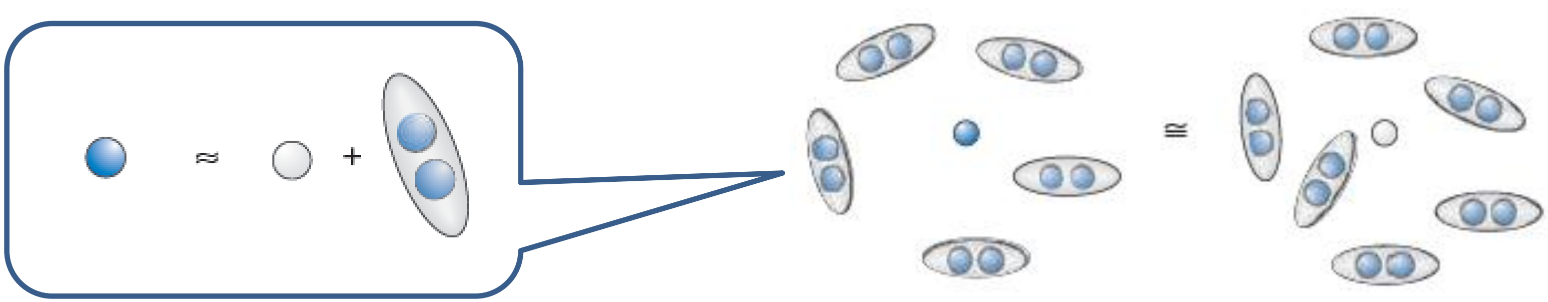
The Majorana condition is imposed by superconductivity

quasiparticle in Nambu rep. quasiparticle anti-quasiparticle

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \\ \psi_\uparrow^\dagger(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix}$$

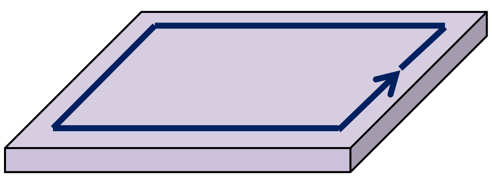
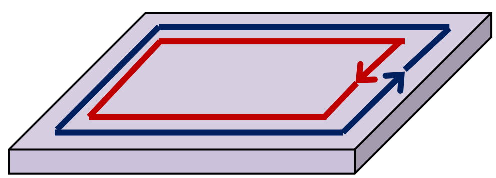
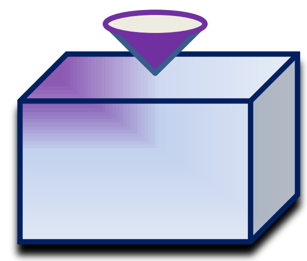
$$\Psi(x) = \mathcal{C}\Psi^*(x), \quad \mathcal{C} = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}$$

Majorana condition



[Wilczek , Nature (09)]

different bulk topological
= different Majorana fermions

2+1D time-reversal breaking SC	2+1D time-reversal invariant SC	3+1D time-reversal invariant SC
1st Chern # (TKNN82, Kohmoto85)	Z_2 number (Kane-Mele 06, Qi et al (08))	3D winding # (Schnyder et al (08))
1+1D **chiral** edge mode 	1+1D **helical** edge mode 	2+1D **helical** surface fermion 
Sr_2RuO_4	Noncentosymmetric SC (MS-Fujimto(09))	^{3}He B

A representative example of topological SC:

Chiral p-wave SC in 2+1 dimensions

[Read-Green (00)]

BdG Hamiltonian

spinless chiral p-wave SC

$$\begin{aligned}\mathcal{H} &= \sum_{\boldsymbol{k}} \epsilon(\boldsymbol{k}) c^{\dagger}_{\boldsymbol{k}} c_{\boldsymbol{k}} + \frac{1}{2}\sum_{\boldsymbol{k}} \left[\Delta(\boldsymbol{k}) c^{\dagger}_{\boldsymbol{k}} c^{\dagger}_{-\boldsymbol{k}} + \text{h.c}\right] \\ &= \frac{1}{2}\sum_{\boldsymbol{k}} \left(c^{\dagger}_{\boldsymbol{k}}, c_{-\boldsymbol{k}}\right) \mathcal{H}(\boldsymbol{k}) \begin{pmatrix} c_{\boldsymbol{k}} \\ c^{\dagger}_{-\boldsymbol{k}} \end{pmatrix} + \text{const.}\end{aligned}$$

with

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} \epsilon(\boldsymbol{k}) & \Delta(\boldsymbol{k}) \\ \Delta(\boldsymbol{k})^{*} & -\epsilon(\boldsymbol{k}) \end{pmatrix}$$

$$\epsilon(\boldsymbol{k}) = -2t_x \cos k_x - 2t_y \cos k_y - \mu$$

$$\begin{aligned}\Delta(\boldsymbol{k}) &= d(\sin k_x + i \sin k_y) \\ &\sim d(k_x + i k_y)\end{aligned}$$

chiral p-wave

Topological number = 1st Chern number

TKNN (82), Kohmoto(85)

$$A_i(\boldsymbol{k}) = i \sum_{a \in \text{filled}} \langle u_a(\boldsymbol{k})| \frac{\partial}{\partial k_i} |u_a(\boldsymbol{k})\rangle$$

$$\begin{aligned} \nu_{\text{Ch}} &= \frac{1}{2\pi} \int d^2k [\partial_{k_x} A_y(\boldsymbol{k}) - \partial_{k_y} A_x(\boldsymbol{k})] \\ &= -\frac{1}{2} \sum_{\Delta(k_0)=0} \text{sgn}\epsilon(k_0) \cdot \text{sgn}[\det(\partial_i R^j(k_0))] \end{aligned}$$

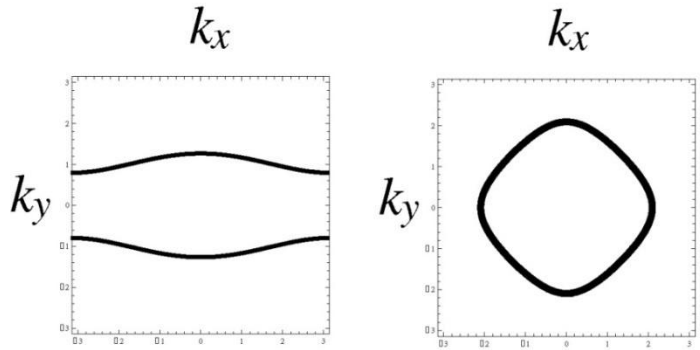
MS (09)

$$(\Delta(\boldsymbol{k}) = R^1(\boldsymbol{k}) + iR^2(\boldsymbol{k}))$$

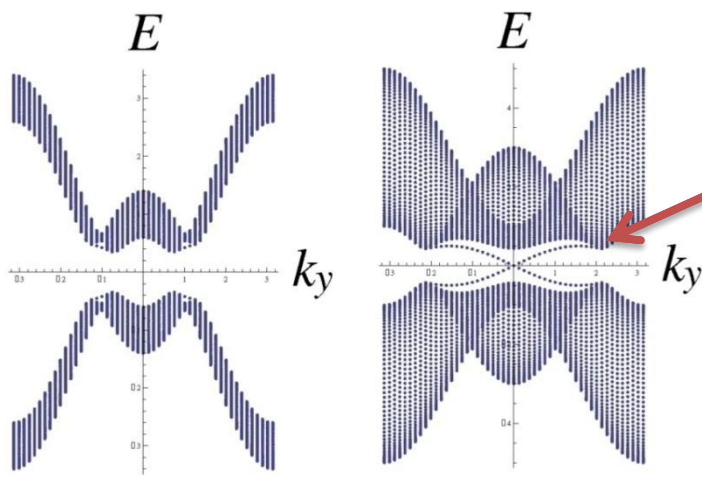
Edge state

$\mu = -1,\ d = 0.5$

Fermi surface

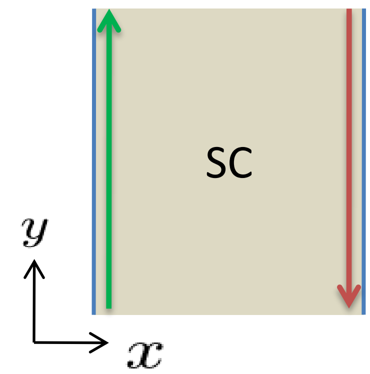


Spectrum



$t_x = 1,\ t_y = 0.2$ $\quad$ $t_x = t_y = 1$

$\nu_{\mathrm{Ch}} = 0$ $\quad$ $\nu_{\mathrm{Ch}} = 1$



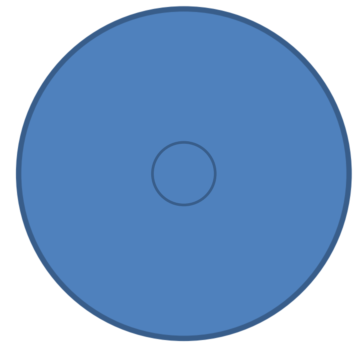
2 gapless edge modes (left-moving , right moving,
on different sides on boundaries)

Majorana fermion

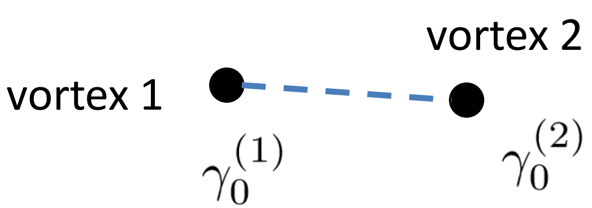
Bulk-edge correspondence

In the second case, there also exist a Majorana zero mode in a vortex

$$\gamma_0^\dagger = \gamma_0$$



We need a pair of the zero modes to define creation op.



$$\gamma^\dagger = \frac{\gamma_0^{(1)} + i\gamma_0^{(2)}}{\sqrt{2}} \qquad \{\gamma^\dagger, \gamma\} = 1$$

non-Abelian anyon

topological quantum computer

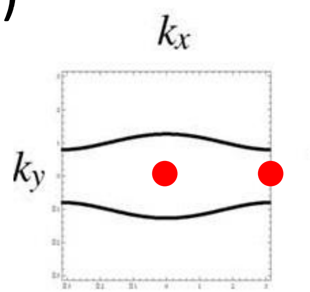
For spin-triplet SCs (or odd parity SCs), there exists a simple criterion for topological phases

If the number of TRIMs enclosed by the Fermi surface is odd, the spin-triplet SC is (strongly) topological.

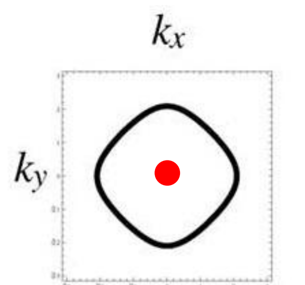
[Sato (09), Sato (10), Fu-Berg (10)]

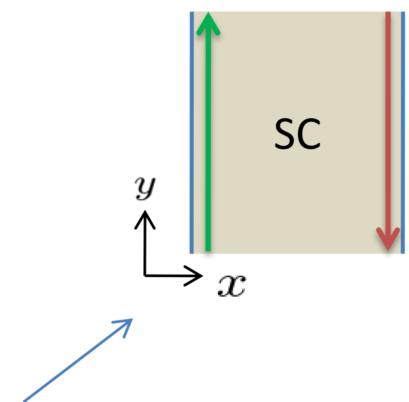
2D spinless SC)

Even



Odd





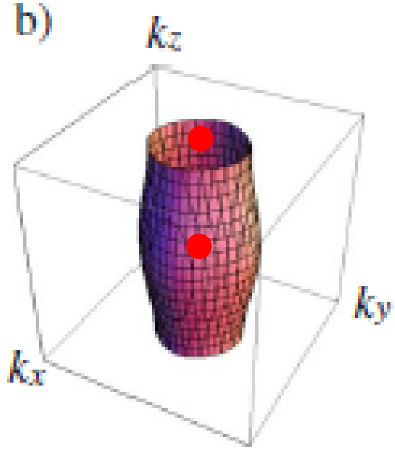
Chiral Majorana mode

$$\Delta(\boldsymbol{k}) = k_x + ik_y$$

3D time-reversal invariant spin-triplet SC)

Even

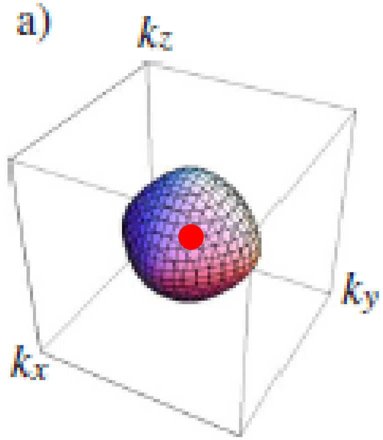
b)



0;[0,0,1]

Odd

a)



2;[0,0,0]

$$d_x(\boldsymbol{k}) = k_x,\ d_y(\boldsymbol{k}) = k_y,\ d_z(\boldsymbol{k}) = k_z$$

With proper topology of the Fermi surface, spin-triplet SCs (or odd-parity SCs) naturally become topological.

Is it possible to realize non-Abelian anyon in s-wave superconducting state?

Yes !

A) MS, Physics Letters B535 ,126 (03), Fu-Kane PRL (08)

B) MS-Takahashi-Fujimoto ,Phys. Rev. Lett. 103, 020401 (09) ;
MS-Takahashi-Fujimoto, Phys. Rev. B82, 134521 (10) (Editor's suggestion),
J. Sau et al, PRL (10), J. Alicea PRB (10)

Key point: Spin-orbit interaction

Majorna fermion in spin-singlet SC

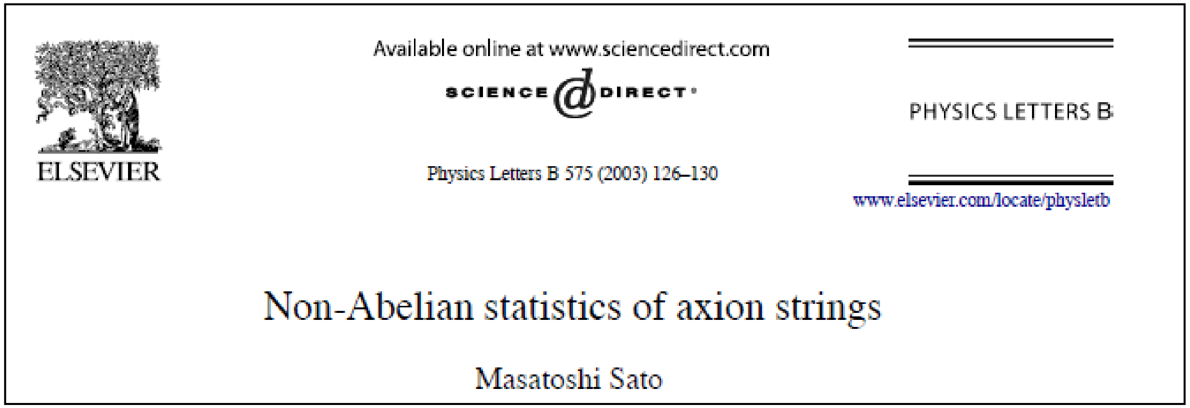
MS, Physics Letters B535 ,126 (03)

① 2+1 dim Dirac fermion + s-wave Cooper pair

$$\mathcal{H} = \begin{pmatrix} -i\sigma_i\partial_i & \Phi^* \\ \Phi & -i\sigma_i\partial_i \end{pmatrix} \qquad \Phi = \Phi_0 f(r) e^{i\theta} \quad \text{vortex}$$

Zero mode in a vortex [Jackiw-Rossi (81), Callan-Harvey(85)]

With Majorana condition, non-Abelian anyon is realized [MS (03)]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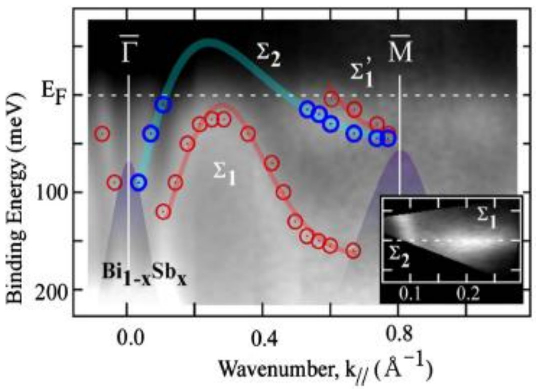
Non-Abelian statistics of axion strings

Masatoshi Sato

On the surface of topological insulator

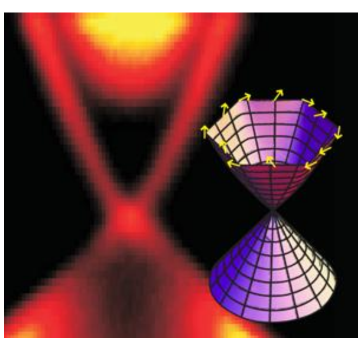
[Fu-Kane (08)]

$Bi_{1-x}Sb_x$ Hsieh et al., Nature (2008)



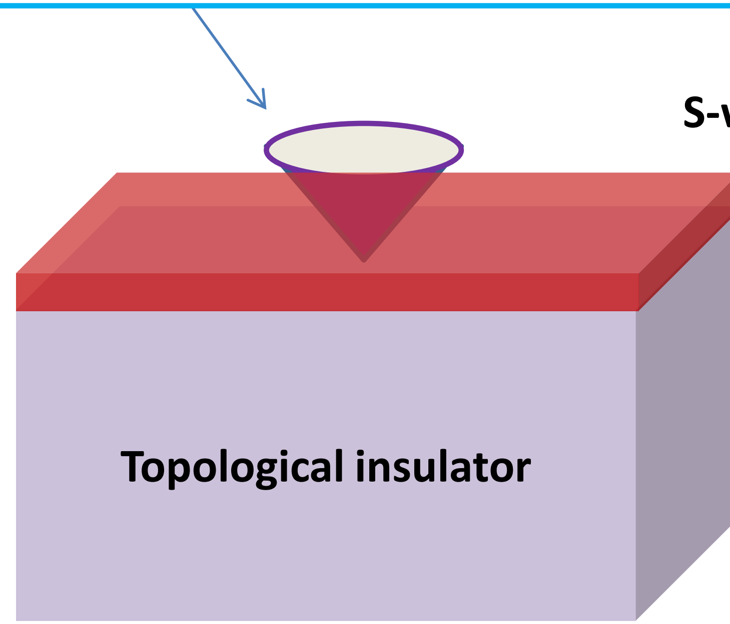
Nishide et al., PRB (2010)

Bi_2Se_3 Hsieh et al., Nature (2009)



Dirac fermion + s-wave SC

S-wave SC



Spin-orbit interaction
=> topological insulator

2nd scheme of Majorana fermion in spin-singlet SC

② 2+1 dim s-wave SC with **Rashba spin-orbit interaction**

[MS, Takahashi, Fujimoto PRL(09) PRB(10)]

Rashba SO

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} \epsilon_{\boldsymbol{k}} - h\sigma_z + \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} & i\psi_{\rm s}\sigma_y \\ -i\psi_{\rm s}\sigma_y & -\epsilon_{\boldsymbol{k}} + h\sigma_z + \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}^* \end{pmatrix}$$

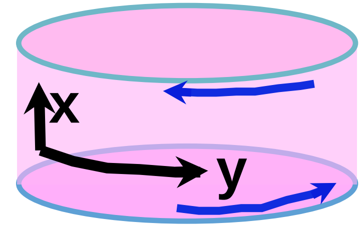
$$\mathcal{H}^{\rm D}(\boldsymbol{k}) = D\mathcal{H}(\boldsymbol{k})D^\dagger, \quad D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i\sigma_y \\ i\sigma_y & 1 \end{pmatrix}$$

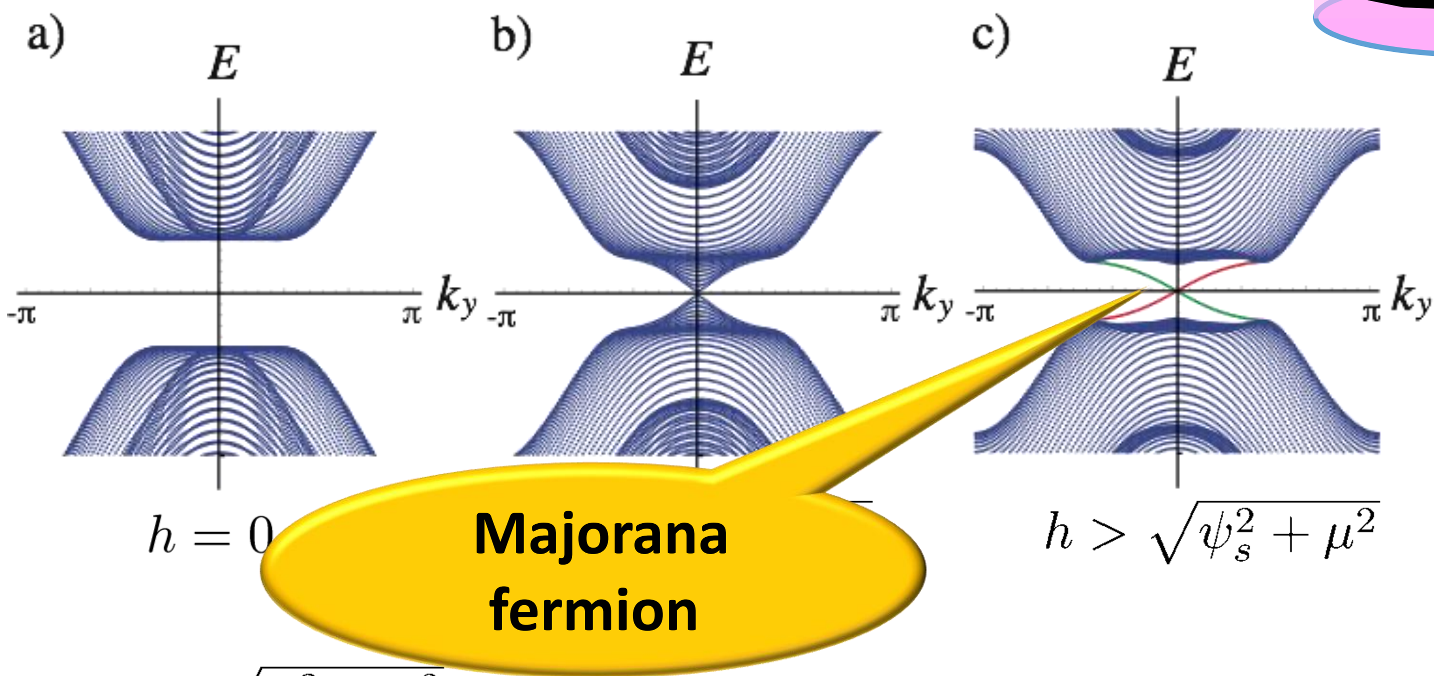
$$\mathcal{H}^{\rm D}(\boldsymbol{k}) = \begin{pmatrix} \psi_{\rm s} - h\sigma_z & -i\epsilon_{\boldsymbol{k}}\sigma_y - i\boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}\sigma_y \\ i\epsilon_{\boldsymbol{k}}\sigma_y + i\boldsymbol{g}_{\boldsymbol{k}}\sigma_y\boldsymbol{\sigma} & -\psi_{\rm s} + h\sigma_z \end{pmatrix}$$

p-wave gap is induced by Rashba SO int.

Gapless edge states

[MS, Takahashi, Fujimoto PRL(09)]





For $h > \sqrt{\psi_s^2 + \mu^2}$

a single chiral gapless edge state appears like p-wave SC !

Chern number

nonzero Chern number $\mathcal{Q} = 1$

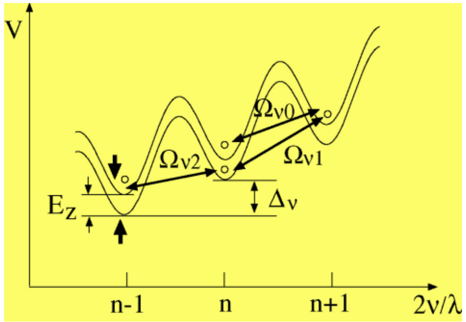
topologically equivalent to spinless chiral p-wave SC

$$h > \sqrt{\psi_s^2 + \mu^2}$$

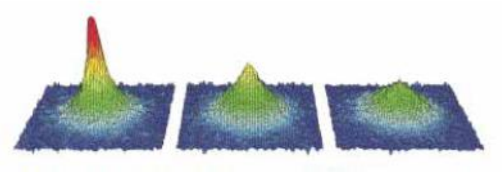
strong magnetic field is needed

How to suppress orbital depairing effect

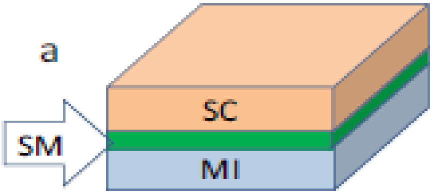
a) s-wave superfluid of cold atoms with laser generated Rashba SO coupling



[Sato-Takahashi-Fujimoto PRL(09)]



b) semiconductor-superconductor interface



[J.Sau et al. PRL(10)
J. Alicea, PRB(10)]

c) semiconductor nanowire on superconductors ....

[R. M. Lutchyn et al. PRL(10), Y. Oreg et al, PRL(10), ...]

Summary (part 1)

With proper topology of Fermi surfaces, topological SCs are naturally realized in spin-triplet (odd-parity) SCs.

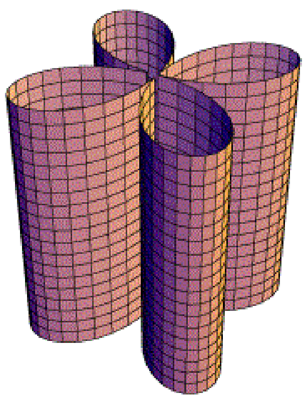
But with SO interaction, spin-singlet SCs can be topological as well.

Gapless topological phase in superconductors

MS-Fujimoto, Phys. Rev. Lett. 105, 217001 (10)
Sasaki, Kriener, Segawa, Yada, Tanaka, MS, Ando, Phys. Rev. Lett. 107, 217001 (11)

Motivation

We usually suppose full-gapped bulk spectrum for topological SCs. However, unconventional SCs often support bulk nodes in the gap function.



High-Tc cuprate

Can we use such nodal SCs to realize Majorana fermion?

Yes !

We find two classes of topological SCs with gap nodes.

1. 2D time-reversal breaking topological nodal SCs

MS-Fujimoto, Phys. Rev. Lett. 105, 217001 (10)

2. 3D time-reversal invariant topological nodal SCs

Sasaki, Kriener, Segawa, Yada, Tanaka, MS, Ando, Phys. Rev. Lett. 107, 217001 (11)

- They support non-zero bulk topological # defined in the entire space of the BZ.
- Existence of Majorana fermions on the boundary

→ **"Strong" topological SC**

2D Time-reversal breaking topological nodal SC

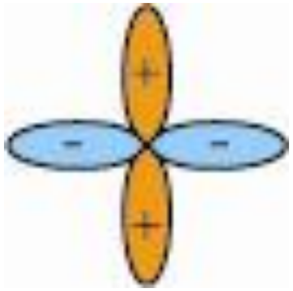
Model: 2d d-wave superconductor with Rashba SO int

[MS, Fujimoto PRL (10)]

Rashba SO

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} \epsilon_{\boldsymbol{k}} - h\sigma_z + \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} & i\Delta_{\boldsymbol{k}}\sigma_y \\ -i\Delta_{\boldsymbol{k}}\sigma_y & -\epsilon_{\boldsymbol{k}} + h\sigma_z + \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} \end{pmatrix}$$

Zeeman

dx^2-y^2 –wave gap function	**dxy –wave gap function**
$\Delta_{\boldsymbol{k}} = \Delta_0(\cos k_x - \cos k_y)$	$\Delta_{\boldsymbol{k}} = \Delta_0 \sin k_x \sin k_y$
	

To understand what happens, we use the dual transformation again

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} \epsilon_{\boldsymbol{k}} - h\sigma_z + \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} & i\Delta_{\boldsymbol{k}}\sigma_y \\ -i\Delta_{\boldsymbol{k}}\sigma_y & -\epsilon_{\boldsymbol{k}} + h\sigma_z + \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma} \end{pmatrix}$$

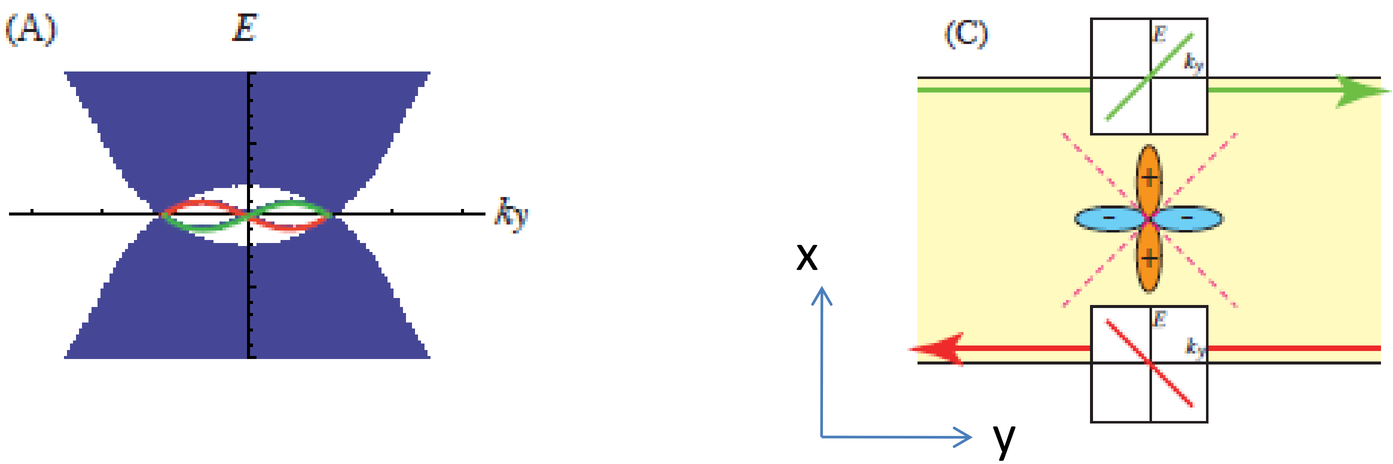
$$\mathcal{H}^{\mathrm{D}}(\boldsymbol{k}) = D\mathcal{H}(\boldsymbol{k})D^{\dagger}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\sigma_y \\ i\sigma_y & 1 \end{pmatrix}$$

$$\mathcal{H}^{\mathrm{D}}(\boldsymbol{k}) = \begin{pmatrix} \Delta_{\boldsymbol{k}} - h\sigma_z & -i\epsilon_{\boldsymbol{k}}\sigma_y - \boxed{i\boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}\sigma_y} \\ i\epsilon_{\boldsymbol{k}}\sigma_y + \boxed{i\boldsymbol{g}_{\boldsymbol{k}}\sigma_y\boldsymbol{\sigma}} & -\Delta_{\boldsymbol{k}} + h\sigma_z \end{pmatrix}$$

p-wave gap is induced by Rashba SO int.

Edge state $(h^2 > \mu^2)$

$d_{x^2-y^2}$ –wave gap function



dxy –wave gap function

There also exist a Majorana zero mode in a vortex

$$\gamma = \int \left[u_\uparrow \psi_\uparrow^\dagger + u_\downarrow \psi_\downarrow^\dagger + u_\uparrow^* \psi_\uparrow + u_\downarrow^* \psi_\downarrow \right]$$

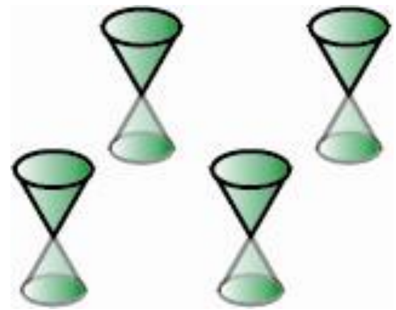
$$u_\uparrow = i e^{i \frac{n-1}{2} \theta} f(r), \quad u_\downarrow = -i e^{i \frac{n+1}{2} \theta} f(r) \qquad f(r) = \sqrt{\frac{h}{\pi \lambda r}} e^{-\frac{h}{2\lambda} r}$$

Zero mode satisfies Majorana condition! $\gamma^\dagger = \gamma$

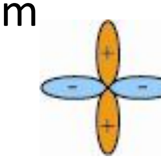
Non-Abelian anyon

The Majorana zero mode is stable against nodal excitations

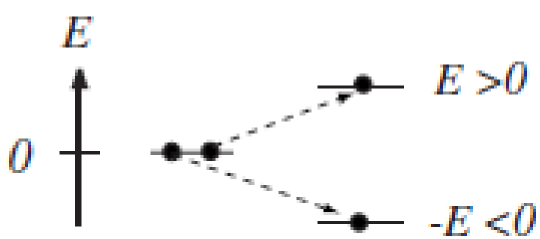
1 zero mode on a vortex



4 gapless modes from gap-node

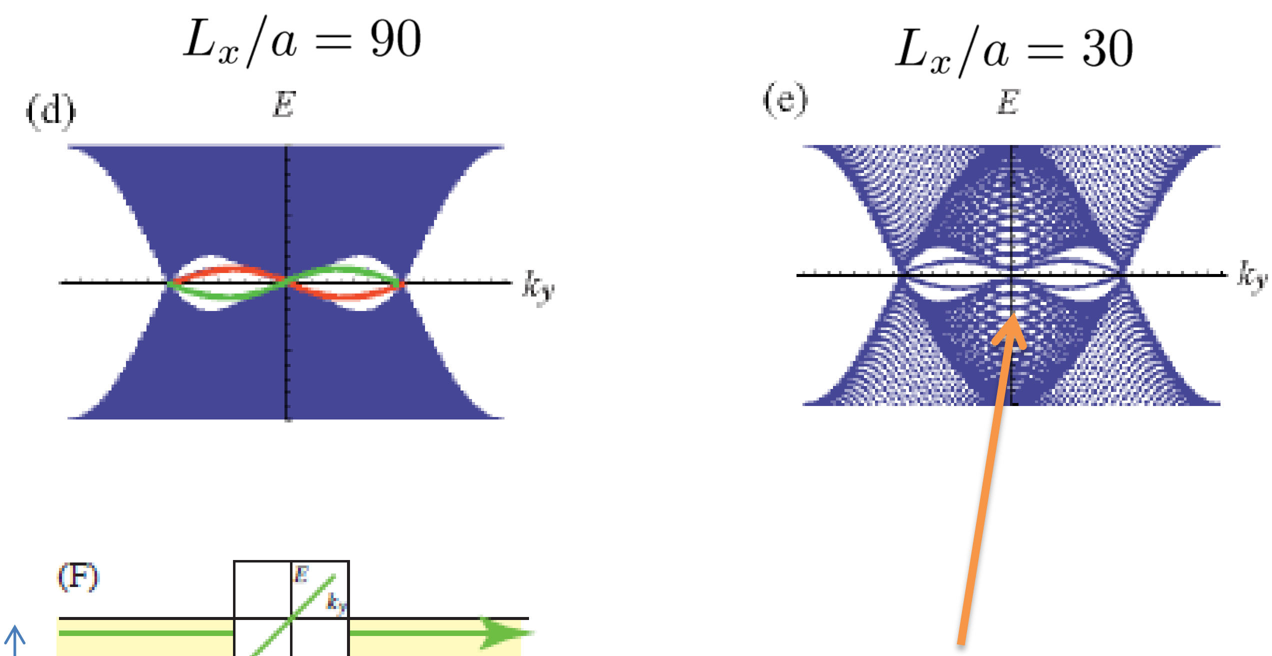


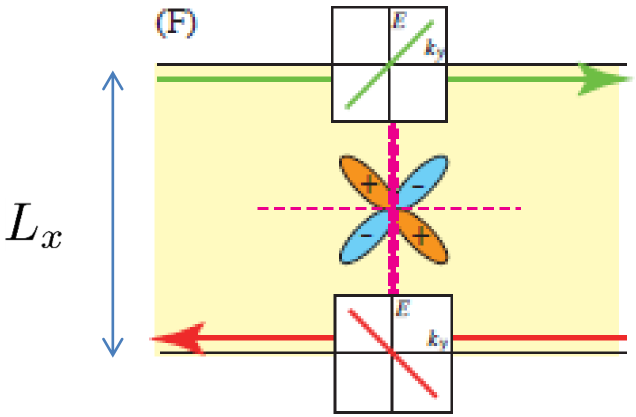
From the particle-hole symmetry, zero modes become massive in pair.



At least one Majorana zero mode survives

The nodal excitations may change the finite size effect





Long-range tunneling between two edges due to the nodal excitations

Topological phase in 2D TRB nodal SCs is characterized by the parity of the Chern number $(-1)^{\nu_{\rm Ch}}$

[MS, Fujimoto PRL (10)]

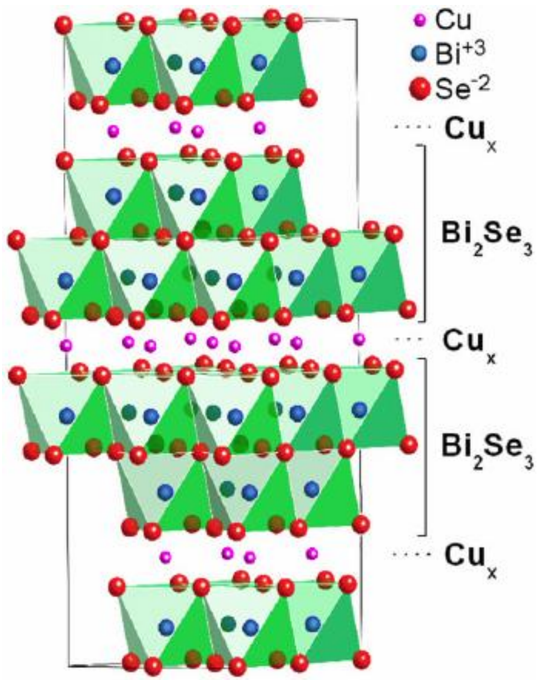
$(-1)^{\nu_{\rm Ch}} = -1$	$(-1)^{\nu_{\rm Ch}} = 1$
There exist an odd number of gapless Majorana fermions	There exist an even number of gapless Majorana fermions
↓ + nodal excitation	↓ + nodal excitation
Topologically stable Majorana fermion	No Majorana fermion survives

3D time-reversal invariant topological nodal SC

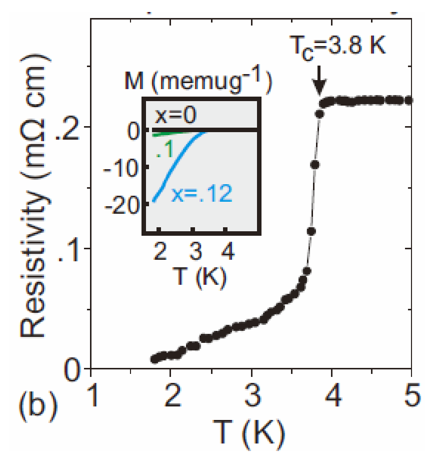
[Sasaki, Kriener, Segawa, Yada, Tanaka, MS, Ando PRL (11)]

$Cu_xBi_2Se_3$

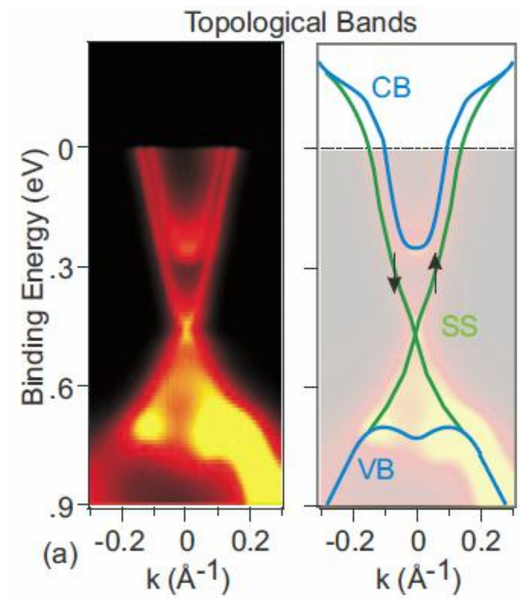
Superconducting only for $0.10 \leq x \leq 0.30$



Hor et al., PRL (2010)



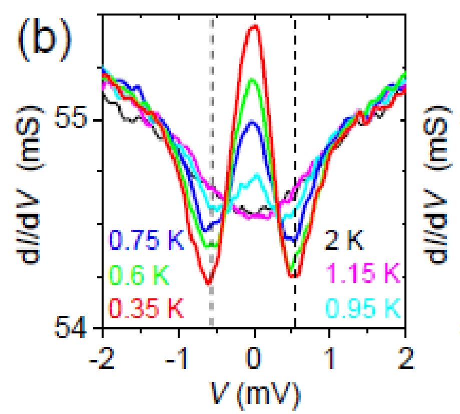
Wray et al.,
Nat. Phys. (2010)



Recent measurement of tunneling conductance shows a pronounced zero-bias conductance peak

Evidence of Majorana fermion on the surface

[Sasaki, Kriener, Segawa, Yada, Tanaka, MS, Ando PRL (11)]



Proposed gap functions

[Fu-Berg (10)]

	gap type	parity	energy gap structure
Δ_1	$\Delta^{11}_{\uparrow\downarrow} = -\Delta^{11}_{\downarrow\uparrow} = \Delta^{22}_{\uparrow\downarrow} = -\Delta^{22}_{\downarrow\uparrow}$ $\Delta^{11}_{\uparrow\downarrow} = -\Delta^{11}_{\downarrow\uparrow} = -\Delta^{22}_{\uparrow\downarrow} = \Delta^{22}_{\downarrow\uparrow}$	even	full gap
Δ_2	$\Delta^{12}_{\uparrow\downarrow} = -\Delta^{12}_{\downarrow\uparrow} = \Delta^{21}_{\uparrow\downarrow} = -\Delta^{21}_{\downarrow\uparrow}$	odd	full gap
Δ_3	$\Delta^{12}_{\uparrow\downarrow} = \Delta^{12}_{\downarrow\uparrow} = -\Delta^{21}_{\uparrow\downarrow} = -\Delta^{21}_{\downarrow\uparrow}$	odd	point node
Δ_4	$\Delta^{12}_{\uparrow\uparrow} = \Delta^{12}_{\downarrow\downarrow} = -\Delta^{21}_{\uparrow\uparrow} = -\Delta^{21}_{\downarrow\downarrow}$ $\Delta^{12}_{\uparrow\uparrow} = -\Delta^{12}_{\downarrow\downarrow} = -\Delta^{21}_{\uparrow\uparrow} = \Delta^{21}_{\downarrow\downarrow}$	odd	point node

- **Δ_2 is full gapped and topological**

[Fu-Berg (10), Sato(10)]

- **Δ_3 and Δ_4 are nodal but topological**

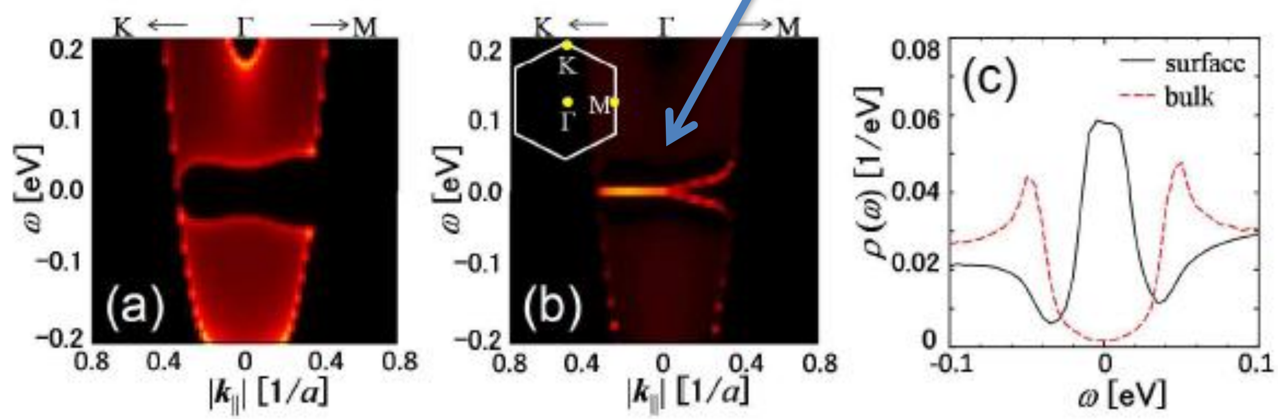
[SKSYTSA(10)]

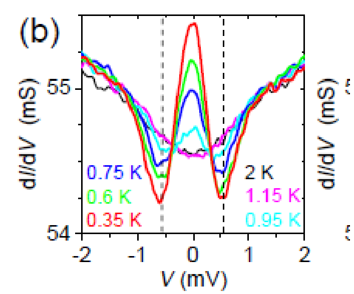
The both cases can be consistent with the experimental result for tunneling conductance.

Surface state of nodal topological SC

Δ_4 (111) surface

Deformed Majorana fermion





[Sasaki et al. PRL (11)]

However, to determine the actual gap function, we need a further theoretical investigation on the tunneling conductance.

Topological phase in 3D TRI nodal SCs is characterized by the parity of 3d winding number (= mod 2 winding number) $(-1)^{\nu_{\rm w}}$

Full gapped SC [Schnyder et al (08)]

			Full gapped SC	Nodal SC
Time-reversal breaking SC	**class D**	**2 dim**	**Chern #** **Z** →	**Parity of Chern #** $\mathbf{Z_2}$
Time-reversal invariant SC	class DIII	2 dim	Z_2 # Z_2	$\mathbf{Z_2}$ **#** $\mathbf{Z_2}$
		3 dim	3d winding # Z →	**mod 2 winding #** $\mathbf{Z_2}$

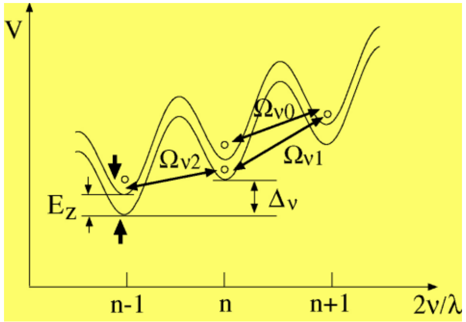
[MS-Fujimoto (10), STSYTSA (11)]

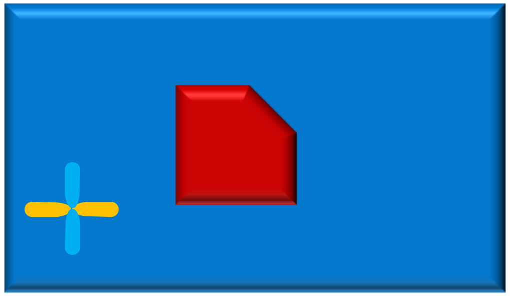
Summary

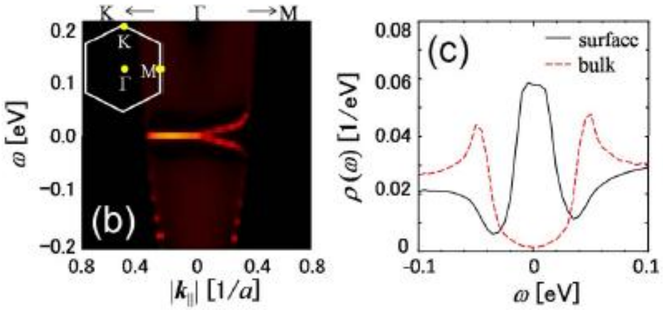
With SO interaction, various superconductors become topological superconductors

1. Majorana fermion in **spin singlet SCs**

2. Majorana fermion in **nodal SC**







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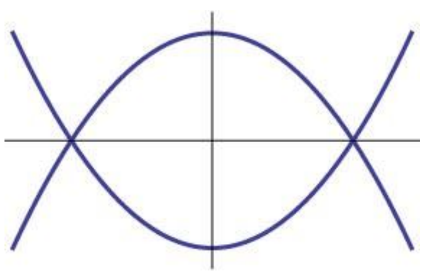
Topological # in nodal SC

Formally, the bulk topological # in nodal SCs can be defined after removing the gap node by perturbation.

$\varepsilon(\boldsymbol{k})$

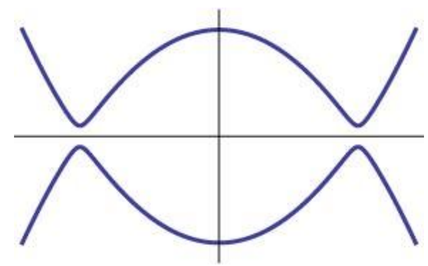


$\longrightarrow \boldsymbol{k}$

perturbation

$$\Delta_{\boldsymbol{k}} \to \Delta_{\boldsymbol{k}} + i\epsilon\Delta'_{\boldsymbol{k}}$$

$\varepsilon(\boldsymbol{k})$



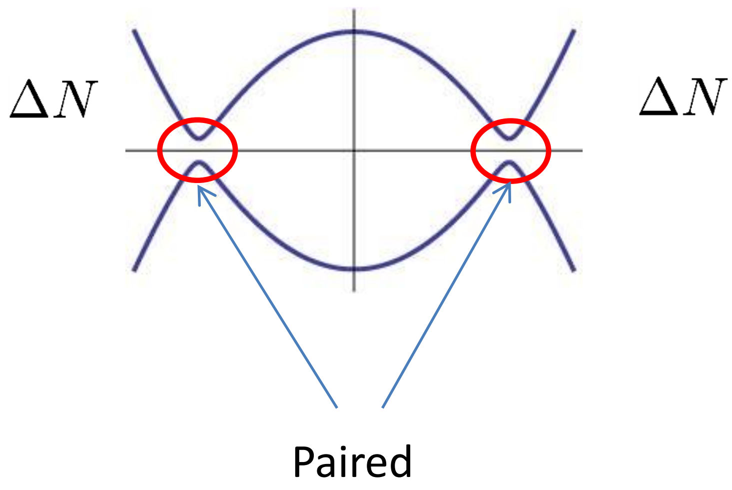
$\longrightarrow \boldsymbol{k}$

$$\text{(e.g.)}\ d_{xy} \to d_{xy} + i\epsilon d_{x^2-y^2},\ d_{xy} \to d_{xy} + i\epsilon s)$$

However, sometimes, the resultant bulk topological # depends on the perturbation.

~~Chern #~~ ~~3D winding #~~

However, the parity of the topological # does not have the ambiguity.



- Because of the particle-hole symmetry, nodes are paired in the momentum space.
- Each node may give an ambiguity, but the total ambiguity of the topological # should be even since nodes are paired.
- We have a unique value of the parity of the topological #