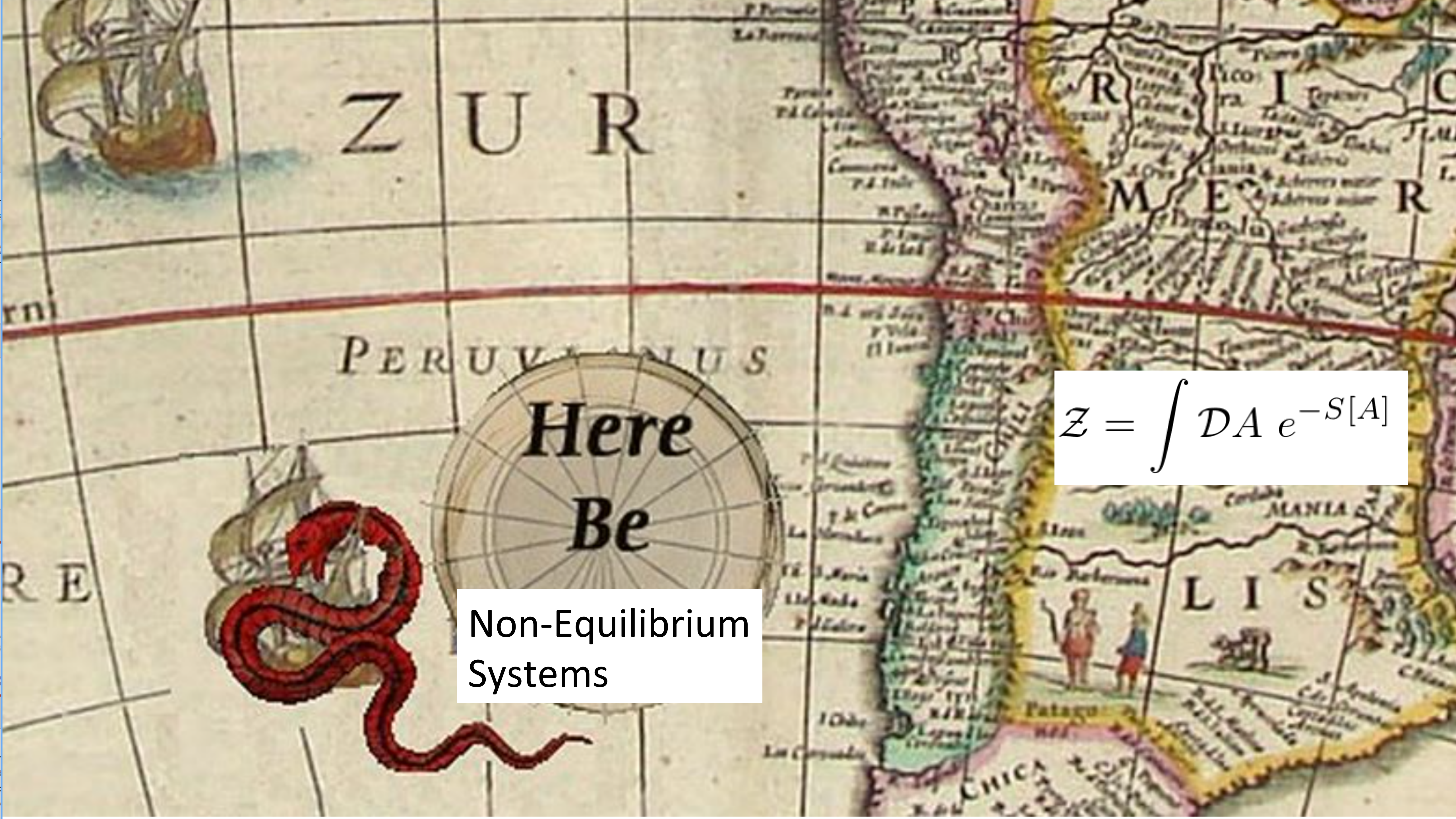
# Time Crystals

D. Else, B. Bauer, C. Nayak

Microsoft Station Q and UCSB

Dominic Else, Bela Bauer, C.N., arXiv:1607.05277
Dominic Else, Bela Bauer, C.N., PRL **117**, 090402 (2016).
Dominic Else, C.N., PRB **93**, 201103 (2016) .


ZUR
Here Be
Non-Equilibrium Systems
$\mathcal{Z} = \int \mathcal{D}A \; e^{-S[A]}$
MER
LIS

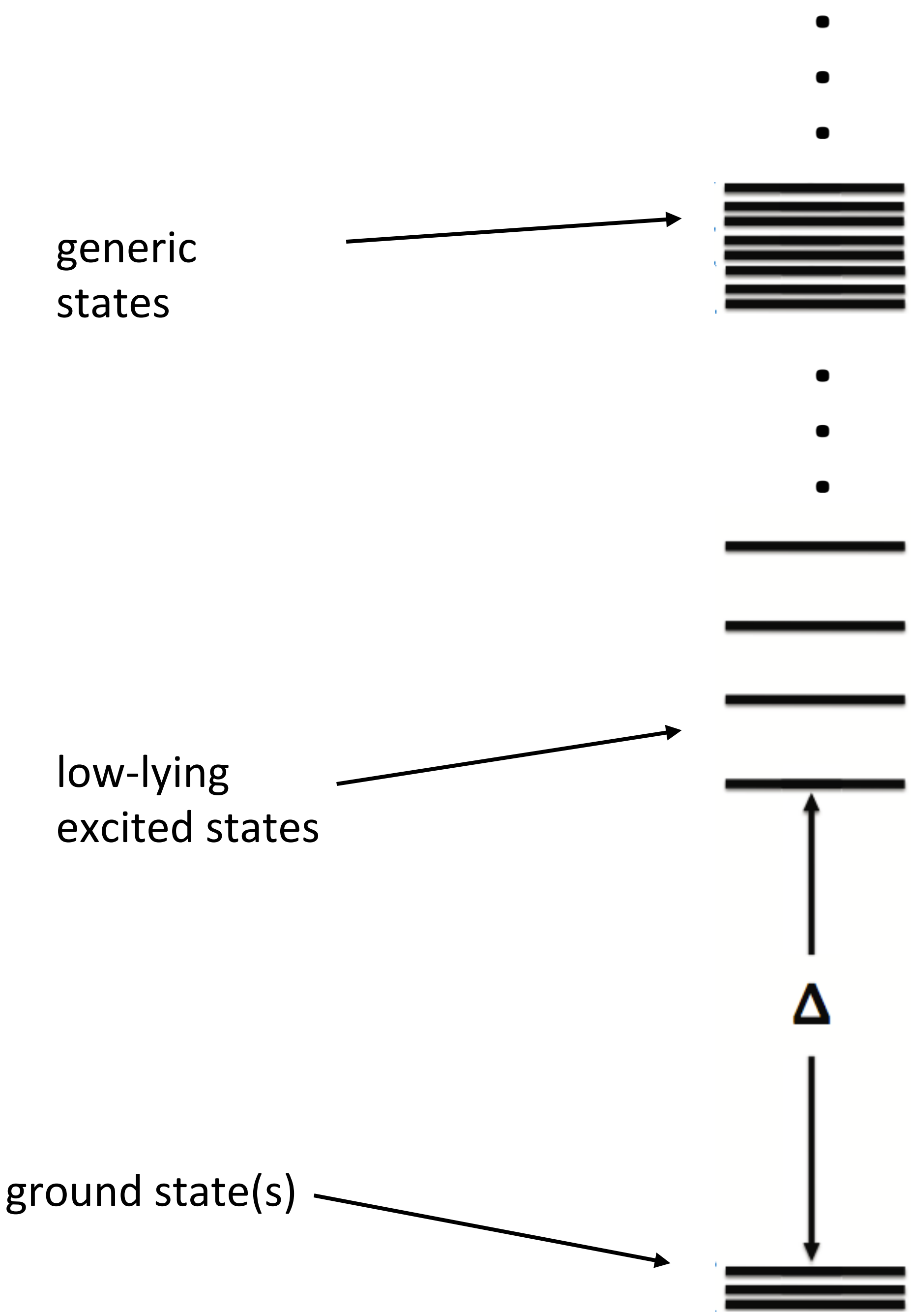
# Universal Non-Equilibrium Quantum Phenomena

- Floquet Topological Phases/SPTs
- Stable Gapless Edge Modes
- Broken-Symmetry Phases
- ***Time Crystals***

Many-Body Localization can stabilize.

*But not necessary.*

Chandran and Sondhi ’16
Khemani et al. ’16; von Keyserlingk and Sondhi ’16
von Keyserlingk and Sondhi ’16; Potter et al. ’16
Roy and Harper ’16; Von Keyserlingk et al. ’16
Po et al. ’16; Potter and Morimoto ’16


generic states
low-lying excited states
ground state(s)
Δ


CHINA
JAPAN
RUSSIA
PACIFIC OCEAN
LOS ANGELES
TEXAS
UTAH
LAS VEGAS
NEBRASKA
KANSAS CITY
MEXICO
CANADA
Chicago
WASHINGTON D.C.
JERSEY
HUDSON RIVER
10TH AVE
9TH AVE
STEINBERG

# Periodically-Driven Systems

TTS: Hamiltonian $H(t) = H(t+T)$

period of the drive

$$i\frac{d|\psi(t)\rangle}{dt} = H(t)|\psi(t)\rangle$$

time-translation operator

$$U(t_2, t_1) \equiv \mathrm{T}\exp\left(-i\int_{t_1}^{t_2} dt\, H(t)\right)$$

Floquet Operator:

$$U_{\mathrm{f}} \equiv U(T, 0)$$

Floquet Eigenstates:
(eigenstates of $U(T, 0)$)

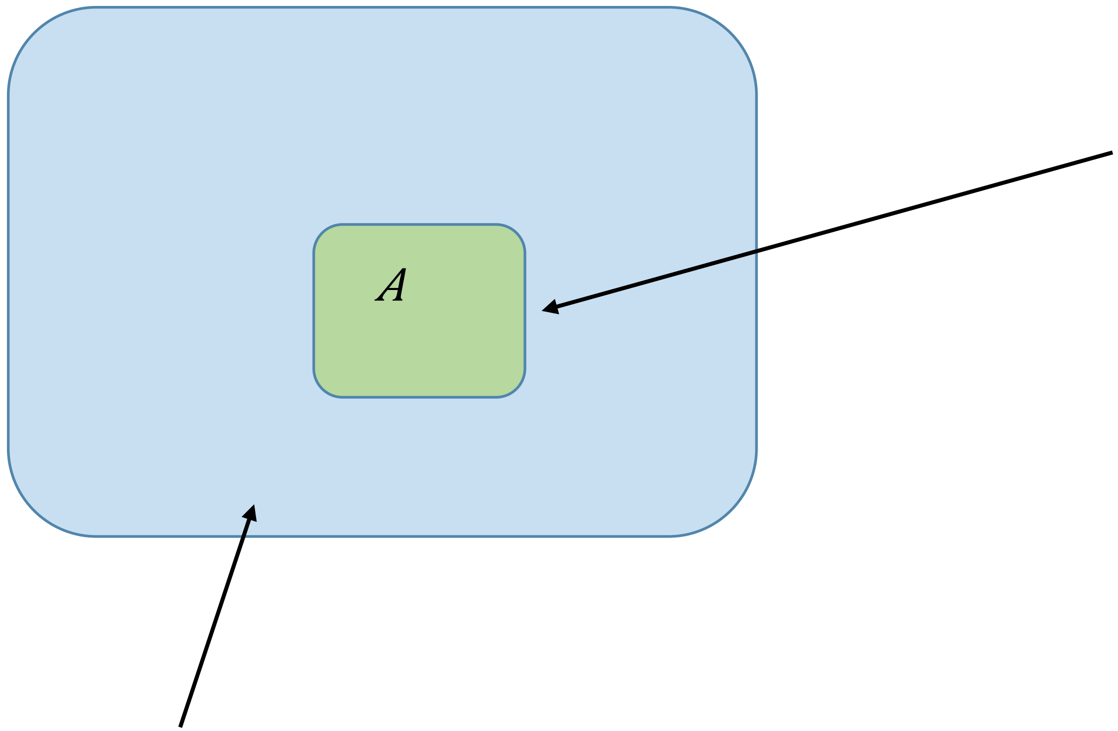
$$|\psi_\alpha(t)\rangle = e^{-i\epsilon_\alpha t}|\phi_\alpha(t)\rangle$$

$\epsilon_\alpha \equiv \epsilon_\alpha + 2\pi/T$

$$|\phi_\alpha(t)\rangle = |\phi_\alpha(t+T)\rangle$$

Floquet Eigenvalues:
“quasienergies”

# Properties of Generic Energy/Floquet Eigenstates




Reduced Density Matrix: $\rho_A \equiv \mathrm{tr}_{A^c} \, |\varepsilon\rangle\langle\varepsilon|$

Entanglement Entropy: $S_A \equiv \mathrm{tr} \, \rho_A \ln \rho_A$

| | Eigenstate Thermalization Hypothesis (ETH) | Many-Body Localization (MBL) |
|---|---|---|
| Static Systems | | |
| Floquet Systems | | |

# What are Time Crystals?

Wilczek '12, Shapere and Wilczek '12: *Spontaneously-Broken Time-Translation Symmetry*

P. Bruno '13; Nozieres '13: *proposals not in ground state; will radiate energy*

$|0\rangle \rightarrow e^{iE_0 t}|0\rangle$

Ground State: *One-point function*: ~~$\langle 0|\Phi(x,t)|0\rangle = f(t)$~~

*Two-point function*: ~~$\langle 0|\Phi(x,t)\Phi(0,0)|0\rangle \rightarrow f(t)$~~

Watanabe and Oshikawa '15

Thermal Equilibrium: ~~$\mathrm{tr}(\Phi(x,t)\Phi(0,0)\, e^{-\beta H}) = f(t)$~~

*'No Go' theorem is a challenge*

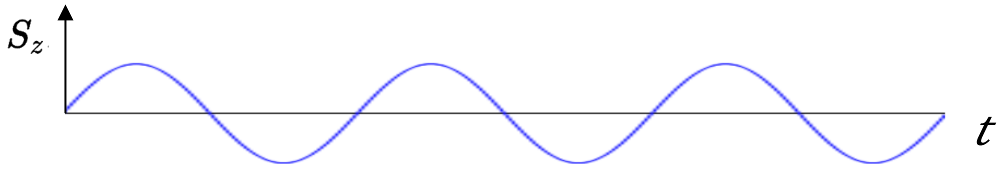
# What systems are *Not* Time Crystals?

# What are Time Crystals?

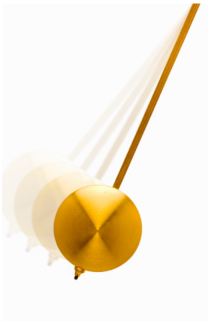
- Some part of the definition may be too restrictive → *assumption of ground state/thermal equilibrium*

- Out of equilibrium, it's too easy to get time dependence

✗ Rabi oscillations: $H = -\gamma \, S_z B$




✗ Pendulum:



✓ Borderline cases: AC Josephson effect, driven BEC

K. Sacha '14

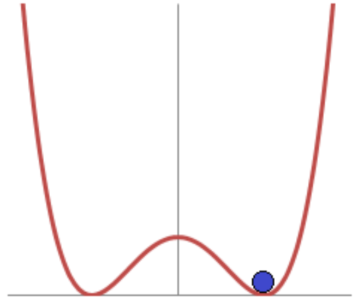
Time Crystal 'piggybacks' on some other broken symmetry

Explicitly break other symmetry, Time Crystal gone

# ‘Ordinary’ Spontaneously-Broken Symmetry

$$H = -\sum_{i=1}^{L}\left(h\sigma_i^x + J\sigma_i^z\sigma_{i+1}^z\right) - \tilde{h}\sigma_1^z$$

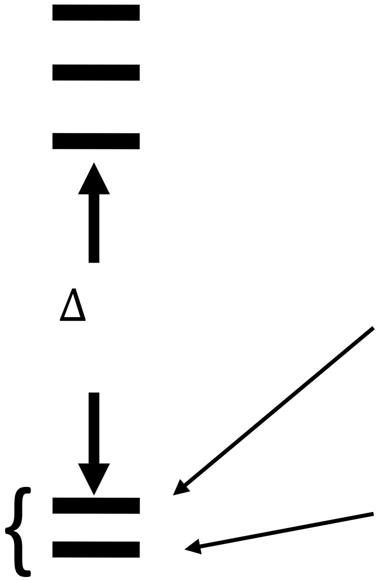
$\langle \sigma_i^z \rangle_{h} \neq 0$



$|\downarrow\downarrow\ldots\downarrow\rangle$ $|\uparrow\uparrow\ldots\uparrow\rangle$

$$H = -\sum_{i=1}^{L}\left(h\sigma_i^x + J\sigma_i^z\sigma_{i+1}^z\right)$$

spectrum




$O(e^{-L/\xi})$

“cat states”

$\frac{1}{\sqrt{2}}(|\uparrow\uparrow\ldots\uparrow\rangle - |\downarrow\downarrow\ldots\downarrow\rangle)$

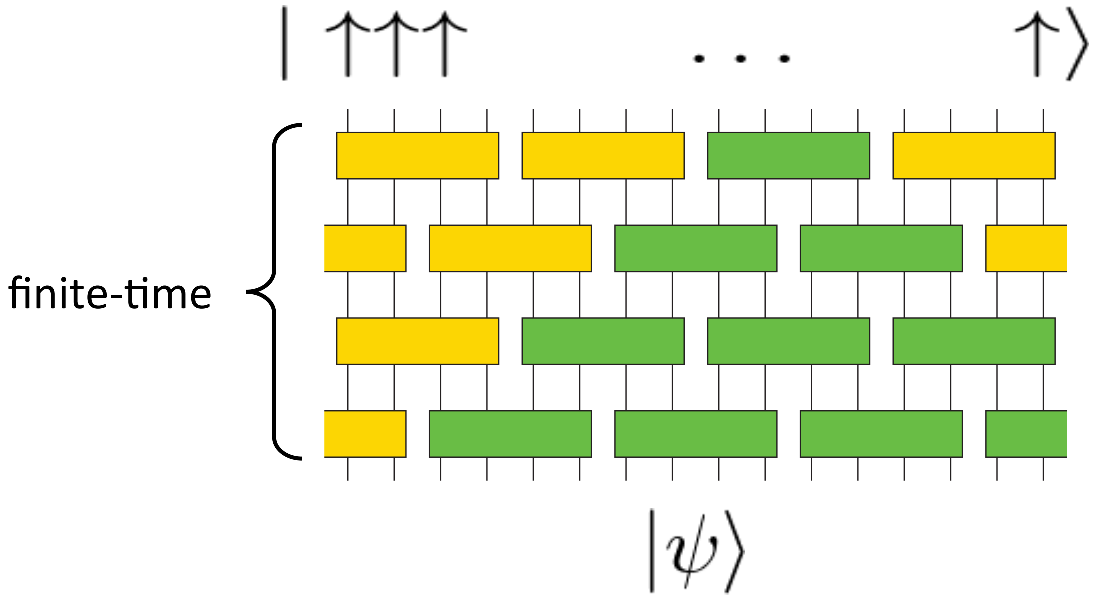
$\frac{1}{\sqrt{2}}(|\uparrow\uparrow\ldots\uparrow\rangle + |\downarrow\downarrow\ldots\downarrow\rangle)$

Low-lying eigenstates of $g$ and $H$ must be “cat states”.

# Short-Range vs. Long-Range Correlated ("Cat")States

- SRC: states such as $|\uparrow\uparrow\cdots\uparrow\rangle$ and states obtained from it in finite time, starting from initial product state.

  *Can be prepared in experiments.*



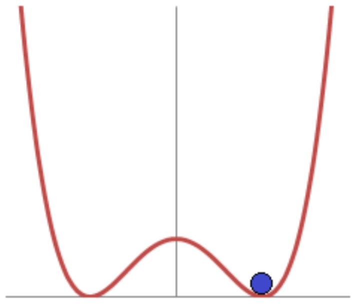

- LRC: states such as $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\ldots\uparrow\rangle + |\downarrow\downarrow\ldots\downarrow\rangle)$

  *Cannot be prepared in finite time from a product state.*
  *Will decohere rapidly, if coupled to environment.*

<u>short-range correlated</u>:
$$\langle\psi|\Phi(x)\Phi(x')|\psi\rangle - \langle\psi|\Phi(x)|\psi\rangle\langle\psi|\Phi(x')|\psi\rangle \rightarrow 0$$

*cluster decomposition holds* $\qquad |x - x'| \rightarrow \infty.$

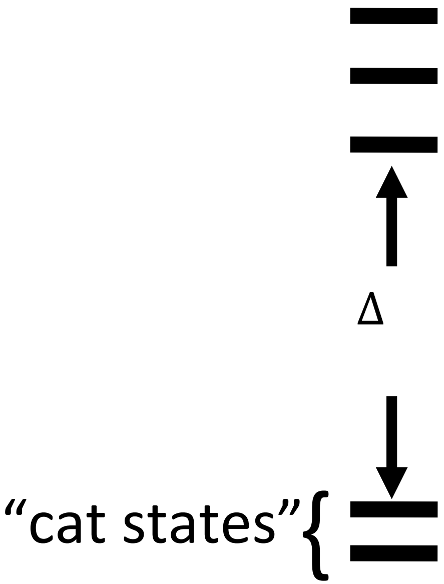
# Definition of Time-Transl. Symm.-Breaking (TTSB)



TTSB-1: TTSB occurs if for each $t_1$, and for every short-range correlated $|\psi(t_1)\rangle$, there exists an operator $\Phi$ such that $\psi(t+T)\Phi\psi(t+T) \neq \psi(t)\Phi\psi(t)$ for all $t$.

TTSB-2: TTSB occurs if the eigenstates of $U(T,0)$ cannot be short-range correlated.

*(no other symmetries assumed)*



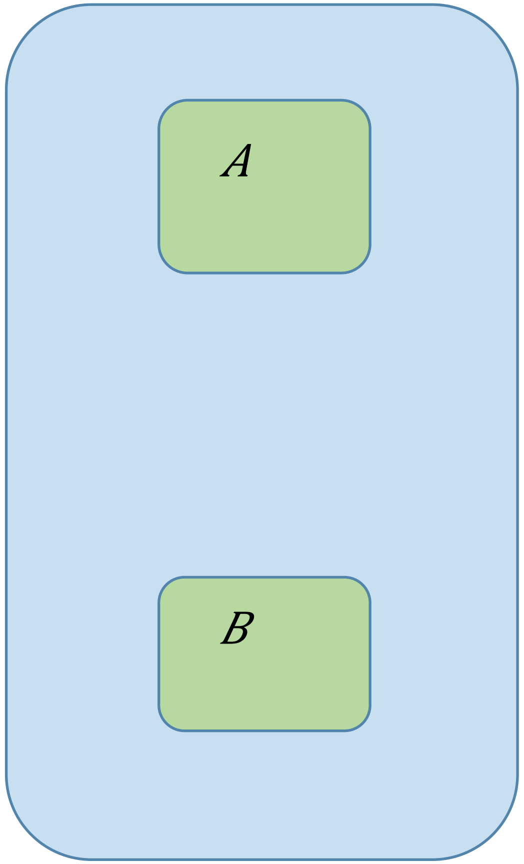

*Note: no restriction to the ground state/low-lying states* at this stage

# Discrete TTSB

- TTS by $T$ spontaneously broken down to TTS by $nT$

signatures

- Some observables will oscillate with period $nT$ *lower harmonics*, not higher harmonics
- Mutual Information: $I(A,B) \equiv S_A + S_B - S_{AB}$

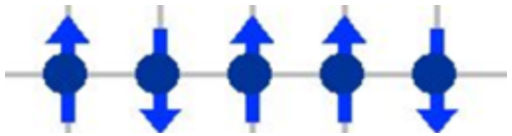


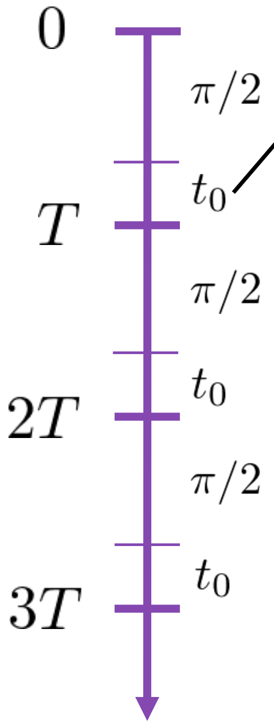

in Floquet eigenstates: $\boxed{I(A,B) \to \ln n}$

*measures violation of cluster decomposition*

# Soluble Model of Discrete TTSB

Chain of Ising Spins






$$U_{\mathrm{f}} = \exp\left(-it_0 H_{\mathrm{MBL}}\right)\, \exp\left(i\frac{\pi}{2}\sum_i \sigma_i^x\right)$$

$$H_{\mathrm{MBL}} = \sum_i \left(J_i \sigma_i^z \sigma_{i+1}^z + h_i^z \sigma_i^z + h_i^x \sigma_i^x\right)$$

$$J_i \in [\tfrac{1}{2}, \tfrac{3}{2}],\ h_i^z \in [0,1],\ h_i^x \in [0,h]$$

See also: Prosen ’98; Alessio and Rigol ’14;
Ponte et al. ’15; Iadecola et al. ’15;
L. Jiang et al. ’11; M. Thakurathi et al. ’13
Khemani et al. ’15; von Keyserlingk and Sondhi ’16

# Solube Model of Discrete TTSB

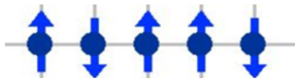
*h*=0 limit: $$H_{\mathrm{MBL}} = \sum_i \left( J_i \sigma_i^z \sigma_{i+1}^z + h_i^z \sigma_i^z \right)$$

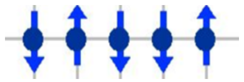
$H_{\mathrm{MBL}}$ eigenstates: $\sigma_k^z |\{s_i\}\rangle \;=\; s_k |\{s_i\}\rangle$

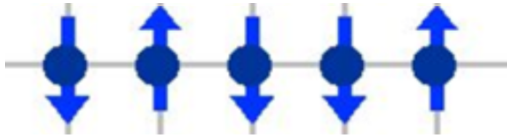
$H_{\mathrm{MBL}}$ eigenvalues: $E^+(\{s_i\}) \;+\; E^-(\{s_i\})$

$$\begin{cases} E^+(\{s_i\}) \;=\; \sum_i (J_i s_i s_{i+1}) \\ E^-(\{s_i\}) \;=\; \sum_i (h_i^z s_i) \end{cases}$$

$$U_{\mathrm{f}} = \exp\left(-it_0 H_{\mathrm{MBL}}\right) \exp\left( i\frac{\pi}{2} \sum_i \sigma_i^x \right)$$



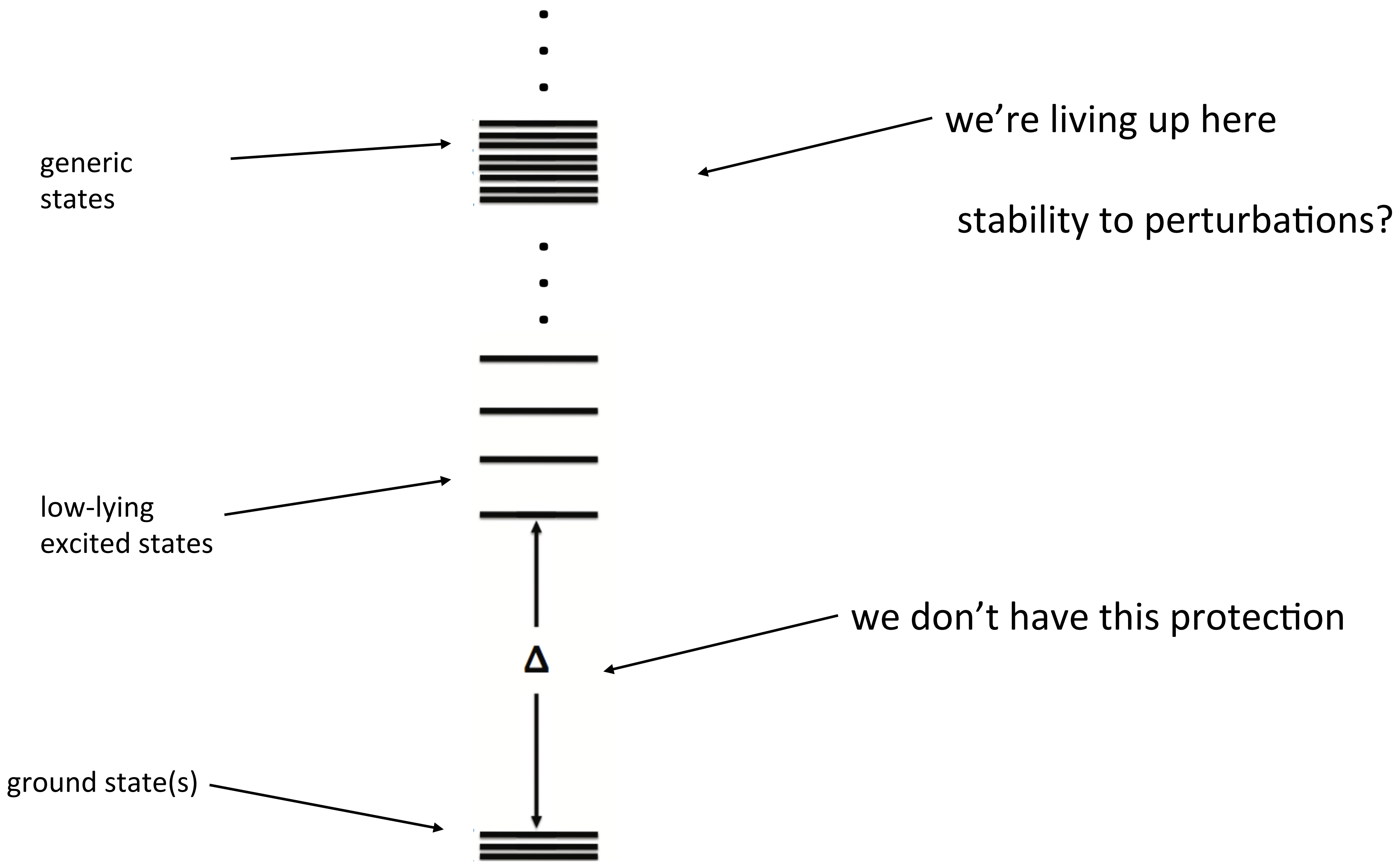




eigenstates of $U_{\mathrm{f}}$: $e^{i\frac{t_0}{2}E^-(\{s_i\})} |\{s_i\}\rangle \pm e^{-i\frac{t_0}{2}E^-(\{s_i\})} |\{-s_i\}\rangle$

eigenvalues of $U_{\mathrm{f}}$: $\pm \exp(it_0 E^+(\{s_i\}))$

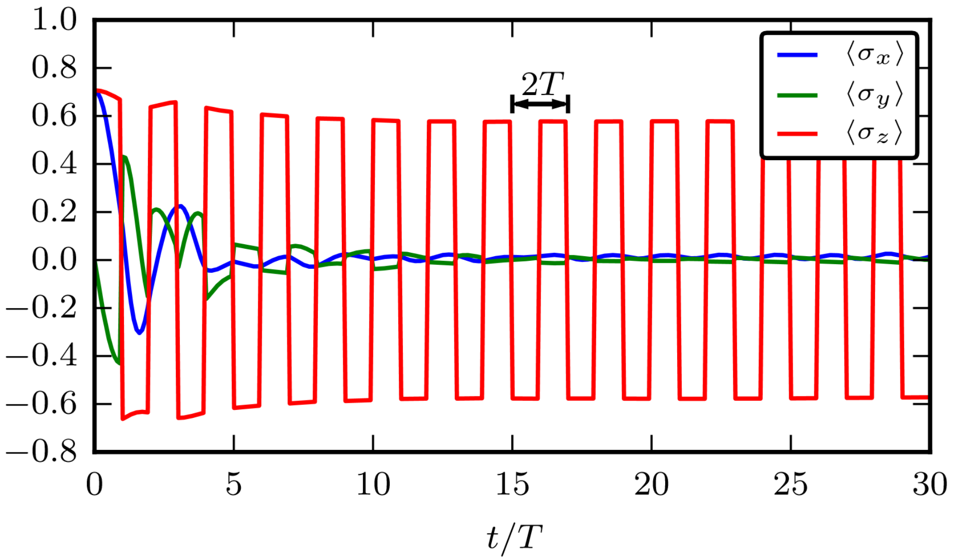
"cat states"


generic states
low-lying excited states
ground state(s)
Δ
we're living up here
stability to perturbations?
we don't have this protection

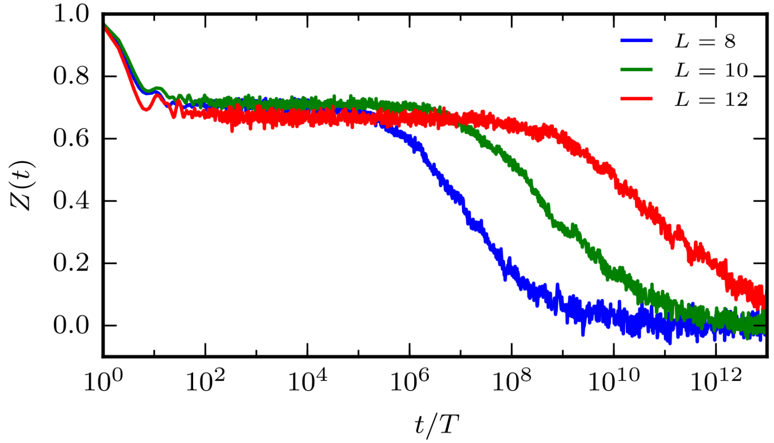
# Stability of TTSB -- Numerics

Compute $U_f(nT)|\psi(0)\rangle$ by
*TEBD* (large systems, shorter times)
and *ED* (smaller systems, longer times)

transient decay, followed
by *persistent oscillations*



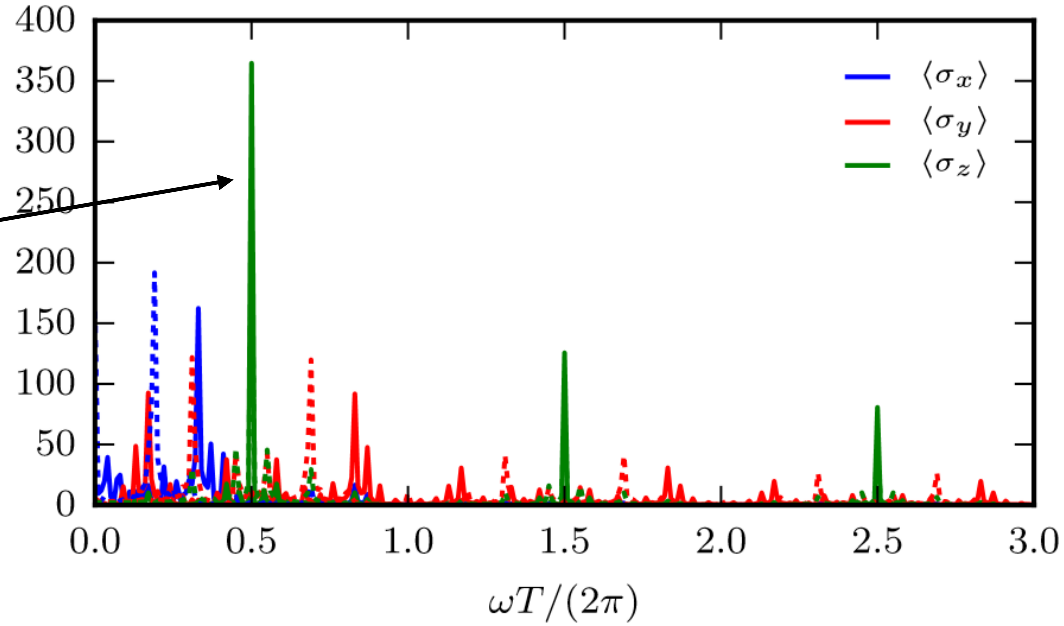

*log scale*
oscillations persist until time ~ $O(e^{aL})$



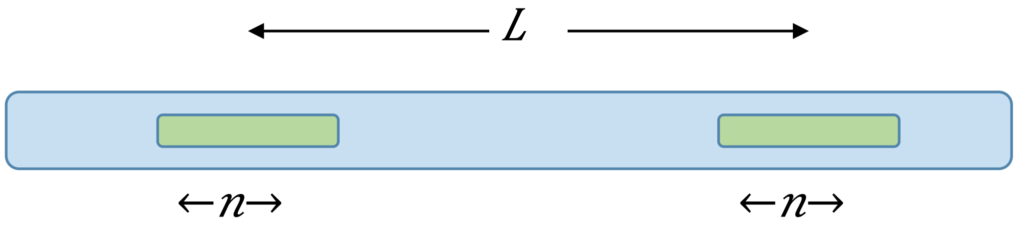
# Stability of TTSB -- Numerics

TEBD: single realization

persistent oscillations,
together with noise



# Stability of TTSB -- Numerics

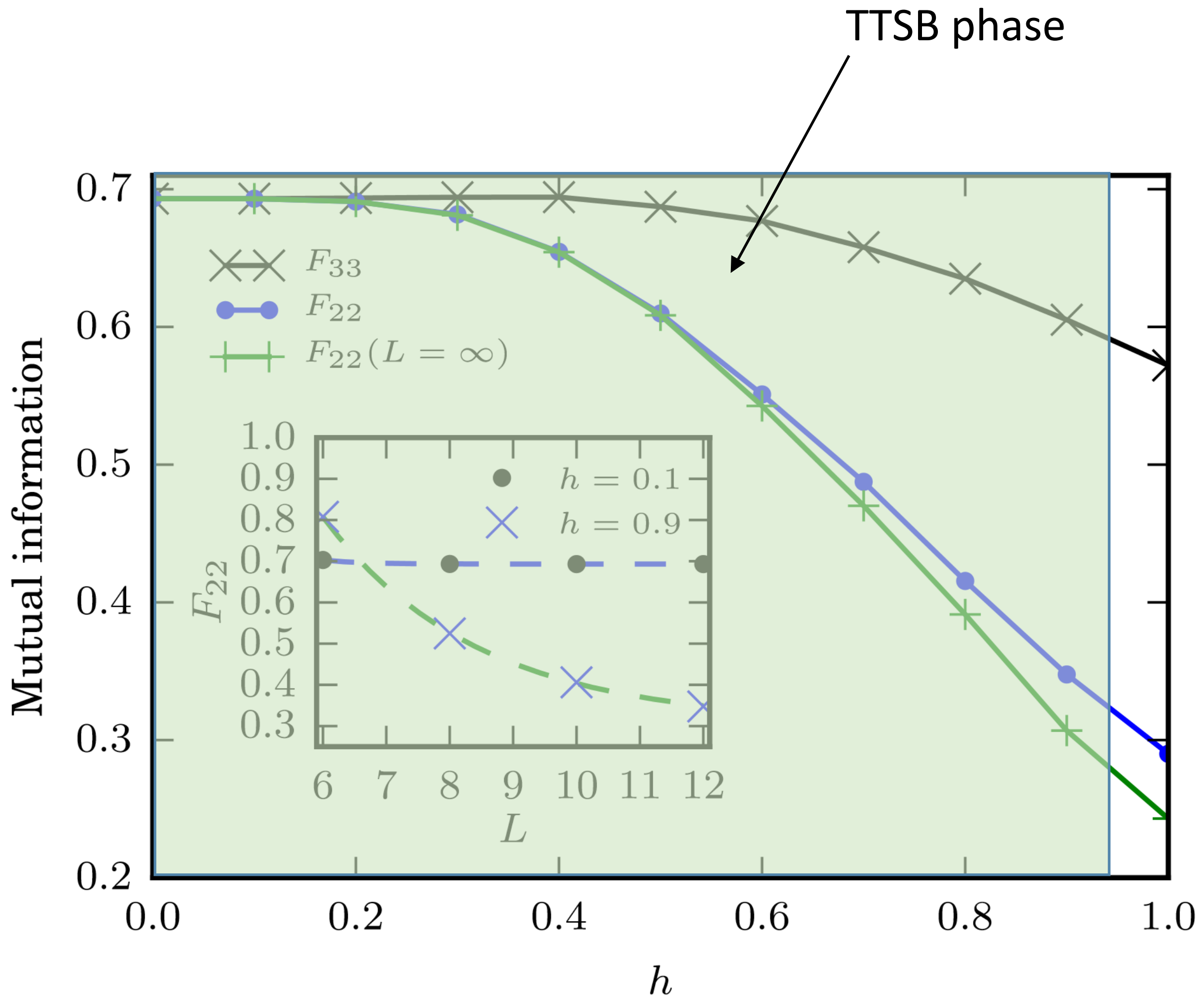



Mutual Information between Green Regions:

$$F_{nn}(h,L) \;=\; F_{nn}(g,\infty) + c_n \exp(-L/\xi(h))$$

≠0 in TTSB phase

$F_{nn}(g,\infty) \to \ln 2$ as $n \to \infty$

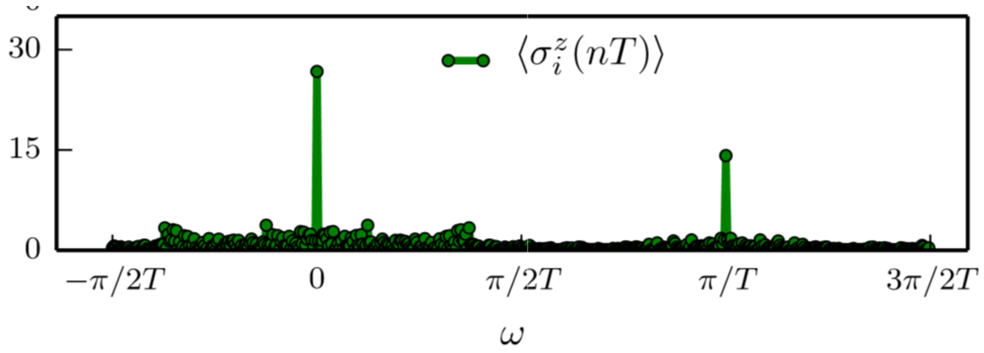



Eigenstates

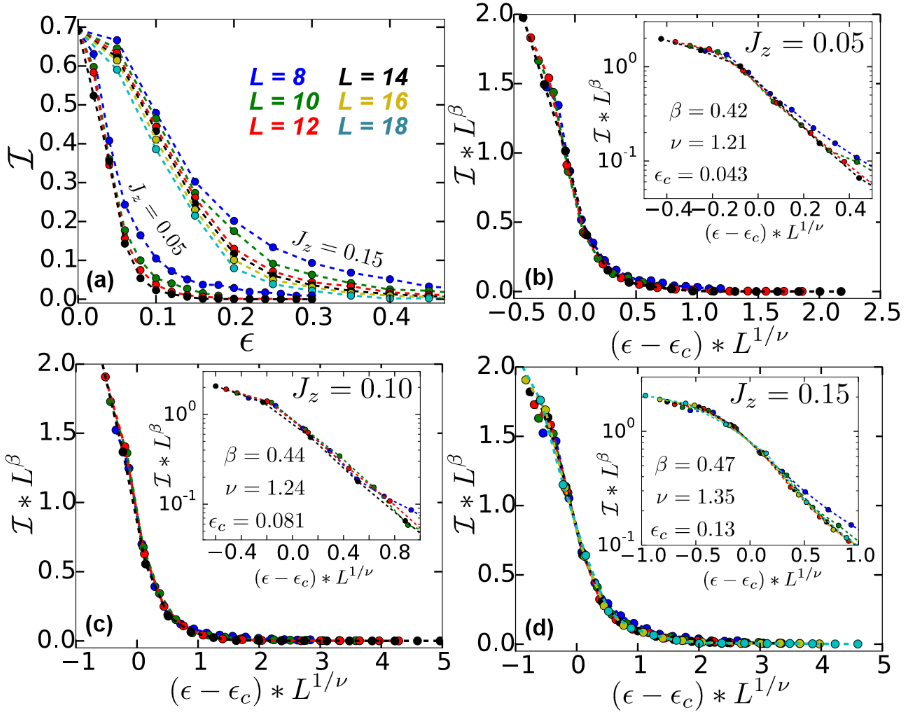
# Further Investigations of Floquet Time Crystals

N. Yao et al. arXiv: 1608.02589

von Keyserlingk et al. arXiv: 1605.00639




noise decays as power-law

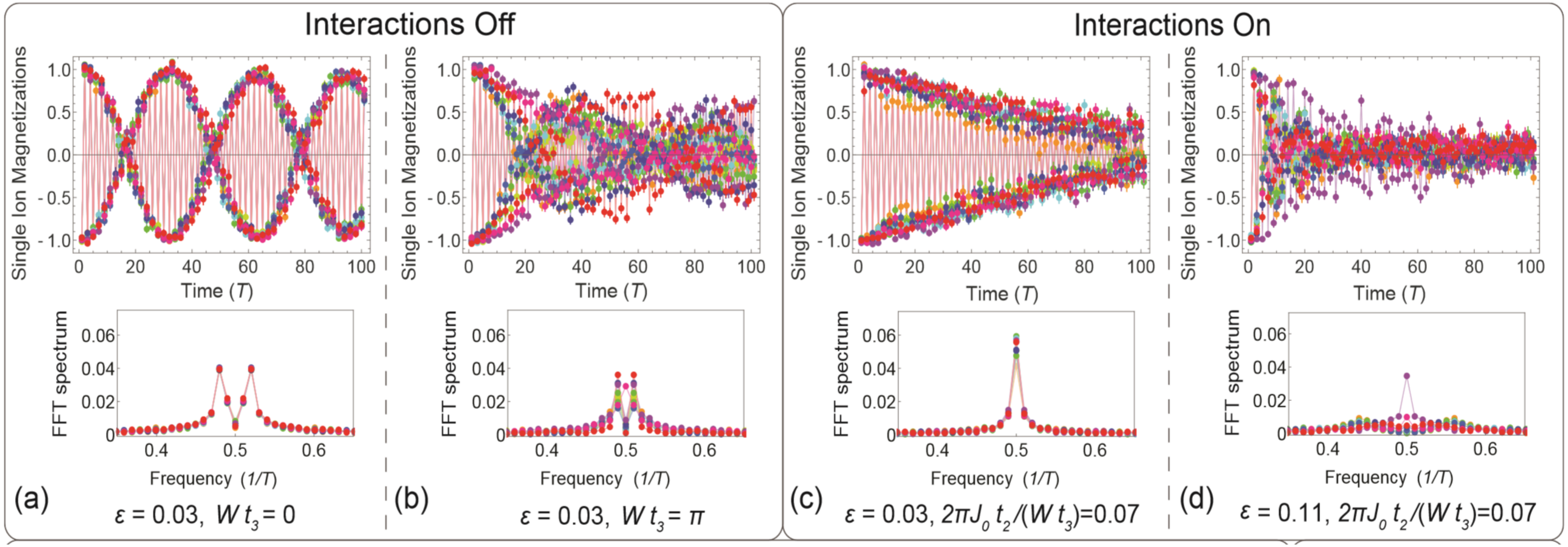



Transition out of Floquet time crystal:
random Ising transition

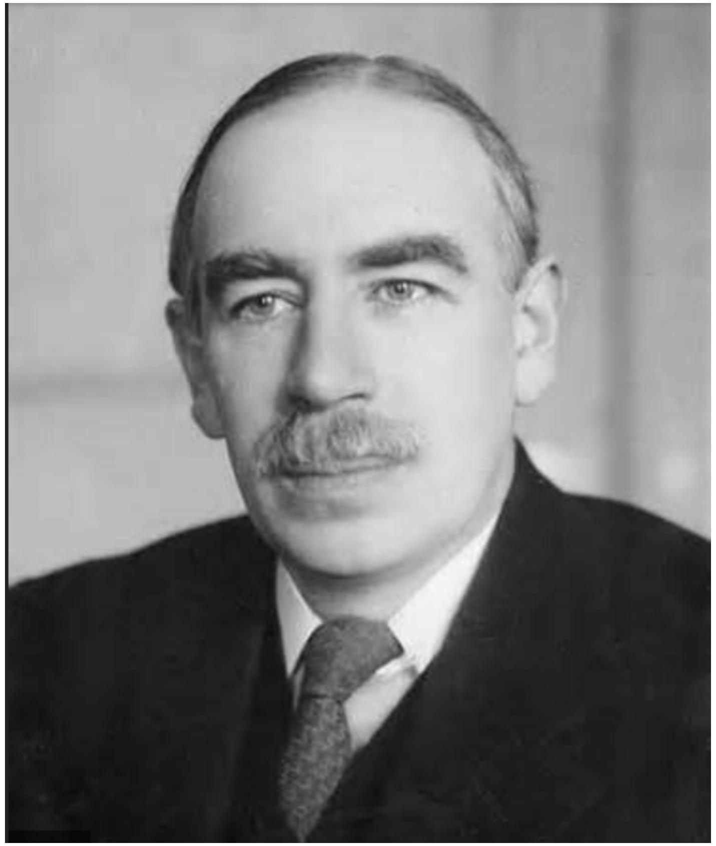
# Experimental Obs. of Floquet Time Crystals

$$H = \begin{cases} H_1 = g(1-\varepsilon)\sum_i \sigma_i^y, & \text{time } t_1 \\ H_2 = \sum_i J_{ij}\sigma_i^x\sigma_j^x, & \text{time } t_2 \\ H_3 = \sum_i D_i\sigma_i^x & \text{time } t_3. \end{cases}$$

J. Zhang *et al.* arXiv: 1609.08684



| | Eigenstate Thermalization Hypothesis (ETH) | Many-Body Localization (MBL) |
|---|---|---|
| **Static Systems** | generic excited energy eigenstate, non-zero energy density<br>$\boldsymbol{\rho_A \approx e^{-\beta H_A}}$ | $\boldsymbol{S_A \propto \lvert \partial A \rvert}$ |
| **Floquet Systems** | ETH at $T=\infty$ :<br>$\boldsymbol{\rho_A \approx 1}$ | doesn't heat up to $\mathrm{T}=\infty$,<br>$\boldsymbol{S_A \propto \lvert \partial A \rvert}$ |



Keynes: “In the long run, we are all dead.”

# **ADHD** Theorems

D. **A**banin, **D**e Roeck, W.-W. **H**o, F. Huveneers arXiv:1509.05386

$$U_f = \mathcal{T}\exp\left(-i\int_0^T H(t)dt\right)$$ for small $\|H\|_{loc} \cdot T$

$\exists\ \mathcal{U}:\quad \mathcal{U}\, U_f \mathcal{U}^\dagger = e^{-iDT}$ + exponentially-small stuff

D. Abanin, De Roeck, W.-W. Ho, F. Huveneers

# **ADHD** Theorems

$$U_f = \mathcal{T}\exp\left(-i\int_0^T H(t)dt\right)$$

$$\begin{cases} H(t) = H_0(t) + V(t) \\ X \equiv U_0(0,T) \qquad X^N = 1 \end{cases}$$

for small $||V||_{\text{loc}} \cdot T$

$\exists \; \mathcal{U} : \quad \mathcal{U} U_f \mathcal{U}^\dagger = X e^{iDT}$ + exponentially-small stuff

where $[D, X] = 0$

pre-thermal Hamiltonian

rapid thermalization with respect to it

Equilibrium for $D$ = non-equil. steady-state for $H(t)$
Universal phenomena: phases, phase transitions, …

# **ADHD** Theorems

**Theorem 1.** *Consider a periodically-driven system with Floquet operator:*

$$U_f = \mathcal{T}\exp\left(-i\int_0^T H(t)dt\right) \tag{3}$$

*where* $H(t) = H_0(t) + V(t)$, *and* $X \equiv U_0(0,T)$ *satisfies* $X^N = 1$ *for some integer* $N$. *We assume that* $H_0(t)$ *can be written as a sum* $H_0(t) = \sum_i h_i(t)$ *of terms acting only on single sites* $i$. *Define* $\lambda \equiv \|V\|_1$. *Assume that*

$$\lambda T \le \frac{\gamma\kappa_1^2}{N+3}, \quad \gamma \approx 0.14. \tag{4}$$

*Then there exists a (time-independent) unitary* $\mathcal{U}$ *such that*

$$\mathcal{U}\,U_f\,\mathcal{U}^\dagger = X\,\mathcal{T}\exp\left(-i\int_0^T [D + E + V(t)]dt\right) \tag{5}$$

*where* $D$ *is local and* $[D, X] = 0$; $D, E$ *are independent of time; and*

where:

$$\|V\|_{n_*} \le \lambda\left(\frac{1}{2}\right)^{n_*} \tag{6}$$

$$\|E\|_{n_*} \le \lambda\left(\frac{1}{2}\right)^{n_*} \tag{7}$$

*The exponent* $n_*$ *is given by*

$$n_* = \frac{\lambda_0/\lambda}{[1+\log(\lambda_0/\lambda)]^3}, \quad \lambda_0 = \frac{(\kappa_1)^2}{72(N+3)(N+4)T} \tag{8}$$

“pre-thermal Hamiltonian”

# Pre-thermal Time Crystals: Example

$$H(t) = H_0(t) + V, \qquad \begin{cases} H_0(t) = -\sum_i h^x(t)\sigma_i^x \\ V = -J\sum_{\langle i,j\rangle} \sigma_i^z \sigma_j^z - h^z \sum_i \sigma_i^z, \end{cases}$$

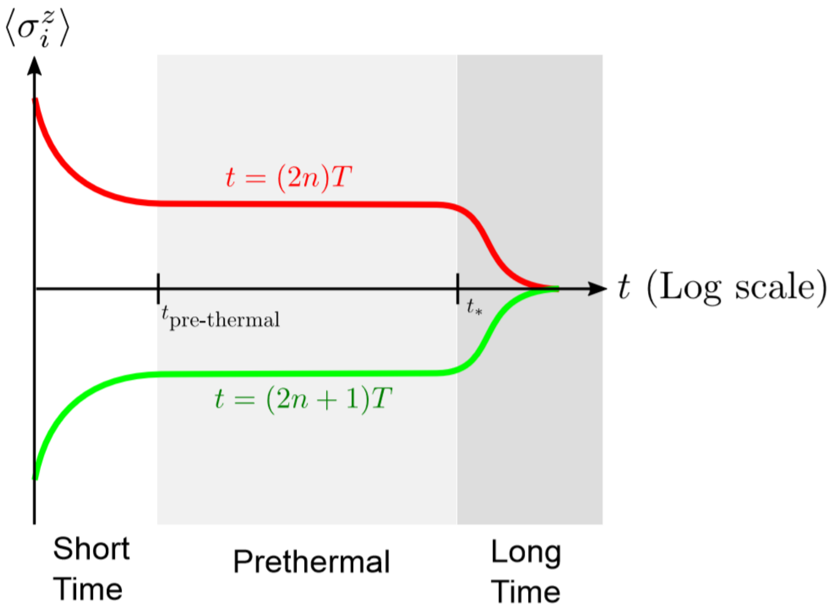
performs a $\pi$-pulse ⟶ $\int_0^T h^x(t)dt = \frac{\pi}{2}$

# Pre-thermal Time Crystals: Example

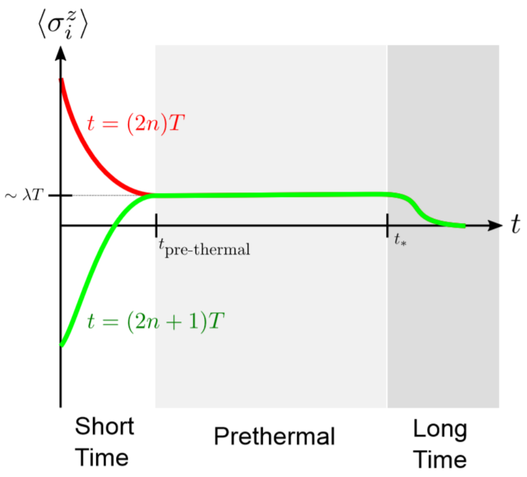
$$h^x(t) = \frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(t-kT) \quad \Rightarrow \quad D = -J\sum_{\langle i,j\rangle}\sigma_i^z\sigma_j^z + \ldots$$

determines intermediate time-evol

(a) Time crystal




(b) Non-time crystal



# Pre-thermal Time Crystals: Effective Field Theory

$$U_{\mathrm{f}} \approx \mathcal{U}(Xe^{-iDT})\mathcal{U}^\dagger \quad \Rightarrow \quad \begin{aligned} \langle\psi_f|\big(U_{\mathrm{f}}\big)^m|\psi_i\rangle &= \langle\psi_f|\mathcal{U}(Xe^{-iDT})^m\mathcal{U}^\dagger\,|\psi_i\rangle \\ &= \langle\tilde{\psi}_f|\,e^{-iDmT}\,|\tilde{\psi}_i\rangle \end{aligned}$$

$$\begin{aligned} \langle\psi|\,\hat{O}(x,kT)\hat{O}(0,kT)\,|\psi\rangle &\equiv \langle\psi|\,\big(U_{\mathrm{f}}\big)^{-k}\,\hat{O}(x,0)\hat{O}(0,0)\,\big(U_{\mathrm{f}}\big)^k\,|\psi\rangle \\ &= \mathrm{tr}(e^{-\beta D}\hat{O}(x)\hat{O}(0)) \\ &= \int\mathcal{D}\varphi\,e^{-\int d^dx\,d\tau\,\left[\frac{1}{2}K(\partial_\tau\varphi)^2+\frac{v^2}{2}K(\nabla\varphi)^2+U(\varphi)\right]} \end{aligned}$$

*Universal prop. of phases and transition (in the rotated basis) ≅ equilibrium Ising model.*

# Discrete TTS and Its Consequences

$$\int \mathcal{D}\varphi \, e^{-\int d^d x \, d\tau \left[ \frac{1}{2} K (\partial_\tau \varphi)^2 + \frac{v^2}{2} K (\nabla \varphi)^2 + U(\varphi) \right]}$$

ADHD: $[D, X] = 0$

Discrete TTS by $T$: $\varphi \to -\varphi$

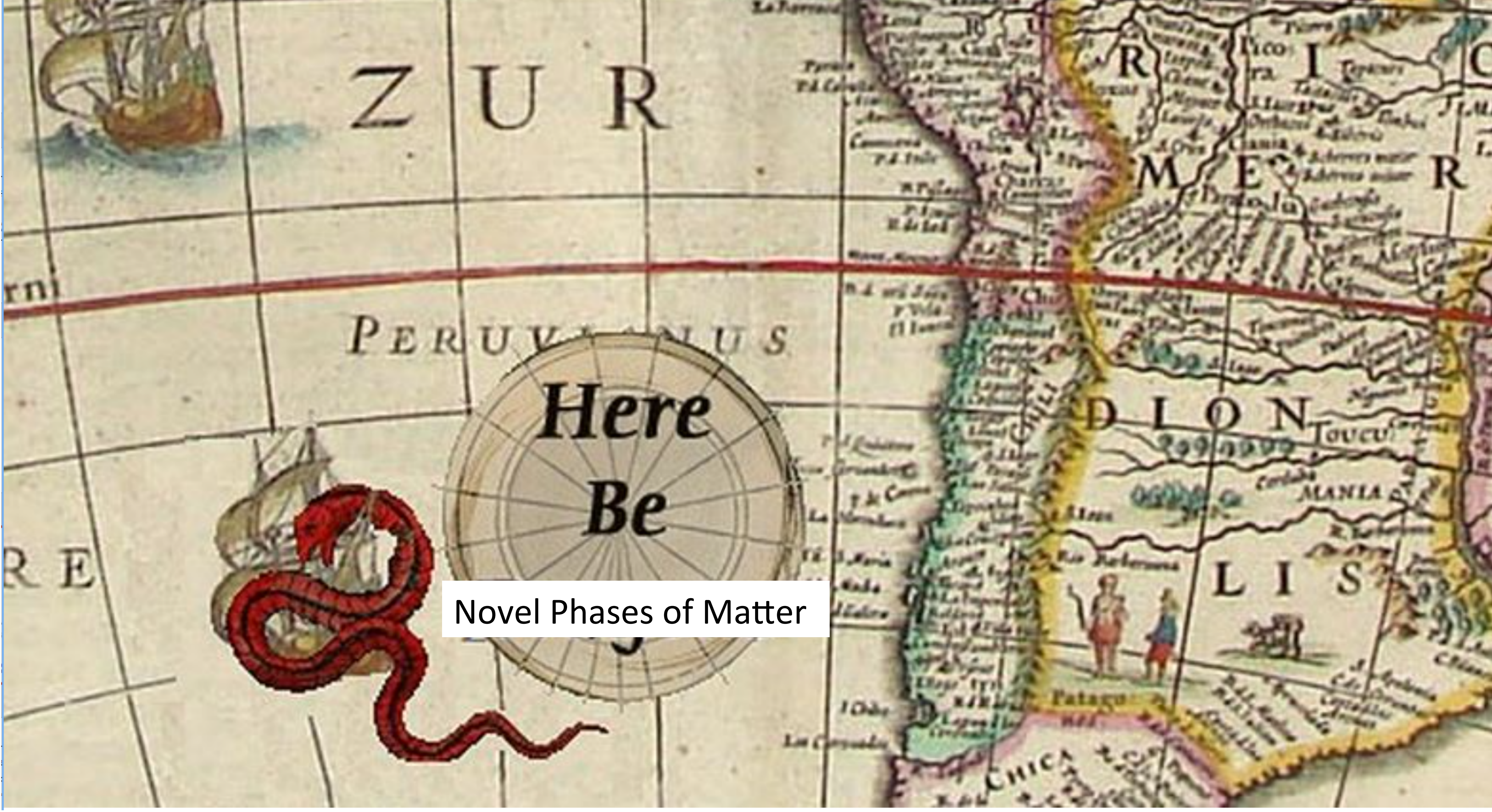
Symm. gen. by $X$: $\varphi \to -\varphi$

analogous to AF: $\mathbb{Z}_{\mathrm{TTS}} \times \mathbb{Z}_2 \to \mathbb{Z}$

Not a symmetry of $H(t)$.
A symmetry of $U\downarrow f$ .
Generated by $\mathcal{U}^\dagger X \mathcal{U}$

<u>But</u>: the symm. gen. by $\boxed{\mathcal{U}^\dagger X \mathcal{U}}$ is a *consequence* of TTS
always present if TTS is present
not present if TTS is not present.

*Protected by Time-Translation Symmetry*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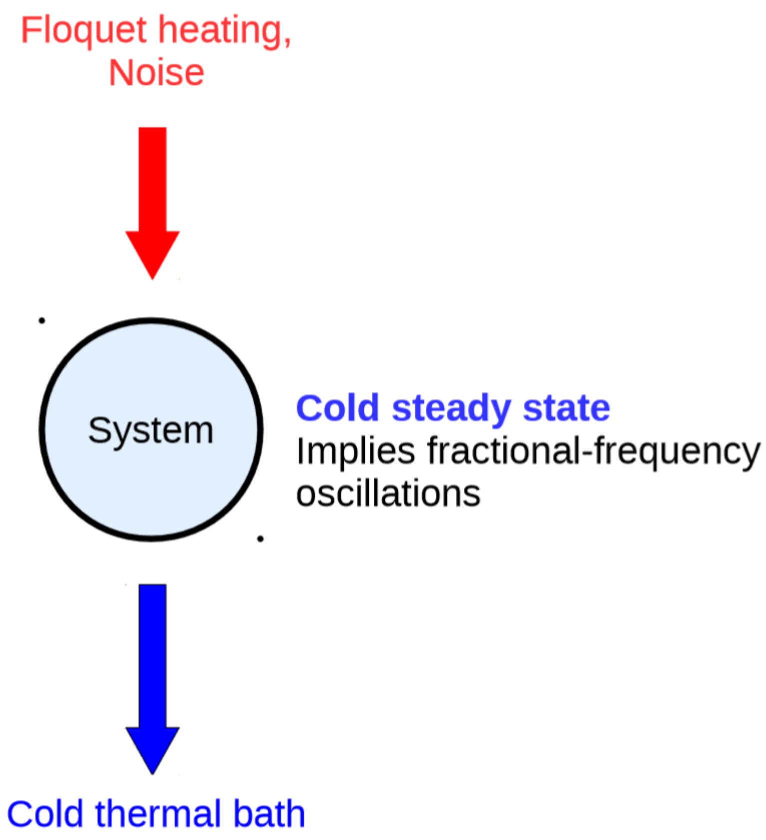
Here
Be
Novel Phases of Matter

# Summary and Future Directions

- Floquet Time Crystals are systems exhibiting TTSB.
- Possibly observed in trapped ions … other driven systems
- Floquet SPTs and Floquet Topological Phases.

# Pre-thermal Time Crystals: Open Systems



# Pre-thermalization in *Undriven* Systems

$$H = -uL + V$$

$\begin{cases} u \text{ large} \\ e^{2\pi L i} = 1 \end{cases}$

$$\mathcal{U} H \mathcal{U}^\dagger = -uL + D + \hat{V}$$

pre-thermal Hamiltonian

where $[D, L] = 0$ and $\hat{V} \sim \lambda\, e^{-O([\log \lambda T]^3/[\lambda T])}$

Abanin et al. ’15

# Pre-thermalization in *Undriven* Systems

large

$$H = -h^z \sum_i S_i^z - h^x \sum_i S_i^x - \sum_{i,j} \left[ J_{ij}^x S_i^x S_j^x + J_{ij}^y S_i^y S_j^y + J_{ij}^z S_i^z S_j^z \right]$$

$$\begin{cases} J_{ij}^x = J + \delta J \\ J_{ij}^y = J_y + \delta J \\ J_{ij}^z = J^z \end{cases}$$

$$\boxed{D = -\sum_{\langle i,j \rangle} \left[ J(S_i^x S_j^x + S_i^y S_j^y) + J^z S_i^z S_j^z \right] + \frac{1}{T} O(\lambda / h^z)^2}$$

# Pre-thermalization in *Undriven* Systems

pre-thermalization

$$\langle\psi_0|e^{-i(-uL+D)t}\Phi e^{i(-uL+D)t}|\psi_0\rangle = \mathrm{tr}\left(\left[e^{-i(-uL+D)t}\Phi e^{i(-uL+D)t}\right] e^{-\beta(D-\mu L)}\right)$$

$$= e^{i(\mu-u)t}\,\mathrm{tr}\left(\left[e^{-i(-\mu L+D)t}\,\Phi\, e^{i(-\mu L+D)t}\right] e^{-\beta(D-\mu L)}\right)$$

$$\phi \sim (S_x + iS_y)e^{i(\mu-u)t}$$

$$\phi(t) \to e^{i(\mu-u)a}\,\phi(t+a)$$

$$S_{\text{eff}} = \int d^d x\, d\tau\ \left[\phi^*\partial_\tau\phi - \mu\phi^*\phi + g(\phi^*\phi)^2 + \ldots\right]$$

# Re-Interpreting Previous Work in terms of TTSB

Floquet SPTs: Classified by $H^{d+1}(G\times\mathbb{Z},U(1))=H^{d+1}(G,U(1))\times H^{d}(G,U(1))$

von Keyserlingk and Sondhi ’16
Else and Nayak ’16
Potter, Morimoto, and Vishwanath ’16
Roy and Harper ’16

stationary classification

in each Floquet cycle, a (d-1)-dim. SPT is “pumped” to the edge

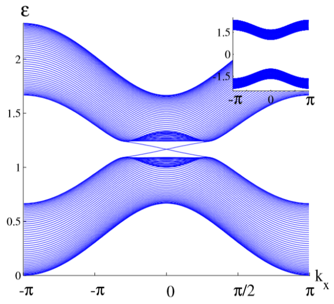
e.g. $U = e^{iHT}V(\mathbb{T})$

also SETs

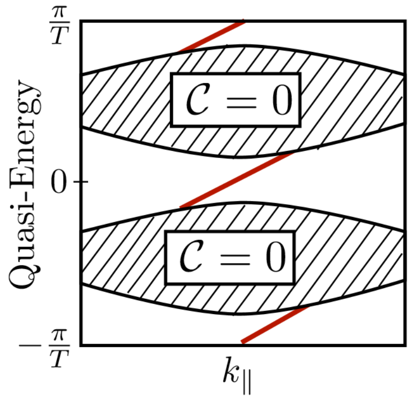
Hamiltonian of stationary MBL system with symmetry group $\mathbb{Z}_n\times G$

symmetry generator

# Exception to ETH: Non-Interacting Floquet Systems



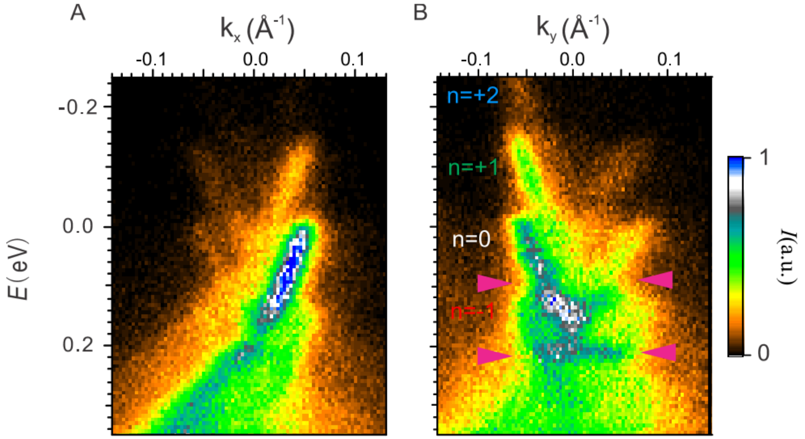
An ordinary insulator can become a topological insulator when driven.
Lindner et al. '11



Bands with vanishing Chern number can have gapless chiral edge states.
Rudner et al. '14

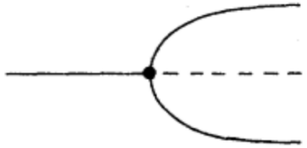
High-intensity MIR radiation opens gap at surface of TI:

Wang et al. '14



# Examples of Systems that Don't Have TTSB

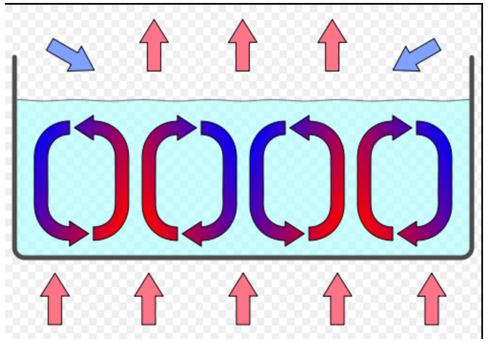
- Classical Period-Doubling Bifurcation: $x_{j+1} = -x_j\,(1+\mu - x_j^2)$



steady state: oscillates between $x = \pm\sqrt{\mu}$

*Doubling is the only generic possibility.*
*Floquet Time Crystals can occur with stable period-doubled, tripled, quadrupled, … phases*

- Rayleigh-Benard Flow:

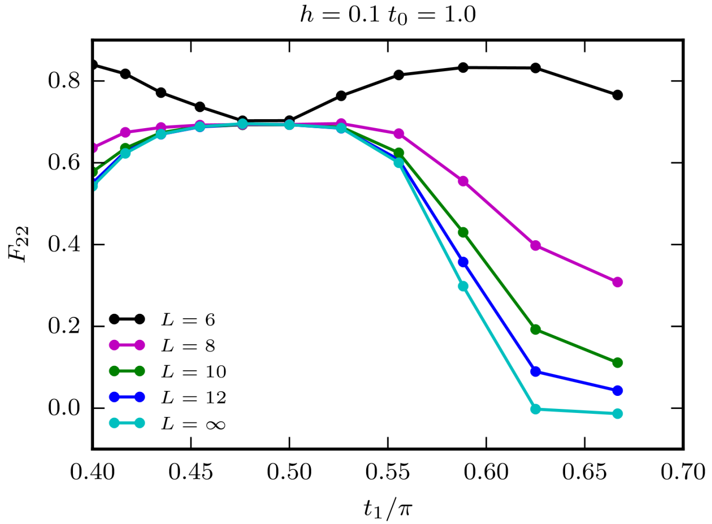


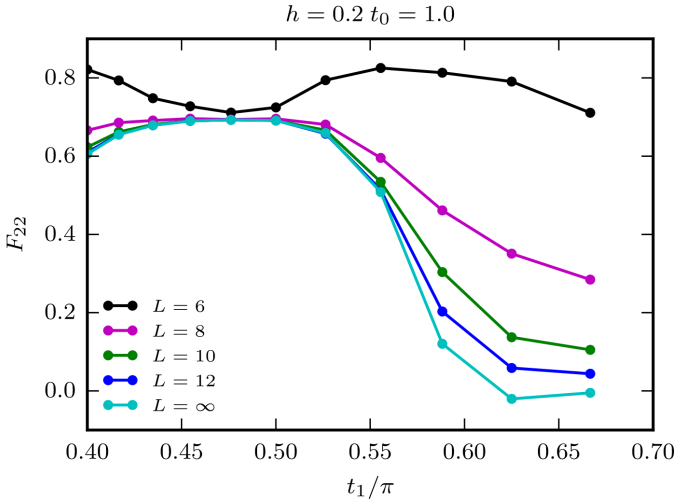
Require dissipation. Occur only in open systems. *Not protected by any generalized rigidity*.

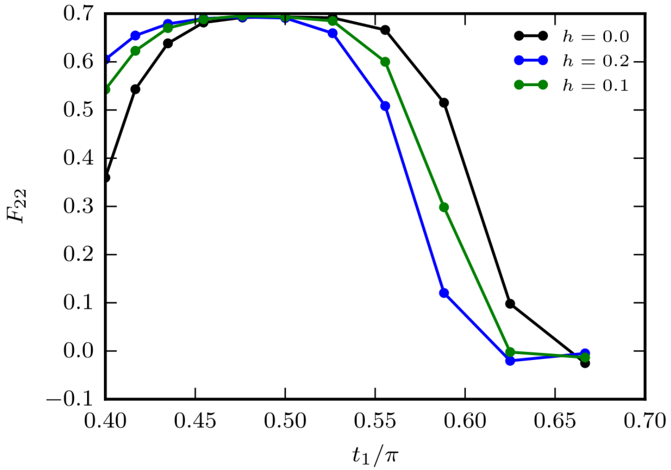
# Stability of TTSB -- Numerics

$$U_{\mathrm{f}} = \exp\left(-it_0 H_{\mathrm{MBL}}\right)\,\exp\left(i\frac{\pi}{2}\sum_i \sigma_i^x\right)$$

what if this isn't exactly $\pi/2$ ?



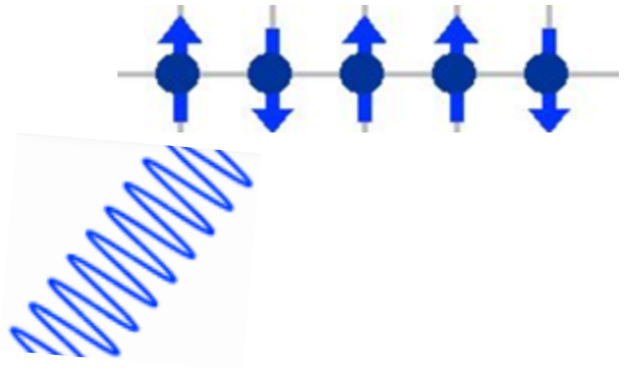




Floquet Eigenstates exhibit TTSB

# Radiation from a Floquet Time Crystal

$$U_{\mathrm{f}} = \exp\left(-it_0 H_{\mathrm{MBL}}\right)\,\exp\left(i\frac{\pi}{2}\sum_i \sigma_i^x\right)$$



$$H_1 = g\sum_i \sigma_i^z (a + a^\dagger)$$

$$A_{m,n} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega + \left[k + \frac{1}{2}\right]\Omega\right)\int_0^T ds e^{i(\Omega/2)s}\langle n|U_0^\dagger(s,0) V U_0(s,0)|m\rangle$$

$$|\{s_i\}\rangle = (|+\rangle \pm |-\rangle)/\sqrt{2}$$

$$A_{+-} = A_{-+} \propto -\frac{2\pi i g N}{T}\delta(0)$$

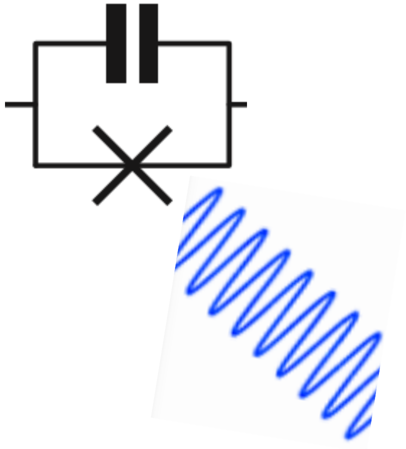
only a phase change when emitting radiation at freq. $\Omega/2$

*quasienergy conserved*

Floquet eigenstates

transition amplitude between Floquet eigenstates

# Similarity between TTSB and AC Josephson Effect



$$H = \omega_J N + \frac{N^2}{2C} - J\cos\theta$$

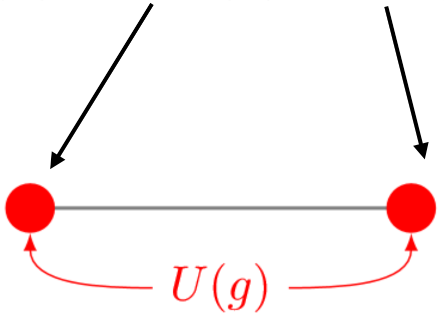
ground state is not a $\theta$ eigenstate;
*not* short-range correlated

- Physical initial states are not eigenstates of $H$
- Junction will radiate, decay into ground state except in open systems (e.g. constant current source)

# Classification of Stationary SPTs

**1D:**

$$U(g) = U_a(g) \otimes U_b(g)$$




$$U(g_1)U(g_2) = U(g_1 g_2) \quad \Rightarrow \quad U_a(g_1)U_a(g_2) = \omega(g_1, g_2)U_a(g_1 g_2)$$

$$\left[U_a(g_1)U_a(g_2)\right]U_a(g_3) = U_a(g_1)\left[U_a(g_2)U_a(g_3)\right]$$

$$\Downarrow$$

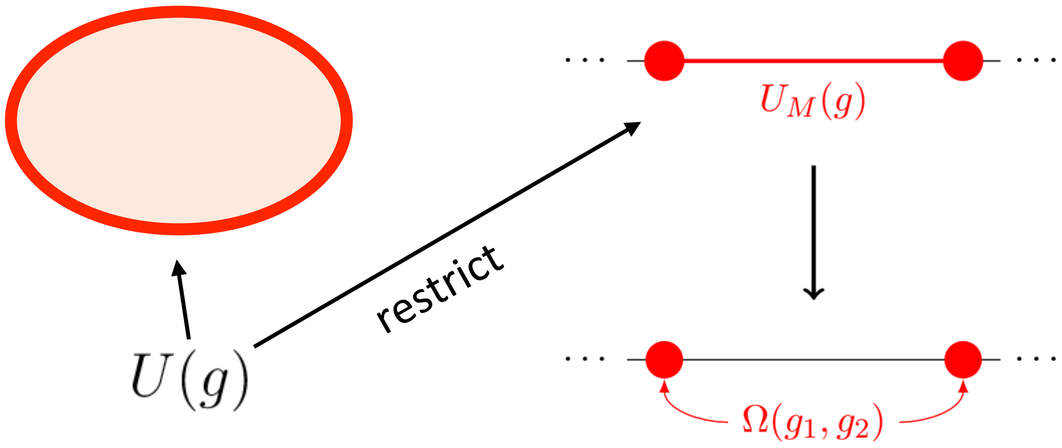
$$\omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_2, g_3)\omega(g_1, g_2 g_3).$$

$$\Downarrow$$

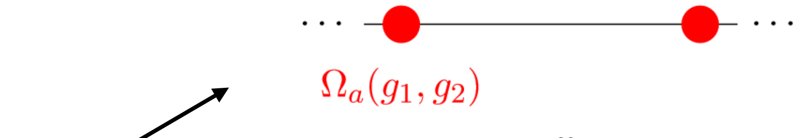
$$H^2(G, U(1))$$

**2D:**

$$\Omega_a(g_1, g_2)\Omega_a(g_1 g_2, g_3) = \omega(g_1, g_2, g_3)\ {}^{U_M(g_1)}\Omega_a(g_2, g_3)\Omega_a(g_1, g_2 g_3)$$




$$U_M(g_1)U_M(g_2) = \Omega(g_1, g_2)U_M(g_1 g_2)$$




$$\Downarrow$$

$$H^3(G, U(1))$$

recover classification of
X. Chen et al.'12

Else and Nayak '14