Congratulations to:

- David Thouless and Mike Kosterlitz:

  In its quantum version (with "2D vortex" mapped to "1+1 D vertex operator") the KT model and its renormalization-group treatment has been ubiqitous in treatments of topological quantum matter (TQM)

- Alexei Kitaev and Xiao-Gang Wen

  Alexei's inspired proposal that TQM could provide a platform for topologically-protected quantum computing, as well as his simple explicit models have driven the explosion of interest in this field

  Xiao-Gang's systematic characterization has allowed us to see the "big picture" that unites different examples of TQM, and the identification of FQHE edge modes with non-trivial chiral Luttinger liquids and conformal field theories made many calculations simple and possible.

- Thanks to all others who have made exciting contributions that advanced our field of TQM!

KITP, UC Santa Barbara, October 17, 2016,

KITP conference: Topological Quantum Matter

Quantum geometry and Composite Fermi liquid states in a partially-filled Landau Level

F. Duncan M. Haldane
Princeton University

- Flux attachment as emergent gauge-field with geometry
- Geometry and energy of the "flux attachment" that forms "composite particles"
- composite fermion fluids

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- thirty years after its experimental discovery and theoretical description in terms of the Laughlin state, the fractional quantum Hall effect remains a rich source of new ideas in condensed matter physics.
- The key concept is "**flux attachment**" that forms "**composite particles**" and leads to topological order.
- Recently, it has been realized that flux attachment also has interesting **geometric** properties

- In a 2D Landau level, we apparently start from a Schrödinger picture, but end with a "quantum geometry" which is no longer correctly described by Schrödinger wavefunctions in real space because of "quantum fuzziness" (non-locality)
- It remains correctly described by the Heisenberg formalism in Hilbert space.

quarks leptons Higgs photons → atomic nuclei electrons → crystalline (rigid) condensed matter → continuum description on lengthscales larger than atoms

Broken Lorentz, Galileian,and continuous rotational invariances

- A very effective approach for understanding the essential physics is to remove all unnecessary non-generic ingredients from the description.

- Only **translation** and (possibly) **inversion** symmetries are generic in a continuum description of phenomena in homogeneous crystalline condensed matter on a larger-than-atomic scale

- Top-level model (Schrödinger):

$$p_i = -i\hbar\boldsymbol{\nabla}_r - e\boldsymbol{A}(\boldsymbol{r})$$

$$\boldsymbol{\nabla}_r \times \boldsymbol{A}(\boldsymbol{r}) = B$$

$$\boxed{H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V_0(\boldsymbol{r}_i - \boldsymbol{r}_j)}$$

not necessarily quadratic (**no** Galilean invariance should be assumed)

bare Coulomb interaction controlled by (possibly anisotropic) dielectric tensor of medium (no rotational invariance should be assumed)

- model has inversion symmetry if $\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$, but even this need not be assumed

$$\boldsymbol{r} = r^a \boldsymbol{e}_a$$

displacement (contravariant index)

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$$

orthonormal basis of tangent vectors of 2D plane: a = 1,2

Euclidean metric of 2D plane

$$p_a = \boldsymbol{e}_a \cdot \boldsymbol{p}$$

dynamical momentum (covariant index)

antisymmetric (2D Levi-Civita) symbol

- Two independent Heisenberg algebras:

$$\boxed{\begin{aligned}[p_a, p_b] &= i\hbar e B \epsilon_{ab} \\ [r^a, p_b] &= i\hbar \delta^a_b \\ [r^a, r^b] &= 0\end{aligned}}$$

organize as $\longrightarrow$

$$\boxed{\begin{aligned}[\bar{R}^a, \bar{R}^b] &= i\ell_B^2 \epsilon^{ab} \\ [R^a, \bar{R}^b] &= 0 \\ [R^a, R^b] &= -i\ell_B^2 \epsilon^{ab}\end{aligned}}$$

$$\bar{R}^a = \hbar^{-1}\epsilon^{ab} p_b \ell_B^2$$

Landau orbit radius vector

$$R^a = r^a - \bar{R}^a$$

Landau orbit guiding-center displacement

$$2\pi\ell_B^2 = \frac{2\pi\hbar}{eB} > 0$$

quantum area (per h/e flux quantum)

- Note: origin of guiding-center displacement has a gauge ambiguity under $\boldsymbol{A}(\boldsymbol{r}) \mapsto \boldsymbol{A}(\boldsymbol{r})$ + constant

- Landau quantization $\varepsilon(\boldsymbol{p})|\Psi_n\rangle = E_n|\Psi_n\rangle$

discrete spectrum of macroscopically-degenerate Landau levels

- Project residual interaction in a single partially occupied "active" Landau level, all other dynamics is frozen by Pauli principle when gap between Landau levels dominates interaction potential

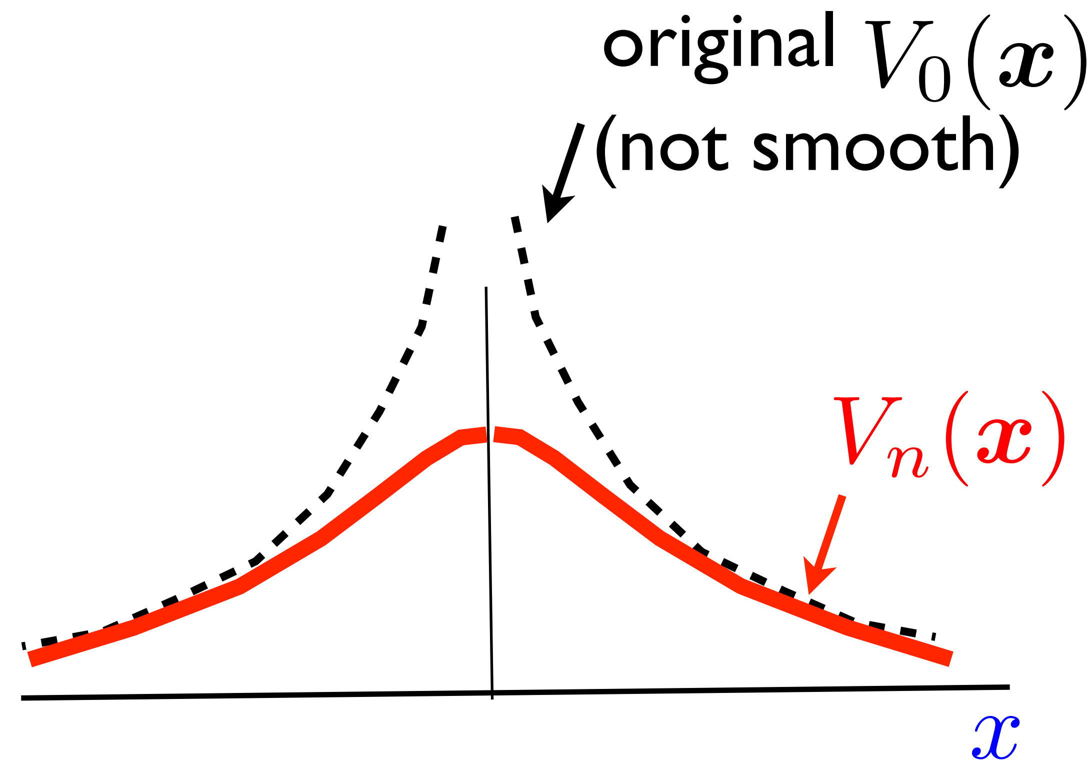
$$\boxed{\begin{aligned}[R^a, R^b] &= -i\ell_B^2\epsilon^{ab} \\ H &= \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)\end{aligned}}$$

residual problem is non-commutative quantum geometry!

$$[R^a, R^b] = -i\ell_B^2\epsilon^{ab}$$

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

Identical quantum particles
(fermions or bosons)

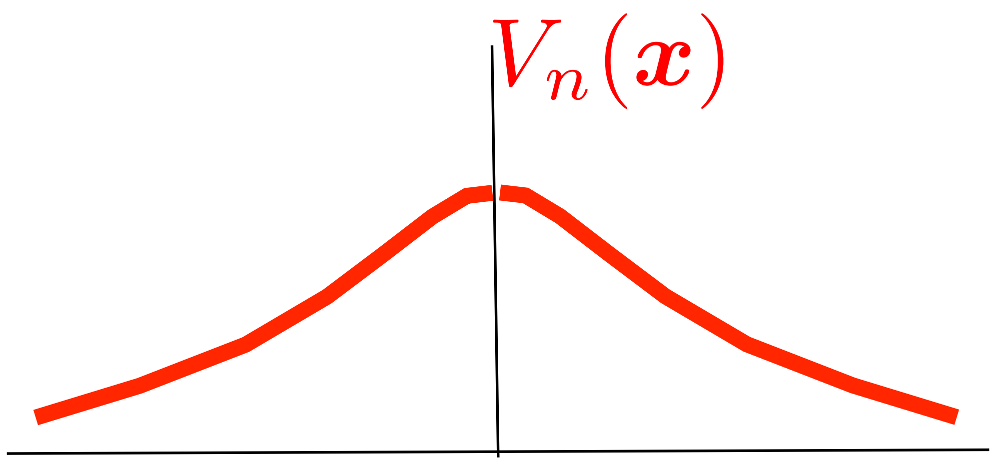


We now have the final form of the problem:

- The potential $V_n(x)$ is a **very smooth** (in fact entire) function that depends on the form- factor of the partially-occupied Landau level
- The essential clean-limit symmetries are translation and inversion: $\boldsymbol{R}_i \mapsto \boldsymbol{a} \pm \boldsymbol{R}_i$

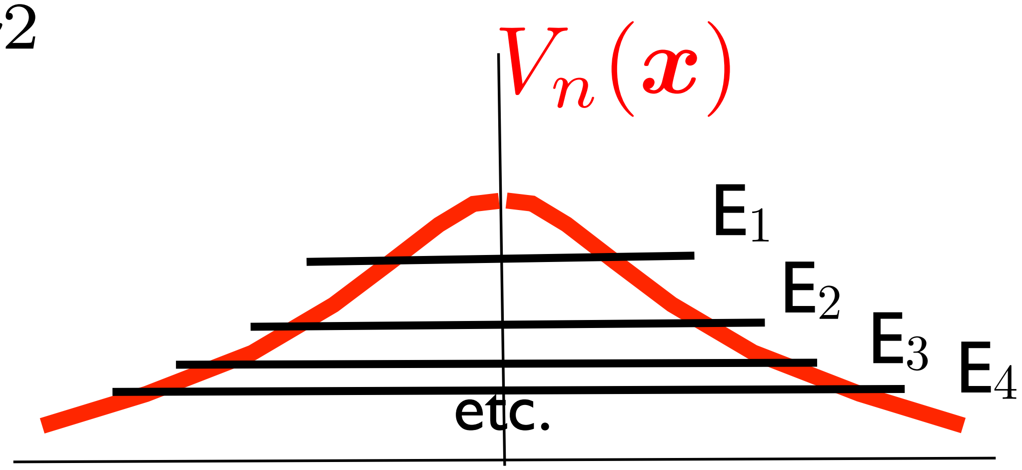
$$[R^a, R^b] = -i\ell_B^2\epsilon^{ab}$$

$$H = \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$



- The quadratic expansion of this even function around the origin defines a natural "interaction metric"
- The problem is often simplified by giving it a continuous rotation symmetry that respects this metric, but this is non-generic, and not necessary.
- This metric and a rotation symmetry are important in model FQH wavefunctions based on cft, which have a stronger conformal invariance property.

- It is straightforward to solve the two-body Hamiltonian: $\boldsymbol{R}_{12} = \boldsymbol{R}_1 - \boldsymbol{R}_2$



$$[R^a_{12}, R^b_{12}] = 2i\ell_B^2 \epsilon^{ab}$$

equivalent to a one-particle problem

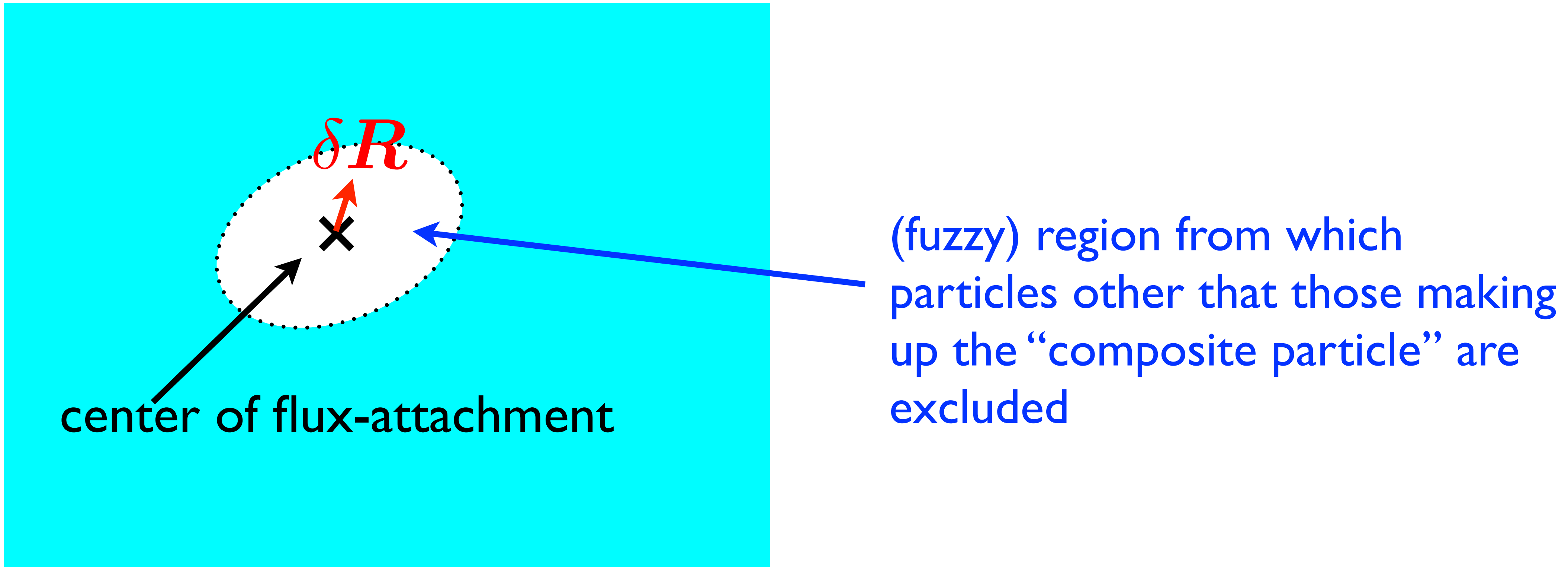
$$H = V_n(\boldsymbol{R}_{12})$$

- If there is a rotational symmetry, the energy levels (called "pseudopotentials") completely characterize the interaction potential.

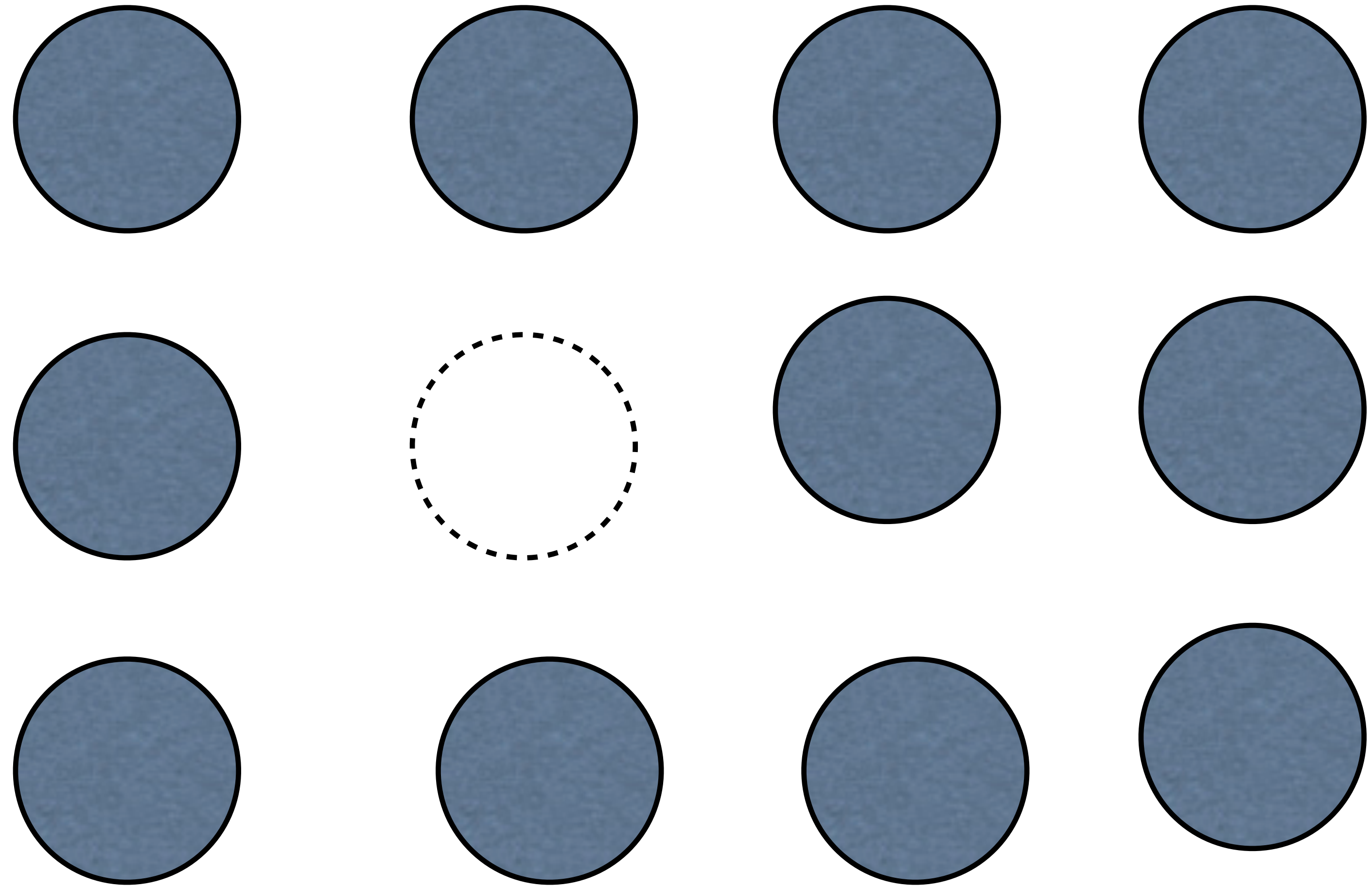
- a large gap between energy levels favors **<u>flux attachment</u>** with a shape close to that of the "interaction metric"

- Flux attachment is a gauge condensation that removes the gauge ambiguity of the guiding centers, giving each one a "natural" origin, so they define a physical electric dipole moment of the "composite particle" in which they are bound by the "attached flux".

- This is analogous to how the "the vector potential becomes an observable" (in a hand-waving way) in the London equations for a superconductor.



- quantum solid
- unit cell is correlation hole
- defines geometry



- repulsion of other particles make an attractive

potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

- In Maxwell's equations, the momentum density is

$$\pi_i = \epsilon_{ijk} D^j B_k \qquad D^i = \epsilon_0 \delta^{ij} E_j + P^i$$

- The momentum of the condensed matter is

$$\boldsymbol{p} = \boldsymbol{d} \times \boldsymbol{B}$$

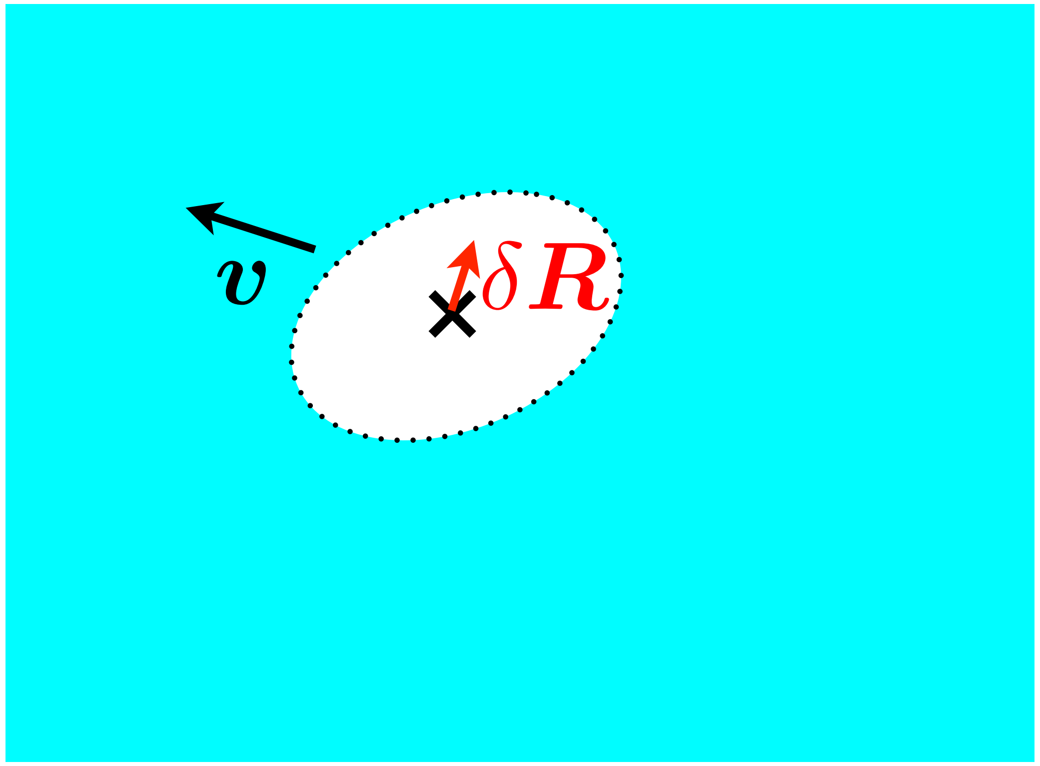
electric dipole moment

- in 2D the guiding-center momentum then is

$$p_a = eB\epsilon_{ab}\delta R^b$$

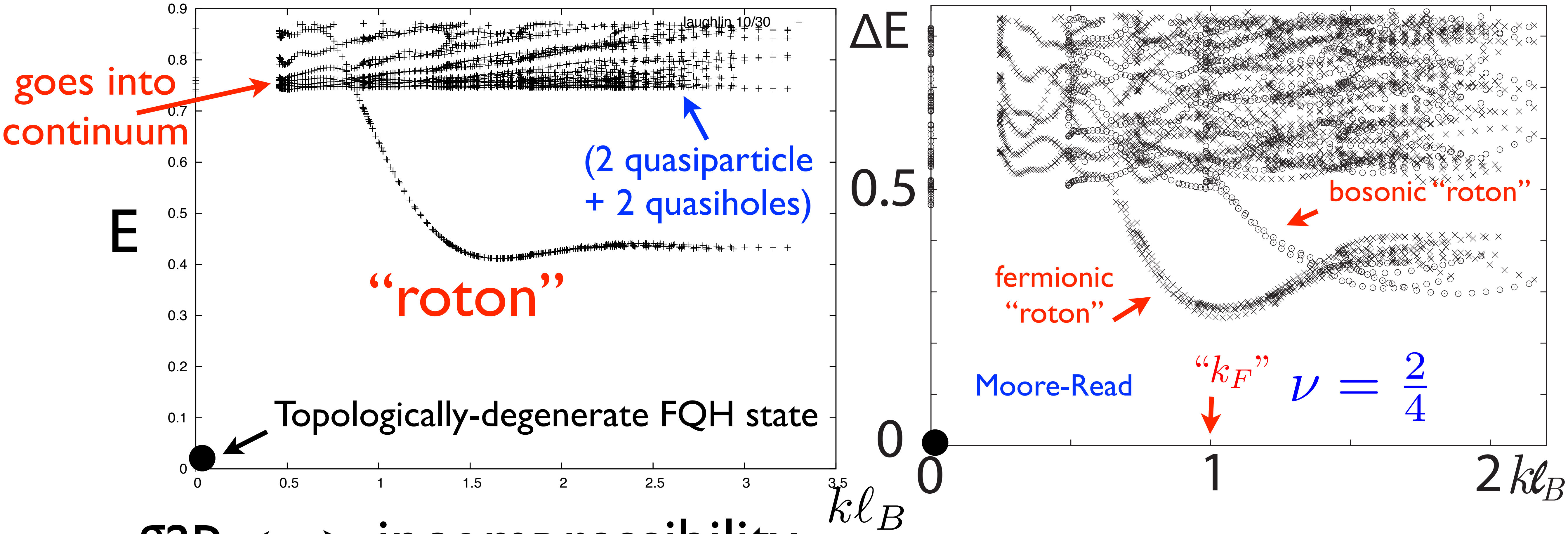
- The electrical polarization energy of the dielectric composite particle then gives its energy-momentum dispersion relation, with no involvement of any "Newtonian inertia" involving an effective mass

- The Berry phase generated by motion of the "other particles" that "get out of the way" as the vortex-like "flux-attachment" moves with the particle(s) it encloses can be formally-described as a Chern-Simons gauge field that cancels the Bohm-Aharonov phase, so that the composite object propagates like a neutral particle.



- If the composite particle is a boson, it condenses into the zero-momentum **(zero electric dipole-moment)** inversion-symmetric state, giving an incompressible-fluid Fractional Quantum Hall state, with an energy gap for excitations that carry momentum or electric dipole moment ("quantum incompressibility", no transmission of pressure through the bulk).

- All FQH states have an elementary unit (analogous to the unit cell of a crystal) that is a composite boson under exchange.
- It may be sometimes be useful to describe this boson as a a bound state of composite fermions (with their own preexisting flux attachment) bound by extra flux (Jain's picture)



gap $\longleftrightarrow$ incompressibility

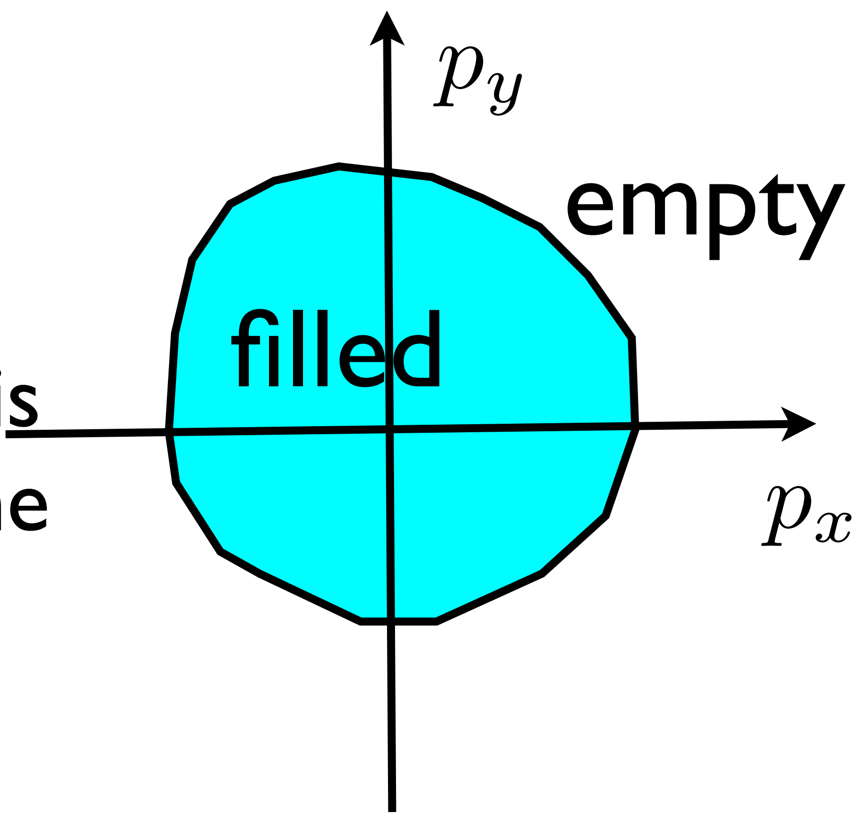
Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

Collective mode with short-range three-body pseudopotential, 1/2 filling (Moore-Read state is exact ground state in that case)

- momentum $\hbar$k of a quasiparticle-quasihole pair is proportional to its **electric dipole moment $\mathbf{p_e}$** $\quad \hbar k_a = \epsilon_{ab} B p_e^b$

gap for electric dipole excitations is a MUCH stronger condition than charge gap: fluid **does not transmit pressure through bulk!**

- The composite particle may also be a fermion. Then one gets a Fermi surface in momentum-space = electric dipole space, and a gapless anomalous Hall effect which is quantized when the Berry phase cancels the Bohm-aharonov phase. (HLR-type state)



- There must be a distribution of dipole moments (or momentum) of the composite fermions, centered at the inversion-symmetric zero-moment state which has lowest energy. These are quantized by a pbc, and no two composite fermions can have the same diople moment.

- Fermi surface quasiparticle formulas for anomalous Hall effect (FDMH 2006)
- in 2D:

$$\sigma_H = \frac{e^2}{2\pi\hbar}\left(n + \frac{\phi}{2\pi}\right)$$

Integer determined at edge

$$e^{i\phi}$$

Berry phase for moving a quasiparticle around Fermi surface (arc)

- holomorphic representations of guiding-center states

$$\frac{R^a}{\sqrt{2}\ell_B} = w^a a^\dagger + w^a a \qquad\qquad [a, a^\dagger] = 1$$

$$(w_a)^* w_a = \tfrac{1}{2}(g_{ab} + i\epsilon_{ab}) \qquad w_a = g_{ab} w^b \quad \det g = 1$$

- This is the Girvin-Jach formalism, except they implicitly assumed the metric g_{ab} was the Euclidean metric of the plane. **In fact, it is a <u>free choice</u>, not fixed by the any physics of the problem.**

- Then, once a metric (i.e., a complex structure) has been chosen, a one-particle state can be described as

$$|\Psi\rangle = \underbrace{f}_{\text{holomorphic}}(a^\dagger)|0\rangle \qquad a|0\rangle = 0$$

- Both the "vacuum" $|0\rangle$ and the function f(z) vary as the metric is changed (a Bogoliubov transformation)

- Normalization/overlap:

$$\langle\Psi_1|\Psi_2\rangle = \int \frac{dz\wedge dz^*}{2\pi i} f_1(z)^* f_2(z) e^{-z^* z}$$

- When compactified on the torus with flux N_Φ , the modular-invariant formulation is

$$f(z) \propto \prod_{i=1}^{N_\Phi} \tilde{\sigma}(z - w_i) \qquad \sum_i w_i = 0$$

Bravais lattice in complex plane

$$\tilde{\sigma}(z|\{L\}) = e^{\frac{1}{2}C_2(\{L\})z^2}\sigma(z|\{L\})$$

"almost holomorphic modular invariant" (Eisenstein series)

Weierstrass sigma function

- In the Heisenberg-algebra reinterpretation

$$|\Psi\rangle = \prod_{i=1}^{N_\Phi} \tilde{\sigma}(a_i^\dagger - w_i)|0\rangle \qquad \sum_i w_i = 0$$

one particle
$N = 1$

- The filled Landau level is

$$|\Psi\rangle = \left(\prod_{i<j} \tilde{\sigma}(a_i^\dagger - a_j^\dagger)\tilde{\sigma}(\textstyle\sum_i a_i^\dagger) \right) |0\rangle$$

filled Level
$N = N_\Phi$

- The Laughlin states are

$\nu = \frac{1}{m}$ Laughlin state
$N_\Phi = mN$

$$|\Psi\rangle = \left(\prod_{i<j} \tilde{\sigma}(a_i^\dagger - a_j^\dagger)^m \right) \prod_{k=1}^{m} \tilde{\sigma}(\textstyle\sum_i a_i^\dagger - w_k)|0\rangle \quad \displaystyle\sum_{k=1}^{m} w_k = 0.$$

- A previously unknown (?) identity that I recently guessed and found was indeed true, and which dramatically transforms calculations on torus (e.g., orders of magnitude Monte-Carlo speedup)

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{\square} \frac{dz \wedge dz^*}{2\pi i} f_1(z)^* f_2(z) e^{-z^* z}$$

$$= \frac{1}{N_\Phi} \sum_z{}' \qquad z \in \{ \tfrac{mL_1 + nL_2}{N_\Phi} \}$$

$(N_\Phi)^2$ points

replace integral over fundamental region by a modular-invariant finite sum

- with Ed Rezayi, I found a remarkable clean composite Fermi liquid model state on the flat torus, inspired by a construction by Jain on the sphere.
- On the torus, the state is precisely equivalent to the usual treatments of the Fermi gas with a pbc.
- It is very accurate as compared to exact diagonalization of the Coulomb interaction, and amazingly "almost" (99.99%) particle-hole symmetric at ν = 1/2.

- Composite Fermi liquid (HLR-like) at $\nu = \dfrac{1}{m}$

gives Chern-Simons

gives bf? / Z2

$$f(\{z_i\}) = \prod_{i<j} \tilde{\sigma}(z_i - z_j)^{(m-2)} \det_{ij} M_{ij} \prod_{k=1}^{m} \tilde{\sigma}(\textstyle\sum_i z_i - w_k)$$

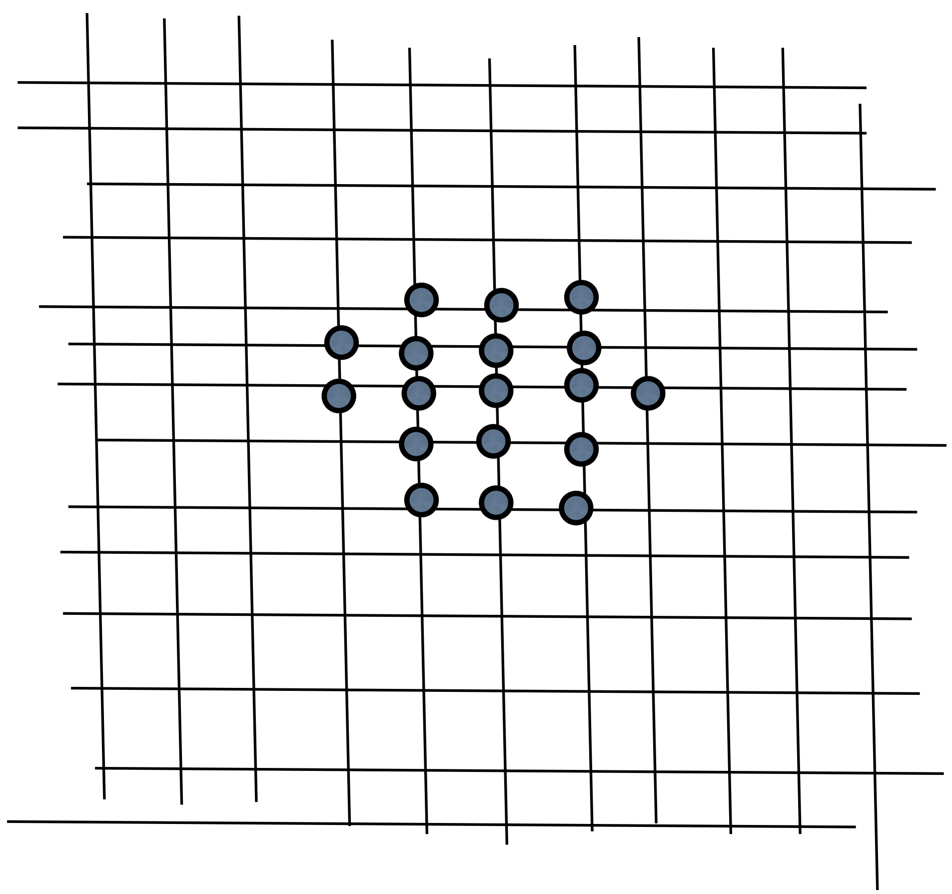
$$\sum_{\alpha=1}^{m} w_\alpha = \sum_{j=1}^{N} d_j = N\bar{d}$$

Fermi (Bose) for m even (odd)

$$M_{ij}(\{z_k\};\{d_k\}) = e^{d_j^* z_i/m} \prod_{k\neq i}{}' \tilde{\sigma}(z_i - z_j - d_i + \bar{d})$$

a set of dipole moments $\quad d_i \in \dfrac{L}{N}$ (particle number, not flux)

- There are vastly more possible choices of dipole "occupations" than independent states: The "good" ones are clusters that minimize $\sum_i |d_i - \bar{d}|^2$

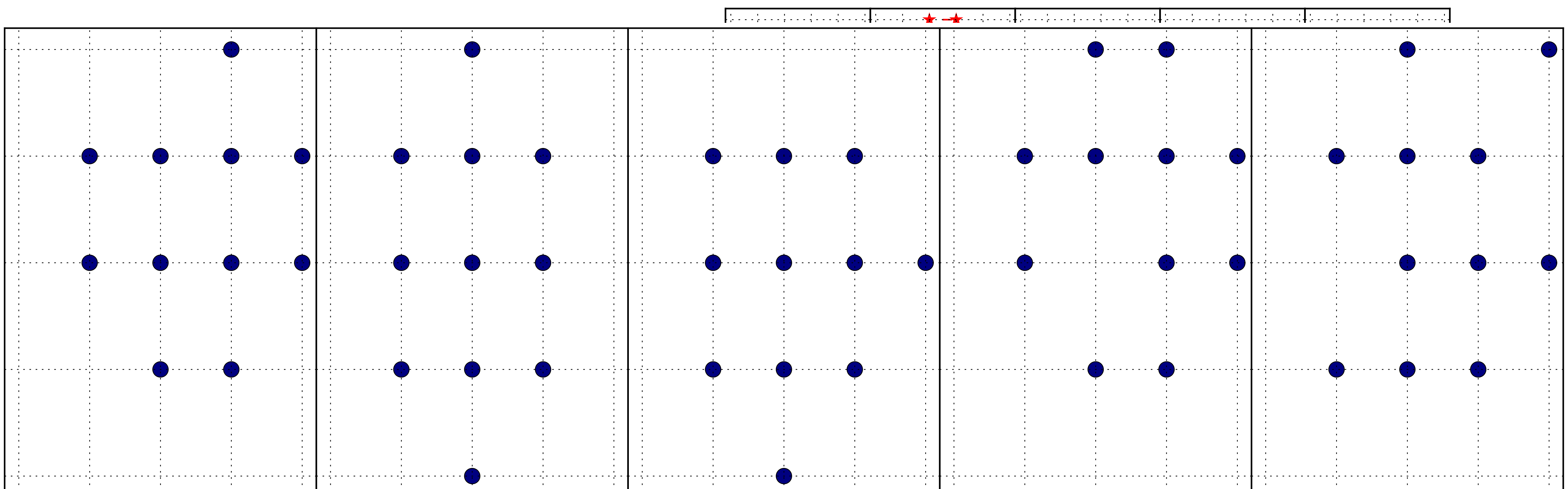


Computing ph symmetry (with Scott Geraedts)

model state is numerically **very** close to p-h symmetry when k's are clustered

```
# Z_{COM}  overlap with PH-conjugate
overlap
0 0.999998870263 1.1297367517e-06
1 0.999999369175 6.3082507884e-07
2 0.99999860296 1.39704033186e-06
3 0.99999860296 1.3970403312e-06
4 0.999999369175 6.30825078063e-07
5 0.999998870263 1.12973675237e-06
6 0.999999369175 6.30825079173e-07
7 0.99999860296 1.39704032942e-06
8 0.99999860296 1.39704032909e-06
9 0.999999369175 6.30825078507e-07
```

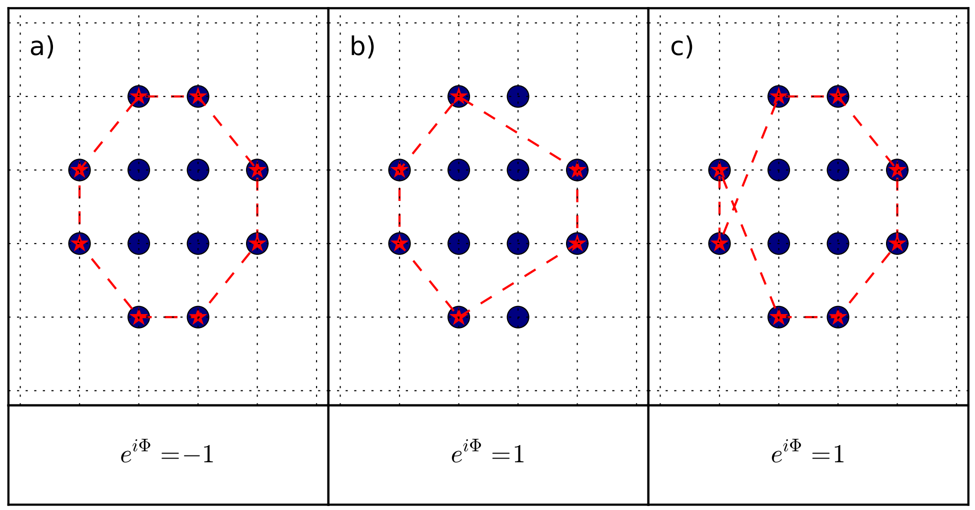
- particle-hole symmetry, and Kramers Z_2 structure (Scott Geraedts and Jie Wang)

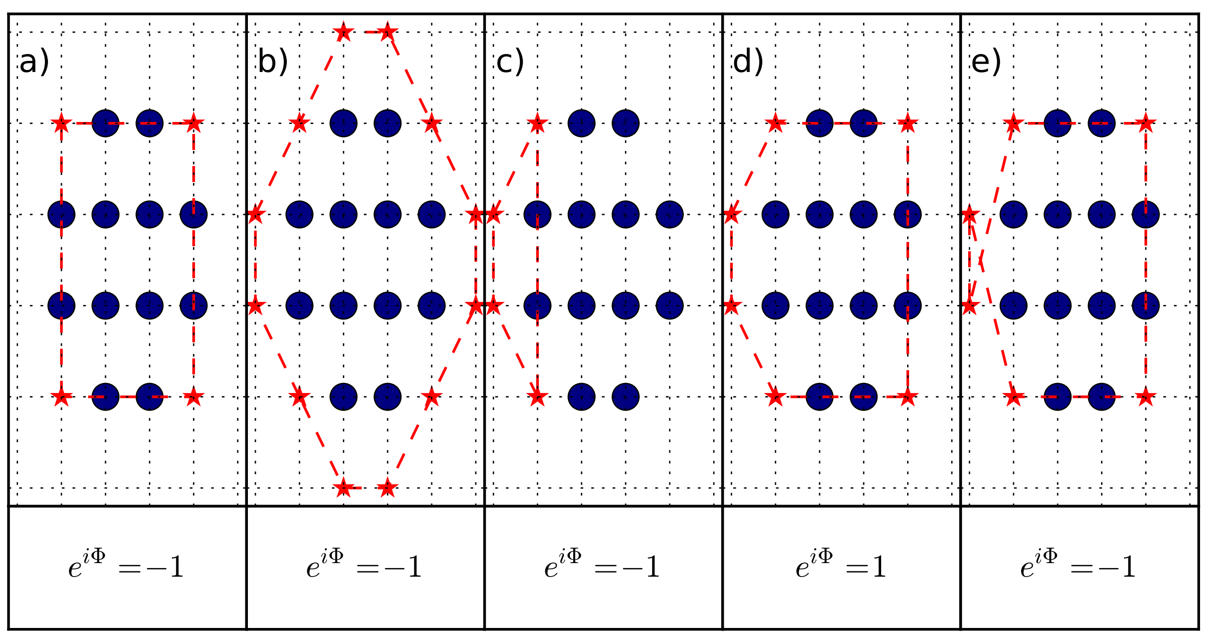


ED ΔE	0.0	ED ΔE	0.000404	ED ΔE	0.000268	ED ΔE	0.005555	ED ΔE	0.0
overlap	0.981955	overlap	0.736717	overlap	0.992142	overlap	0.226429	overlap	0.276141
ΔE	0.000446	ΔE	0.001419	ΔE	0.000531	ΔE	0.009271	ΔE	0.011863
$\langle H^2 \rangle - \langle H \rangle^2$	6.81819e-06	$\langle H^2 \rangle - \langle H \rangle^2$	6.5675e-06	$\langle H^2 \rangle - \langle H \rangle^2$	4.88251e-06	$\langle H^2 \rangle - \langle H \rangle^2$	1.58755e-05	$\langle H^2 \rangle - \langle H \rangle^2$	3.72072e-05
PH symmetry	0.993543	PH symmetry	0.987538	PH symmetry	0.995312	PH symmetry	0.856664	PH symmetry	0.964223

A many-body ansatz for Berry phase

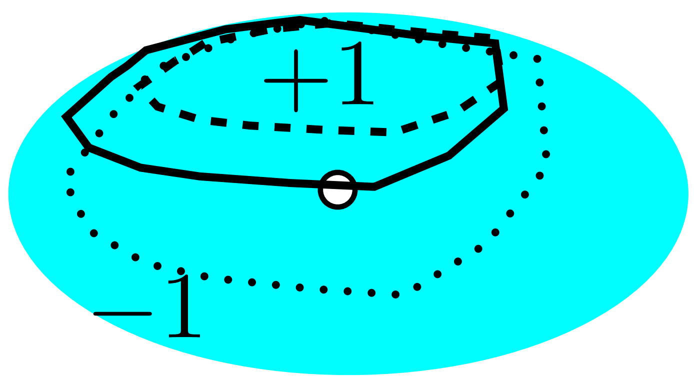
we confirmed all paths are real, by Kramers





These are ED results on exact Coulomb interaction states, with the exact particle hole symmetry, with occupation patterns obtained by finding the model states they have high overlap with

- is there an analog of Dirac cone point ?



State with the quantum numbers of an inversion symmetric Fermi sea with a single hole at the center (has an even number of particles)

A state on the Torus with these quantum numbers is a parity doublet

- as a hole is moved into the bulk, the ansatz must fail as if goes through the inversion-symmetric point!