Tensor Network States for the study of strongly correlated quantum systems

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Introduction

- conventional matter: low entanglement $\rightarrow$ **mean field** description
- **conventional phases** of matter $\leftrightarrow$ **local action of symmetries**

- topological matter: **global entanglement** $\leftrightarrow$ local characterization possible?

- Tensor Network States: **local description** unifying
  physical and entanglement degrees of freedom

  $\rightarrow$ **local modelling** of strongly correlated physics

  $\rightarrow$ description of **topological order** via **local symmetries**

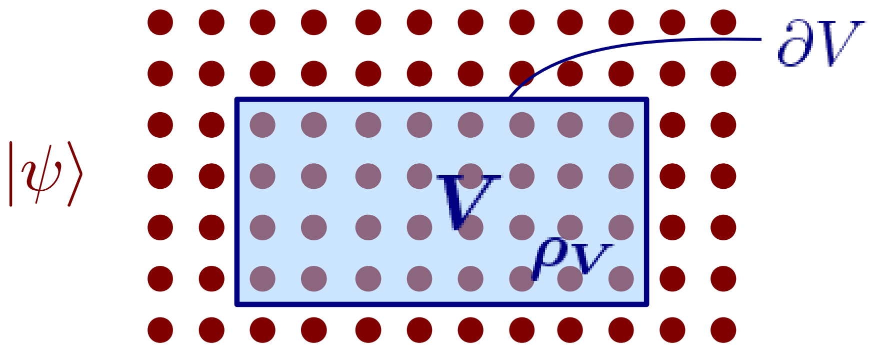
  $\rightarrow$ flesh out **role of boundary** for entanglement

  $\rightarrow$ framework to **study topologically ordered systems**

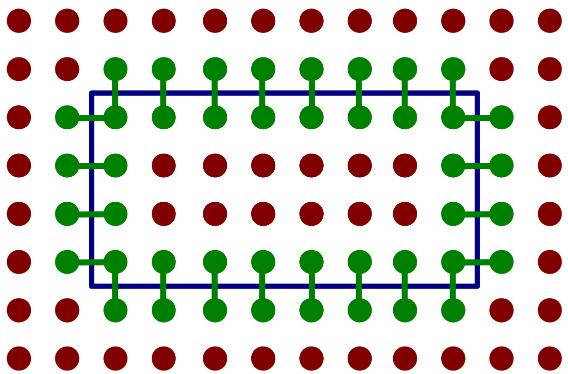
Entanglement structure: The area law

- What is the **entanglement structure** of quantum many-body systems?

- **Area law** for ground states: $S(\rho_V) \sim \partial V$ [Hastings '07]



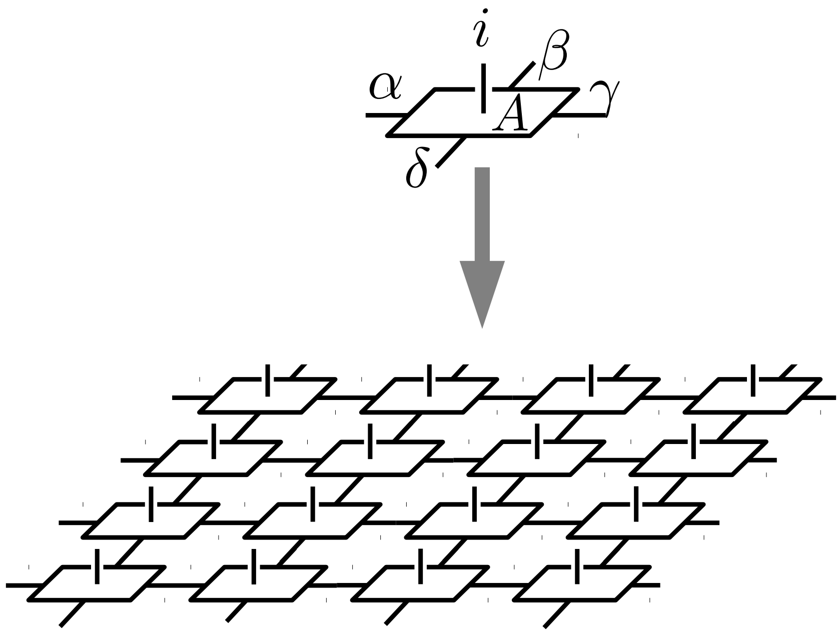
- Entanglement is **distributed locally**



Projected Entangled Pair States

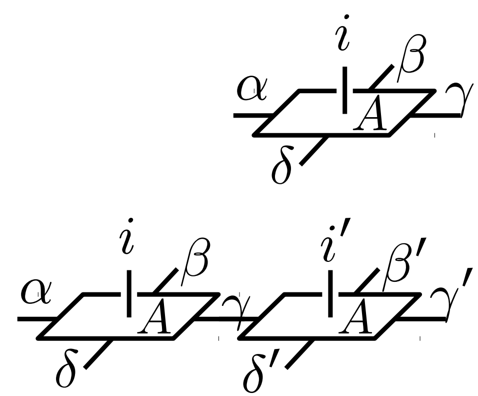
- **Projected Entangled Pair States (PEPS):** [Verstraete & Cirac, PRA '04]

  local description of strongly correlated many-body states



$$|\Psi\rangle = \sum c_{i_1 \cdots i_N} |i_1, \ldots, i_N\rangle$$

Tensor Network Notation:



$$= A^i_{\alpha\beta\gamma\delta}$$

$$= \sum_\gamma A^i_{\alpha\beta\gamma\delta} A^{i'}_{\gamma\beta'\gamma'\delta'}$$

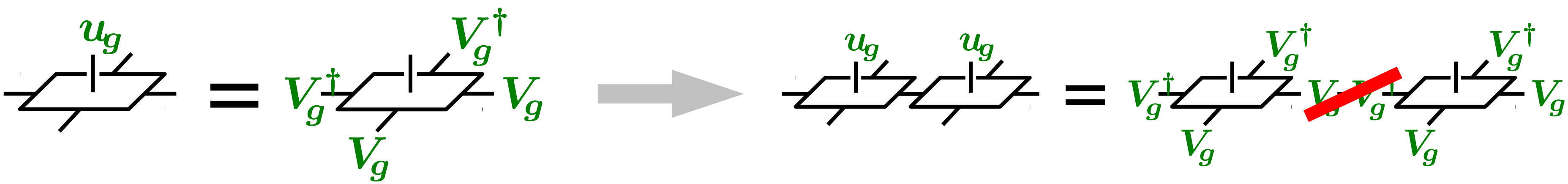
- **faithful approximation** of low-energy states of local Hamiltonians
  [Hastings PRB '06; Molnar, Schuch, Verstraete, Cirac, PRB '14]

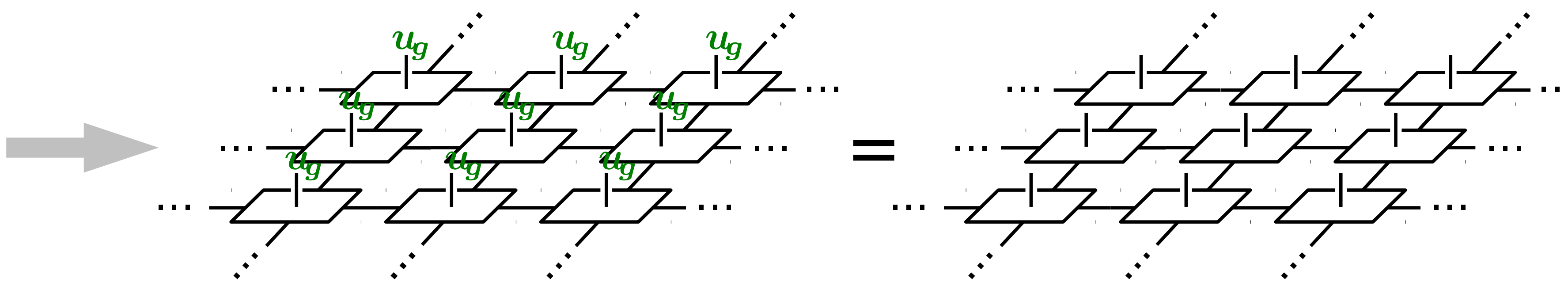
- powerful ansatz for **numerical simulations** [Verstraete & Cirac '04]

PEPS: Encoding physics locally

- PEPS allow to **encode physical structure (symmetries) locally**

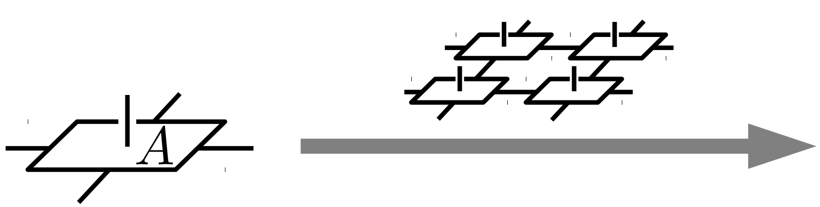
[Perez-Garcia *et al.*, NJP '10]





- local **parent Hamiltonian**: ensure that states looks "locally correct"

  $\Rightarrow$ **inherits all symmetries!**

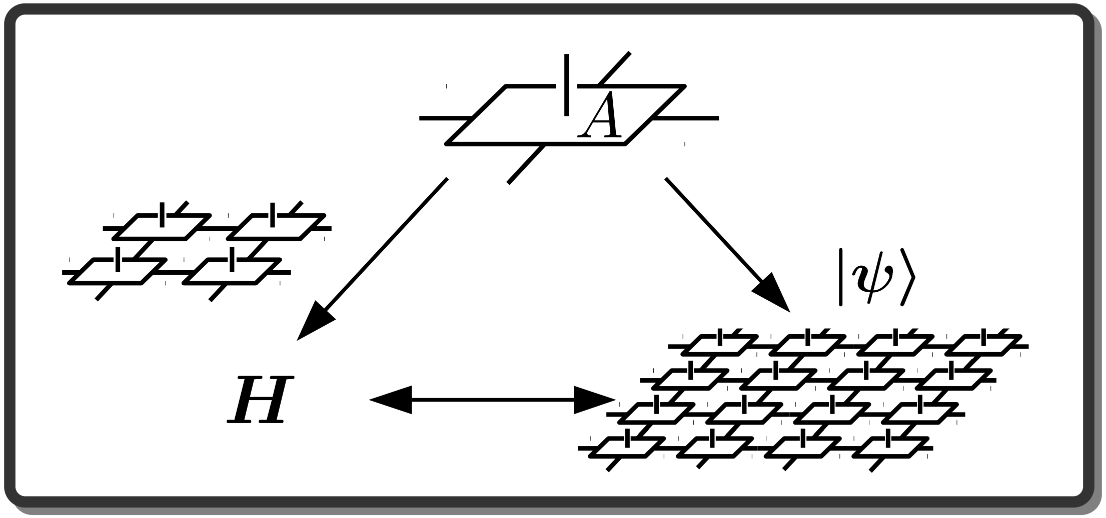


$$H = \sum h_i$$

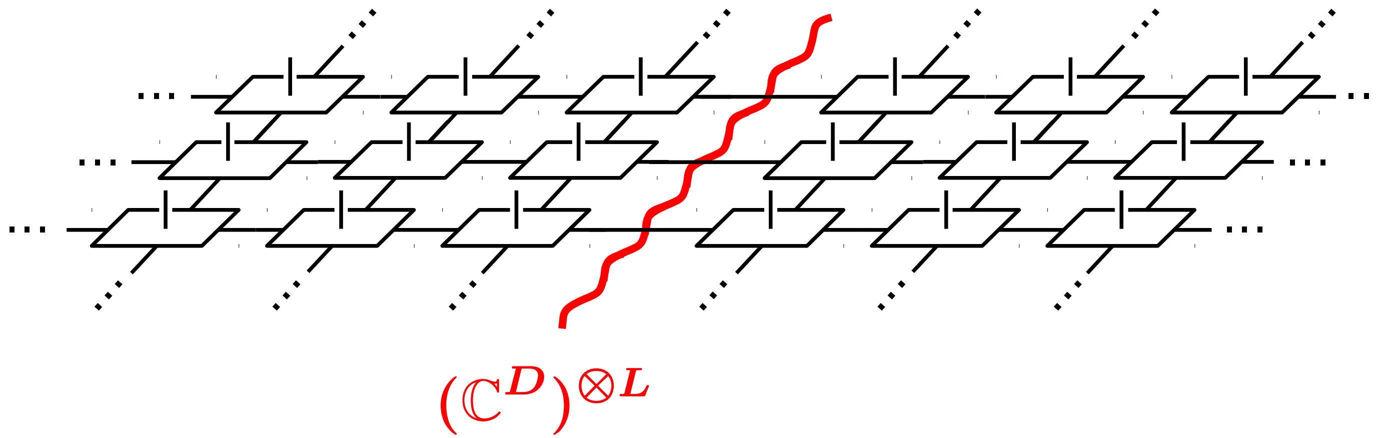
PEPS models

Message 1: PEPS allow to encode local physics locally.

⇒ Framework to construct **solvable PEPS models:**



Bulk-edge correspondence in PEPS



- Bipartition: **entanglement** carried by **degrees of freedom at boundary**

- Allows for direct derivation of **entanglement Hamiltonian**

$$e^{-H_\text{ent}} = \sigma$$ ← lives on **entanglement degrees of freedom**

→ H_ent has **natural 1D structure**!

- $\boldsymbol{H_\text{ent}}$ **inherits all symmetries** from tensor

[Cirac, Poilblanc, Schuch, Verstraete, PRB '11]

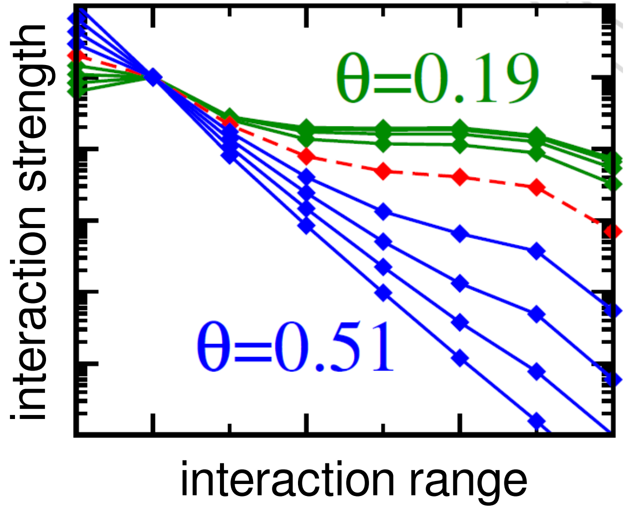
Structure of entanglement Hamiltonian

- gapped phase: H_{ent} **short-ranged** (exp. decay)
  (and mostly few-body)

  e.g. 2D AKLT model:

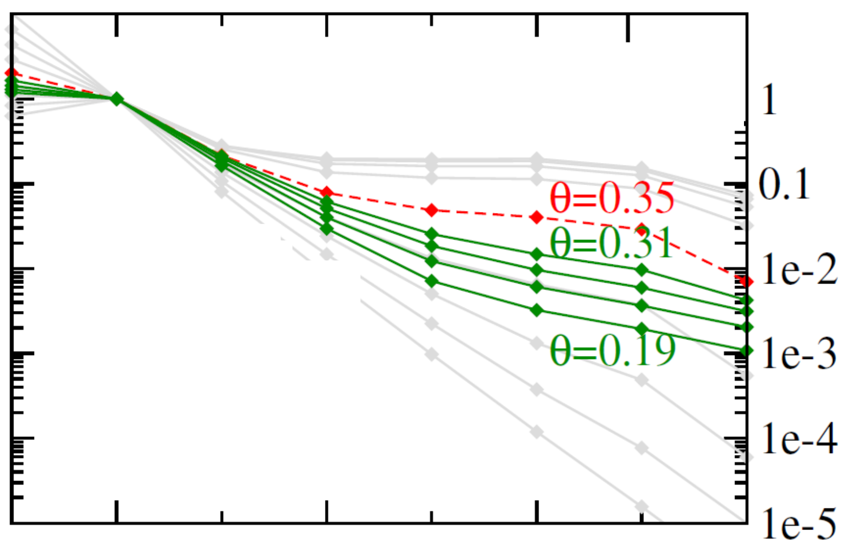
$$H_{\text{ent}} = \sum_i J_1 \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + J_2 \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+2} + \ldots$$

Ising-like PEPS model



- H_{ent} diverges at phase transition

- symmetry broken phase:
  → **short-range** H_{ent} restored by considering **symmetry broken states**

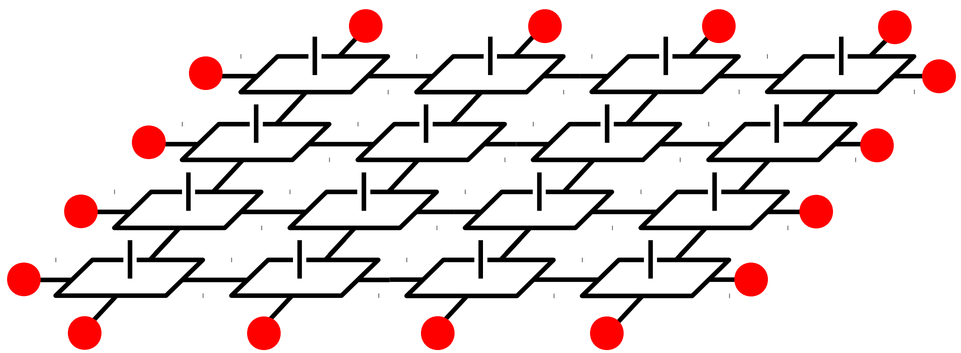


- topological: in one moment ...

[Cirac, Poilblanc, Schuch, Verstraete, PRB '11]
[Rispler, Duivenvoorden, Schuch, PRB '15]

Edge physics

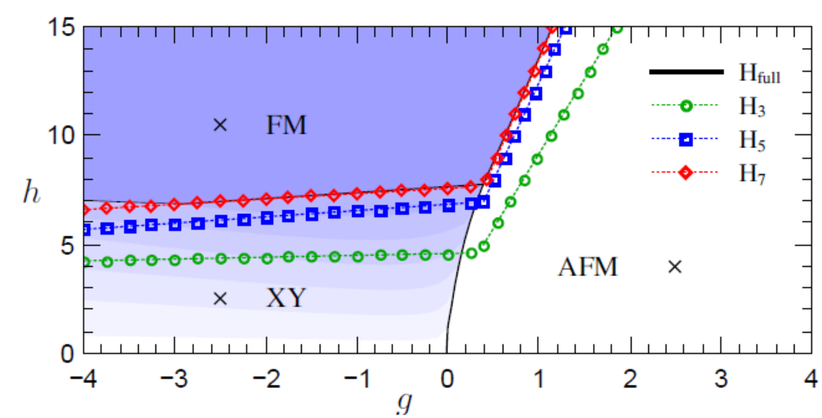
• Low-energy degrees of freedom at edge:
parametrized by imposing boundary conditions on **entanglement DoF**



→ gapless **edge modes** live on entanglement degrees of freedom

→ parent Hamiltonian: completely flat edge physics

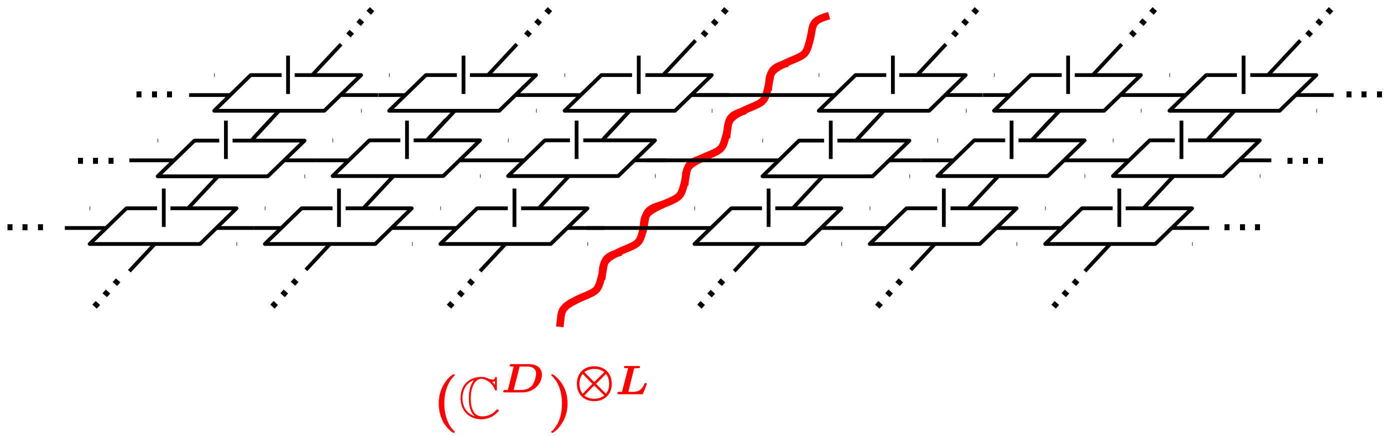
→ perturbations: **different phases** at edge



[Yang, Lehman, Poilblanc, Van Acoleyen, Verstraete, Cirac, Schuch, PRL '14]

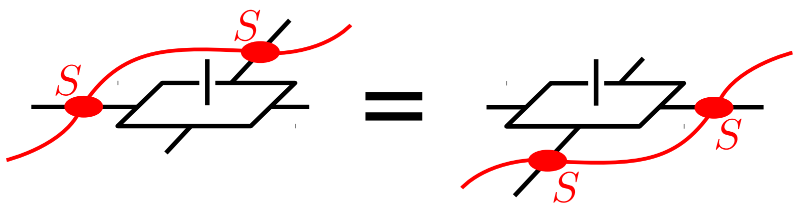
PEPS and the entanglement space

<u>Message 2:</u> PEPS provide an explicit 1D Hilbert space for the entanglement and boundary degrees of freedom.



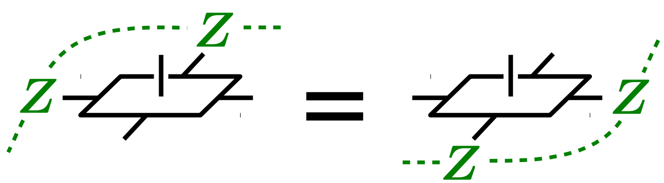
Topological order and local symmetries

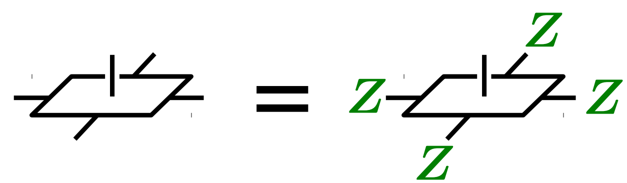
- Topological order in PEPS: Rooted in **entanglement symmetry**



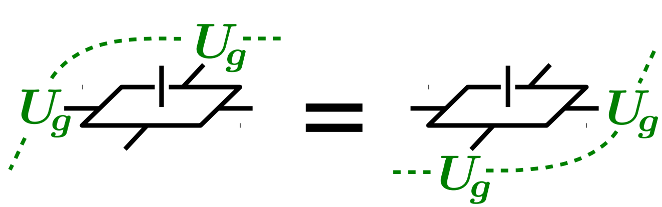
"pulling through condition"

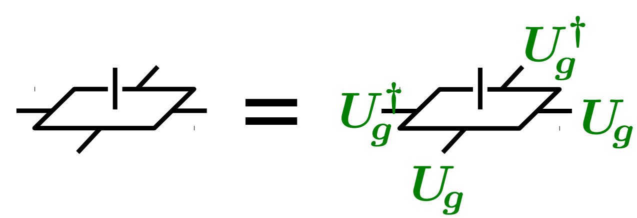
- **Toric Code**

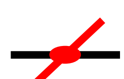




- **Double Models** of finite group G



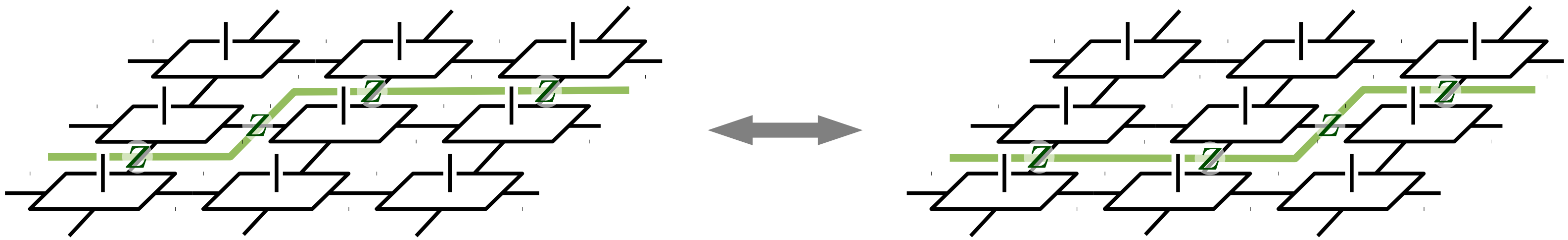


- **String-net models** (with  given by the F-symbol)
- **chiral fermionic PEPS**
- ...

[Schuch, Cirac, Pérez-García '10]
[Buerschaper '13]
[Wahl, Haßler, Tu, Cirac, Schuch '14]
[Sahinoglu *et al.* '14]

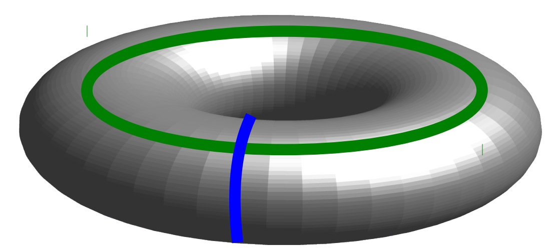
Symmetry vs. ground space structure

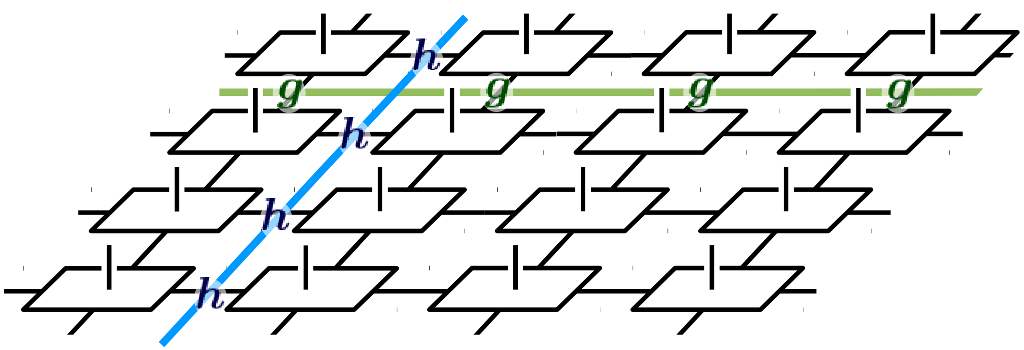
- pulling-through condition $\Rightarrow$ **Strings can be freely moved!**



$\Rightarrow$ Strings are **invisible locally** (e.g. to Hamiltonian)

- Torus: **closed strings** yield **different ground states**





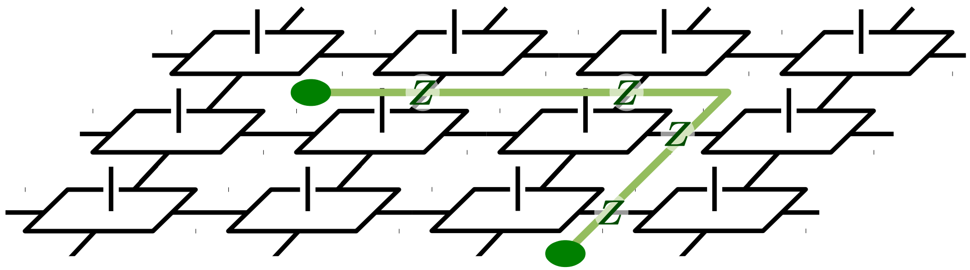
$(g, h = \mathbb{1}, Z)$

- parametrization of **ground space** based on **symmetry of tensor**
- allows to **explicitly construct & study ground states**

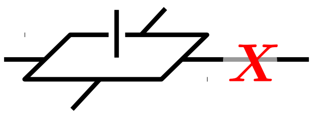
[Schuch, Cirac, Perez-Garcia, Ann. Phys. '10]

Symmetry vs. excitations

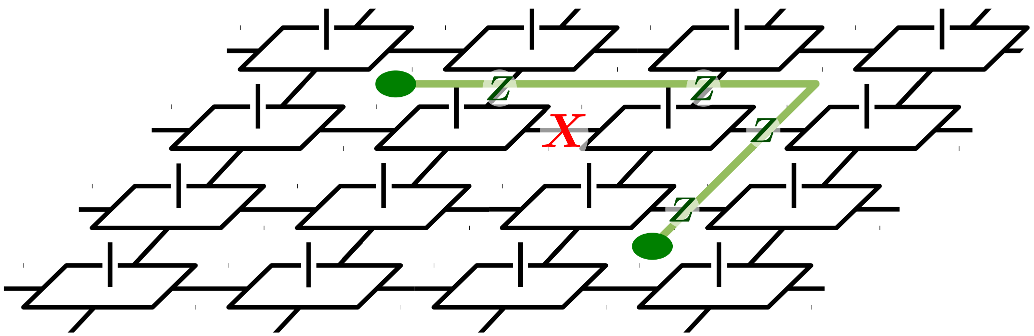
- **Strings w/ open ends:**
  $\rightarrow$ endpoints = **excitations**
  $\rightarrow$ excitations come in **pairs**



- dual excitations:
  anti-commuting with string



- $XZ = -ZX$: mutual
  fermionic statistics!



- virtual symmetries: comprehensive **modeling of anyonic excitations**
- fully **local description** also at finite correlation length

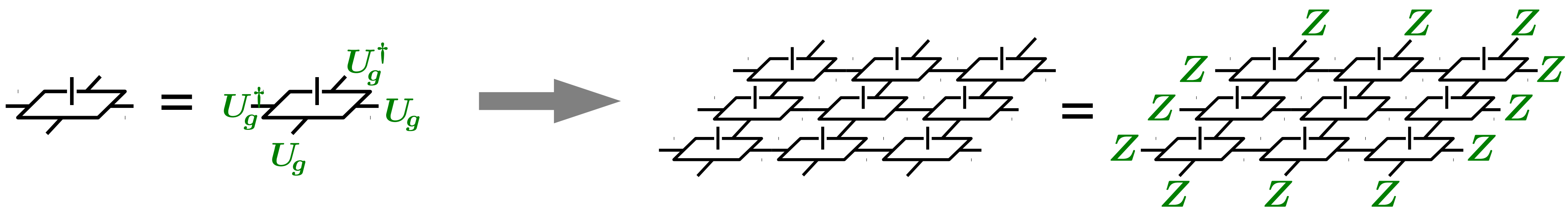
[Schuch, Cirac, Perez-Garcia, Ann. Phys. '10]

PEPS and topological order

Message 3: **In PEPS, topological order originates from a local symmetry on the entanglement degrees of freedom.**

Topological symmetries at the edge

- **Entanglement symmetry** inherited by the **edge**:



- global constraint (e.g. parity) on entanglement DoF:
  only **states in trivial sector** appear!

  $\rightarrow$ **topological correction** to entanglement entropy

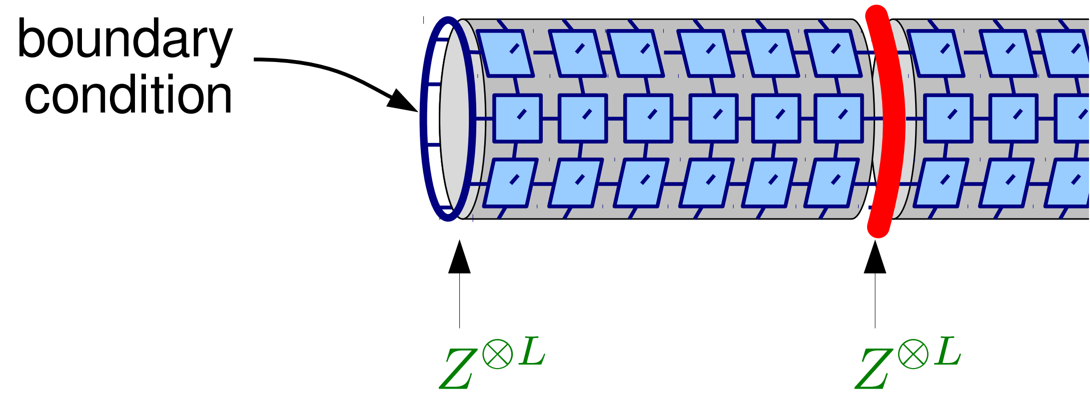
  $\rightarrow$ topological anomaly in **edge physics:**
  edge dynamics restricted to superselection sector

[Yang, Lehman, Poilblanc, Van Acoleyen, Verstraete, Cirac, Schuch, PRL '14]

Entanglement Hamiltonian

- Entanglement spectrum:

$$\rho = \begin{pmatrix} w_e \rho_e & \\ & w_o \rho_o \end{pmatrix}$$

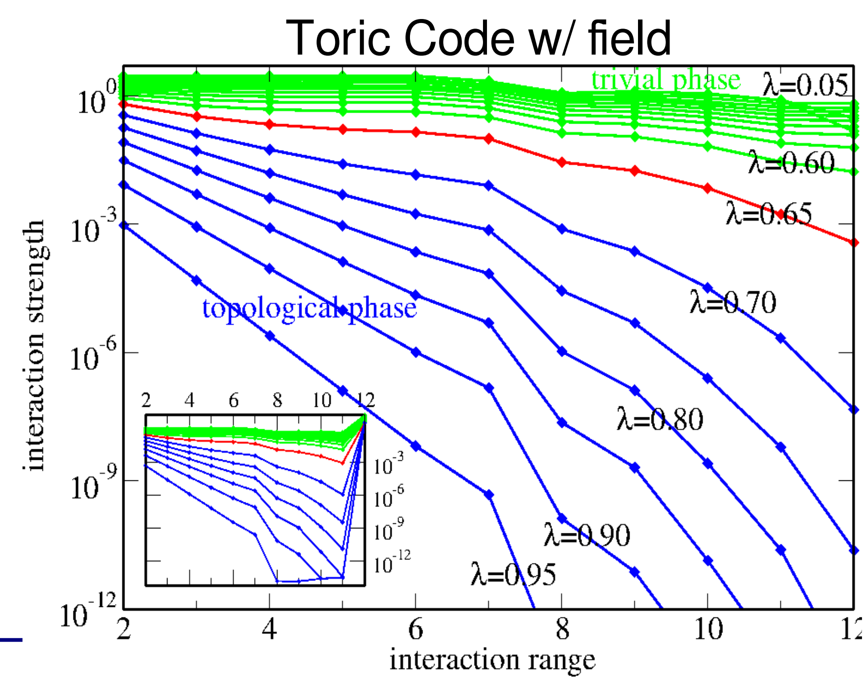


- **Entanglement Hamiltonian** has **anomaly**:

$$\boldsymbol{H_{\mathrm{bnd}} = H_{\mathrm{loc}} + \beta_{\mathrm{topo}}\, H_{\mathrm{topo}}} \qquad ([H_{\mathrm{loc}}, H_{\mathrm{topo}}] = 0)$$

H_{loc}: local, non-universal

H_{topo}: universal, non-local

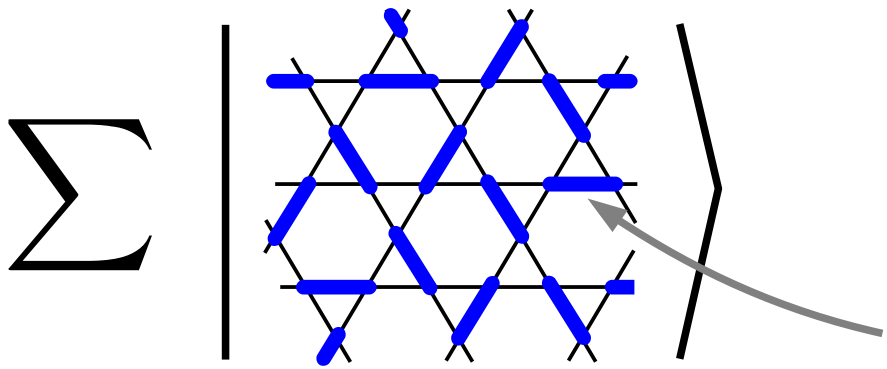


- H_{loc} additionally **couples to flux**

[Schuch, Poilblanc, Cirac, Perez-Garcia, PRL '13]

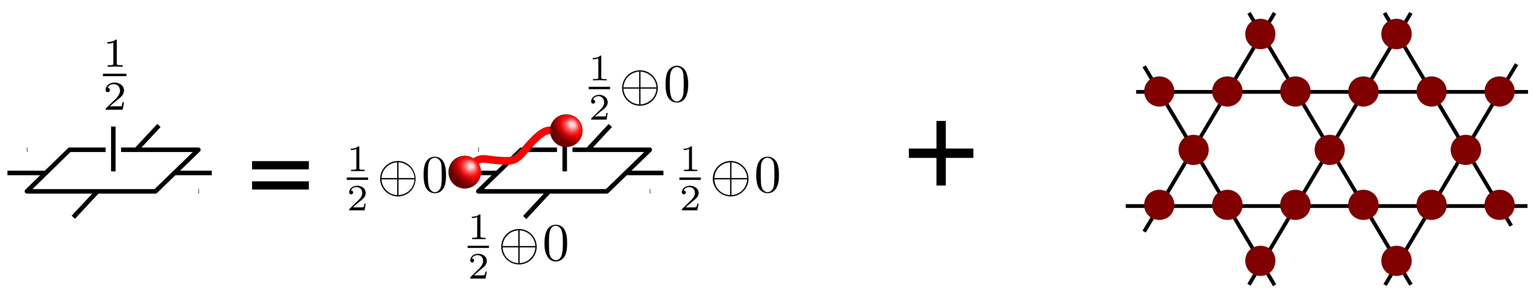
Resonating Valence Bond states

- **Resonating Valence Bond** (RVB) state: candidate kagome spin liquid



singlets $\frac{1}{\sqrt{2}}|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle$

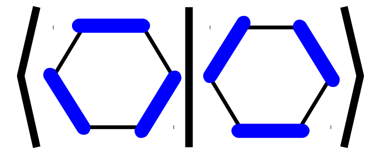
- RVB has **exact PEPS representation** with $\mathbb{Z}_2$ entanglement symmetry (in fact, it is the most natural SU(2)-invariant PEPS!)



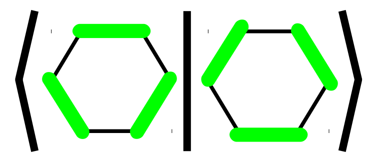
[Verstraete, Wolf, Perez-Garcia, Cirac, PRL '06; Schuch, Poilblanc, Cirac, Perez-Garcia, PRB '12]

RVB and dimer models

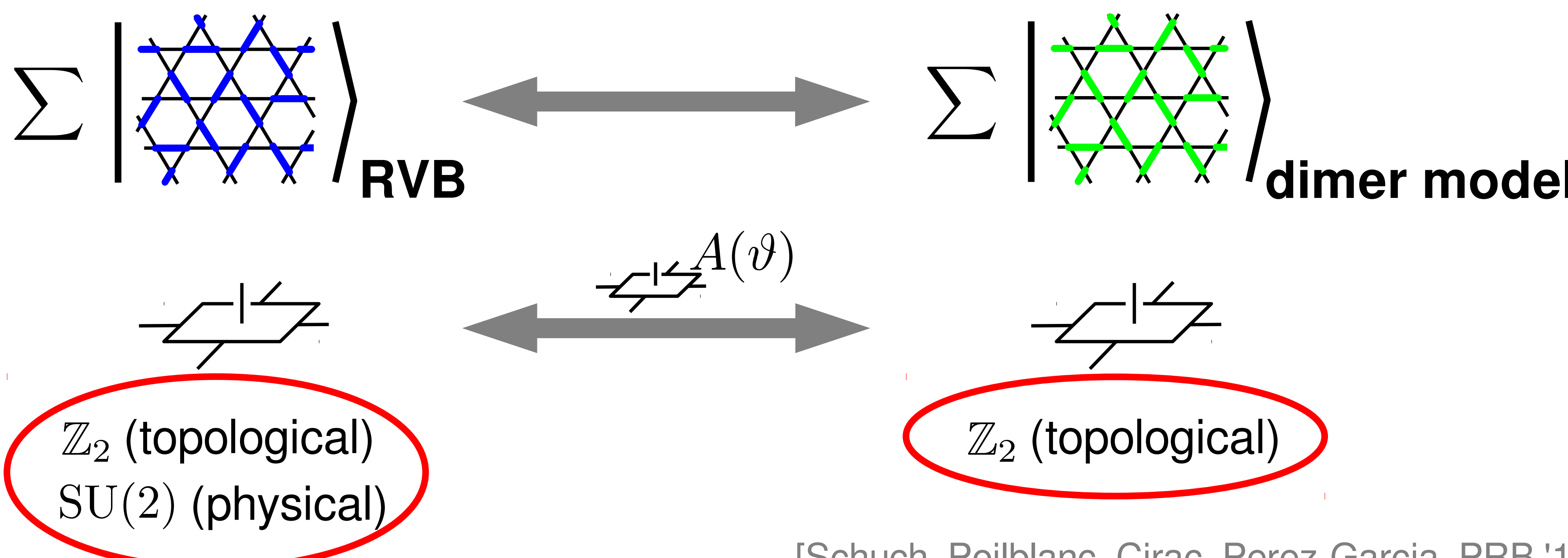
- RVB difficult to study:
  - configurations **not orthogonal**, negative signs
  - Topological? Magnetically ordered?

 $= \frac{2}{2^{\ell/2}}$

- resort to **dimer models** with orthogonal dimers
  - can be exactly solved
  - topologically ordered (≡ toric code)

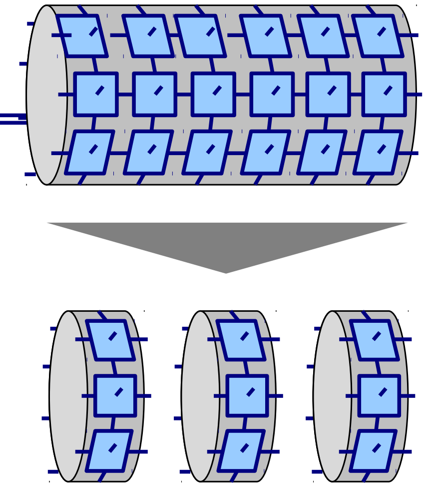
 $= 0$

- **Interpolation** in PEPS (w/ smooth Hamiltonian!):

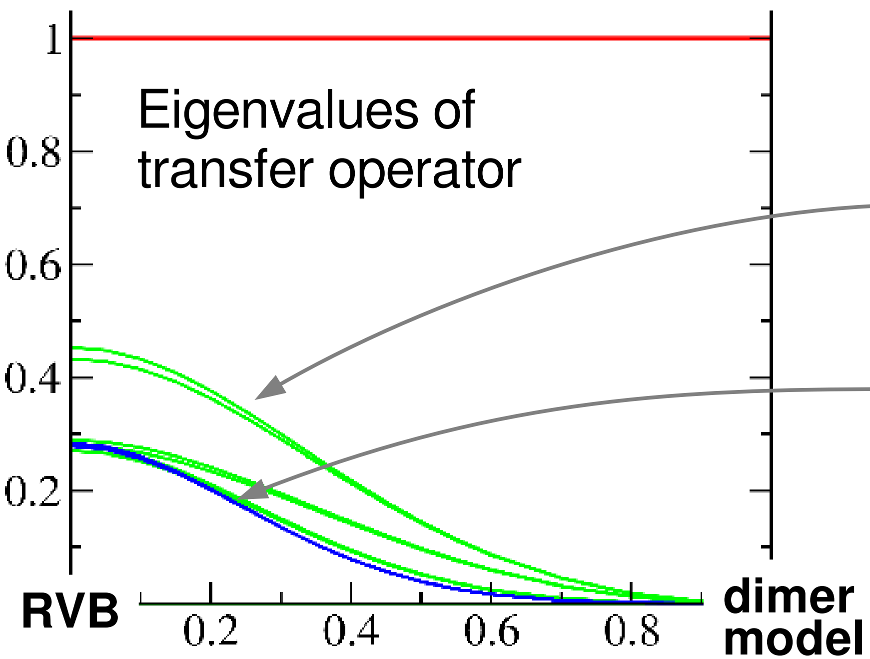


[Schuch, Poilblanc, Cirac, Perez-Garcia, PRB '12]

Numerical study of the RVB state



- numerical study of interpolation RVB ↔ dimer model
- "transfer operator": - governs **all correlation functions**
  - **topological sector** labeled by symmetry



no overlap of topological sectors
⇒ **topologically ordered**

Finite correlation length
⇒ **no long range order**
⇒ spin liquid

[Schuch, Poilblanc, Cirac, Perez-Garcia, PRB '12]

⇒ RVB state on kagome lattice is a $\mathbb{Z}_2$ **topological spin liquid**

- can prove: **RVB** is (topo. degenerate) ground state of **parent Hamiltonian**

Numerical study of

- numerical study of interpolation RVB ↔ dim

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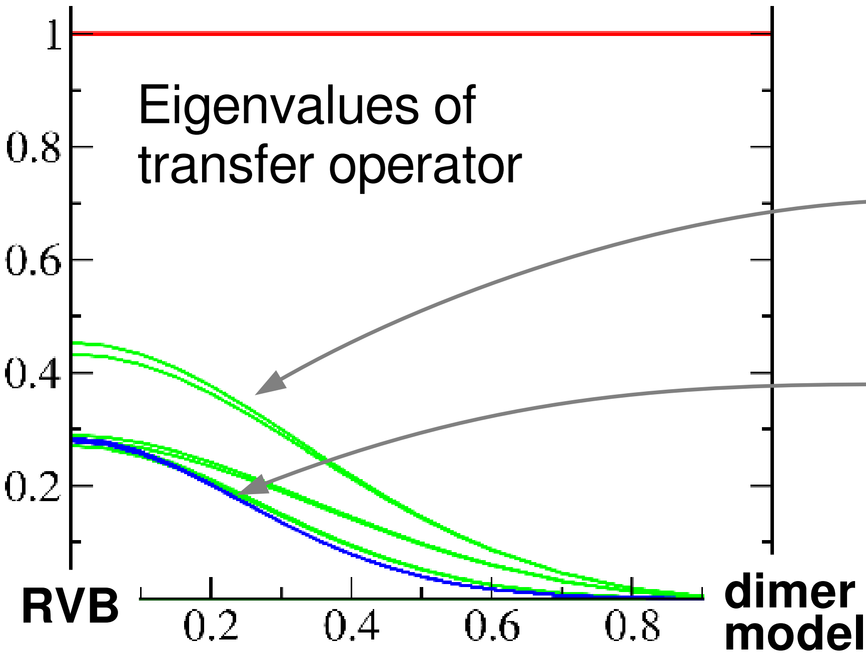
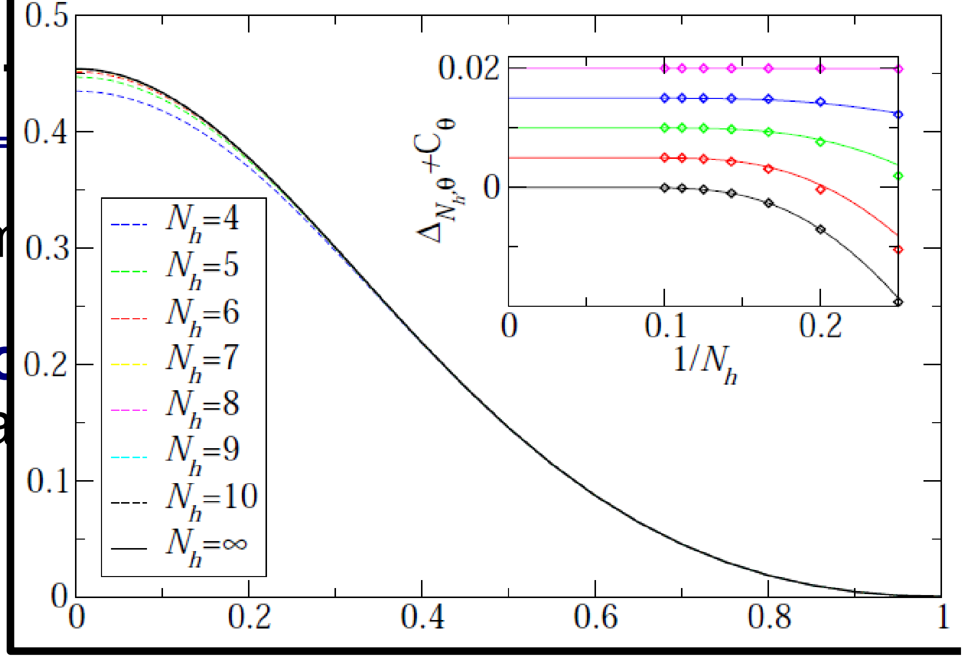
  - **topological sector** la





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Entanglement Hamiltonian of RVB

- What is the **entanglement spectrum** + Hamiltonian of RVB?

- H_{ent} inherits on-site & topological **symmetries of PEPS**

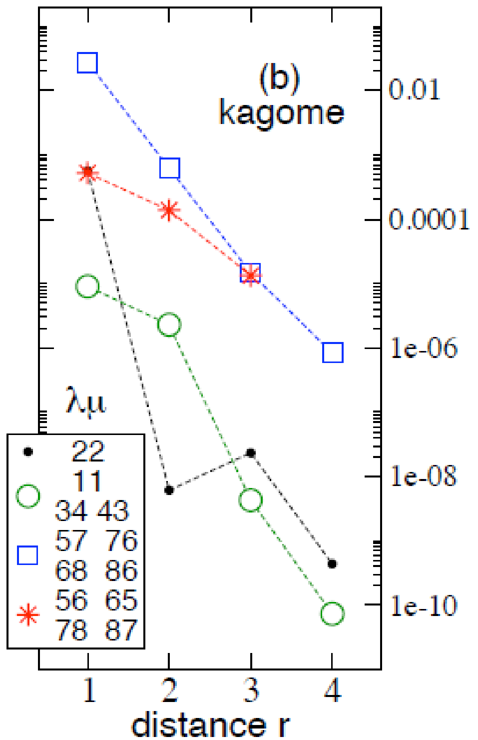
  - topological symmetry: $Z = \mathrm{diag}(-1,-1,1)$

  "fermionic" parity constraint at boundary!

  - local $\mathrm{SU}(2)$ symmetry: $\frac{1}{2}\oplus 0 \equiv \{|\uparrow\rangle, |\downarrow\rangle, |\mathrm{vac}\rangle\}$

$\rightarrow H_{\mathrm{ent}}$ has **t-J-model type structure** (+pairing)

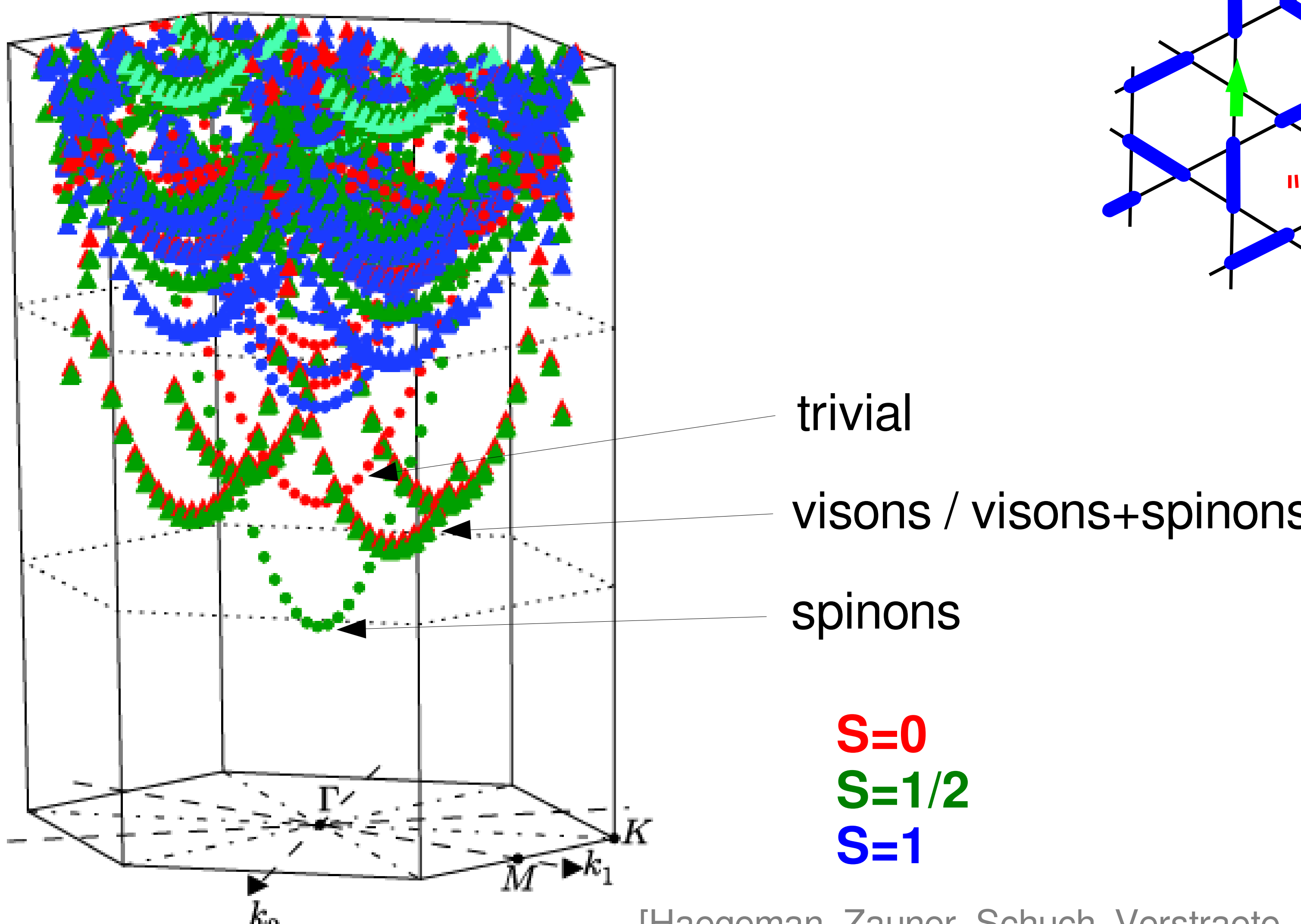
- numerical study:
  - H_{ent} is **approximately local** (+topological term):
    dominant NN hopping/pairing,
    smaller Heisenberg & repulsion,
    longer-range terms bosonic

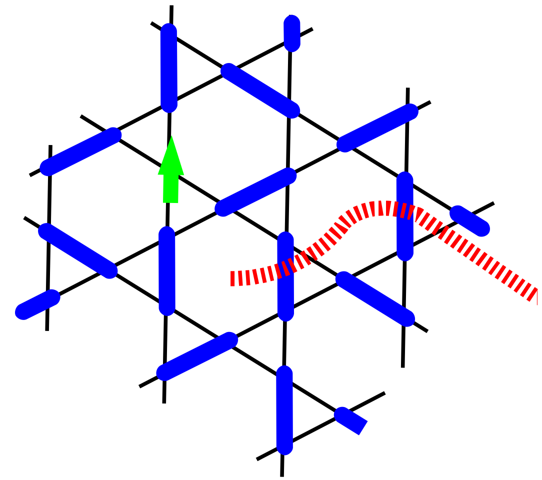


[Poilblanc, Schuch, Perez-Garcia, Cirac, PRB '12]

Dispersion relations

- Ansatz for excitations → extract information about **dispersion relation** for different **topological excitations** from correlation functions



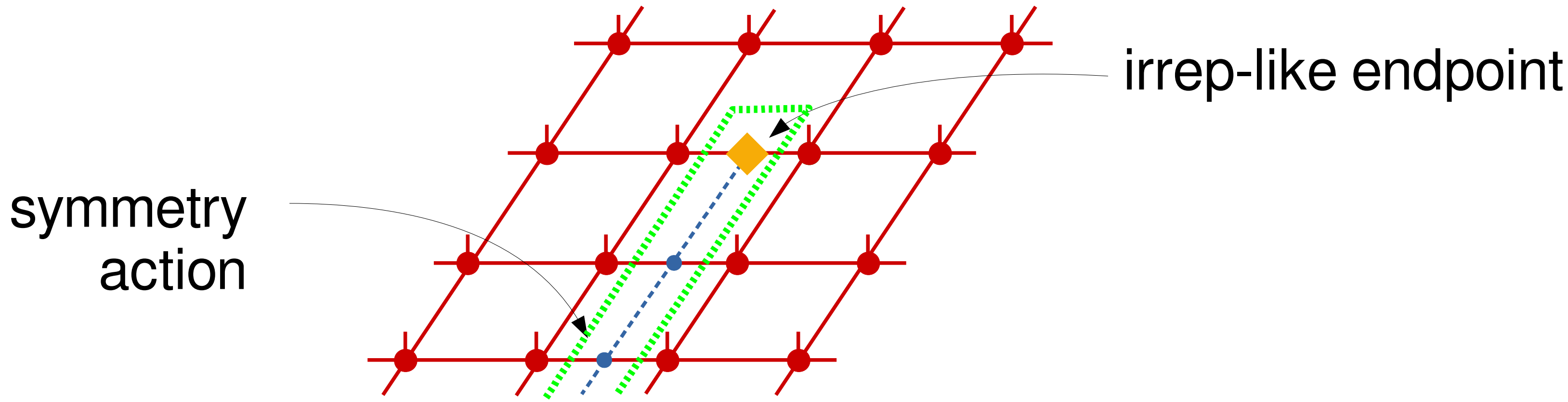


S=0
S=1/2
S=1

[Haegeman, Zauner, Schuch, Verstraete, Nat. Comm. '15]

Topological phases and anyon condensation

- **Description of anyon** on entanglement degrees of freedom:



- Does this describe a **physically observable excitation**?

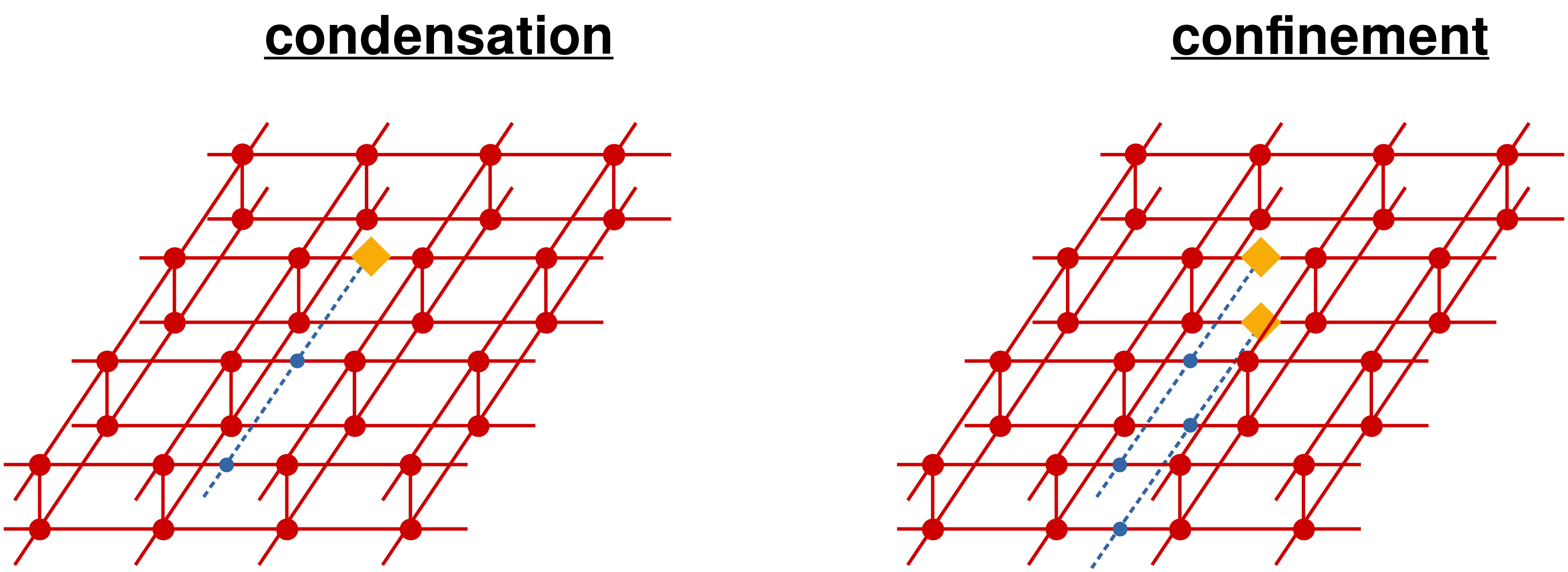
  $\rightarrow$ depends on **environment**!

- What could "go wrong"?

  - environment absorbs string $\rightarrow$ equal to original state: **condensed**
  - environment orthogonal to string $\rightarrow$ ill-defined: **confined**

Virtual vs. physical anyons

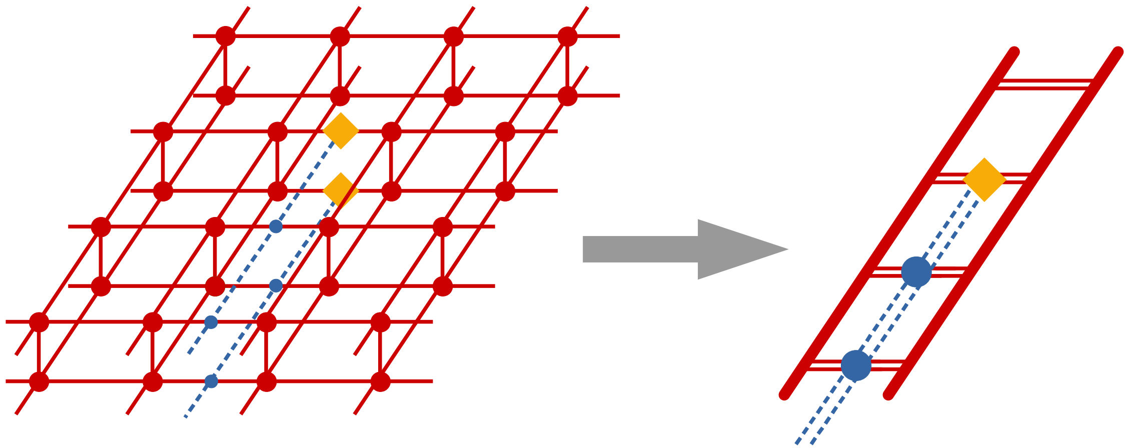
• Express as expectation values:



• can serve as **order parameters for topological phases**

The transfer operator

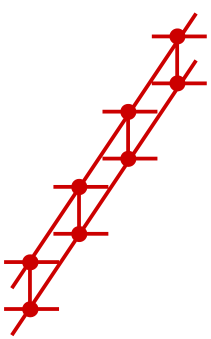
- **condensation/confimement** order parameters $\rightarrow$ **string order** at boundary

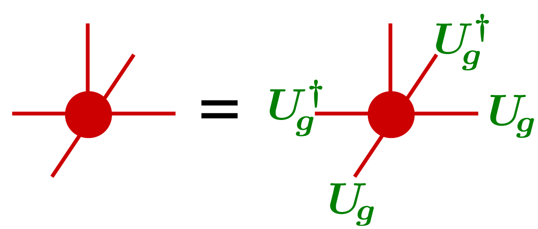


- boundary state = fixed point of **transfer operator** $\mathbb{T}$

$\mathbb{T}$ **inherits symmetries** from tensor:

$\mathbb{T} :=$





$\Rightarrow \ [\mathbb{T}, U_g^{\otimes N} \otimes \bar{U}_{g'}^{\otimes N}] = 0$

symmetry group $G \times G$

- classification of **anyon behavior** $\leftrightarrow$ classification of **1D phases of** $G \times G$

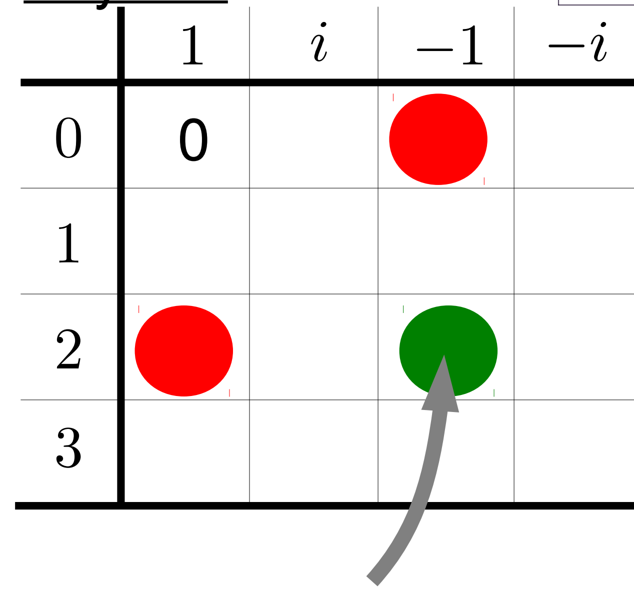
(symmetry breaking *and* SPT)

[Duivenvoorden, Iqbal, Haegeman, Verstraete, Schuch, in preparation]

Case study: $\mathbb{Z}_4$ symmetry

- $\mathbb{Z}_4$ double: symmetry group $\mathbb{Z}_4$
- Symmetries of $\mathbb{T}$ ($\leftrightarrow$ 1D classification): $\mathbb{Z}_4 \times \mathbb{Z}_4$

Anyons:



double semion phase:
need to **condense dyon**

$\mathbb{Z}_4$ double model
symmetry: $\mathbb{Z}_4 \cong \{00, 11, 22, 33\}$

Toric Code
$\mathbb{Z}_2 \cong \{00, 22\}$

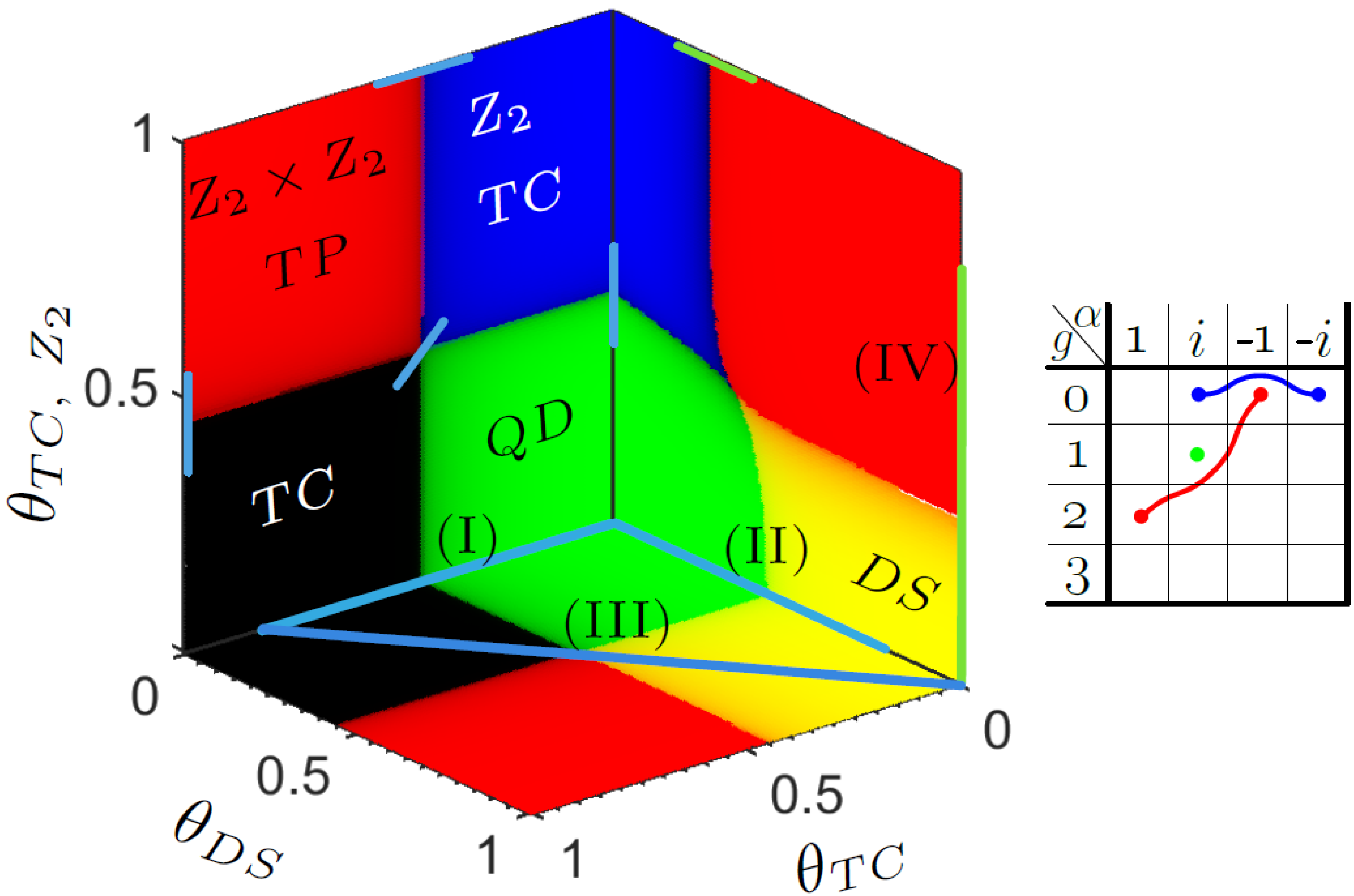
Toric Code or **Double Semion**
$\mathbb{Z}_4 \times \mathbb{Z}_2 \cong \{00, 11, 22, 33, 02, 13, 20, 31\}$

trivial
$\mathbb{Z}_1 \cong \{00\}$

trivial
$\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \{00, 02, 20, 22\}$

trivial
$\mathbb{Z}_4 \times \mathbb{Z}_4$

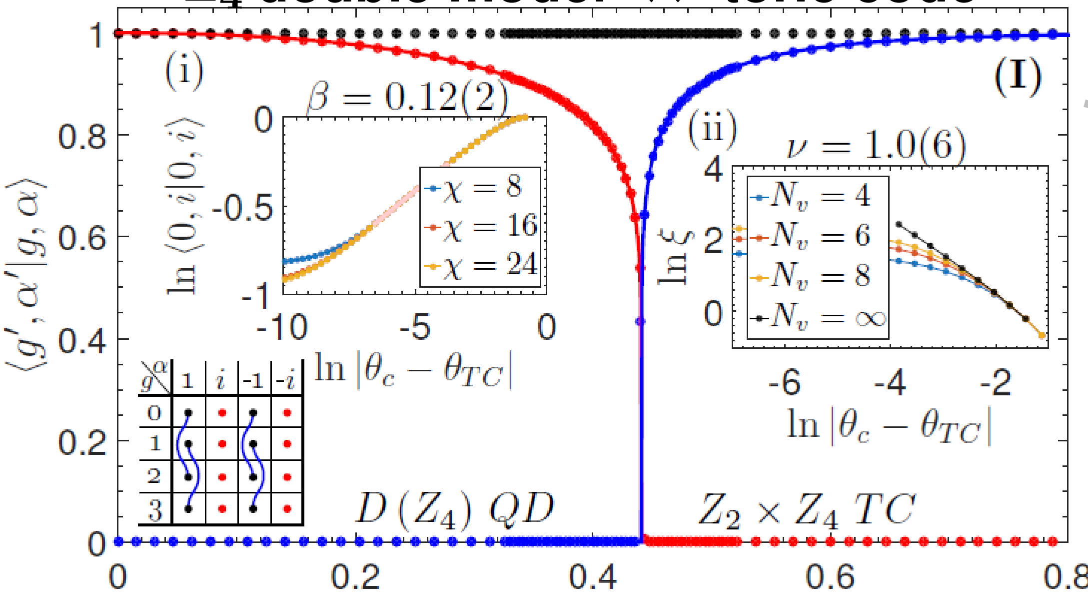
Phase diagram of $\mathbb{Z}_4$-invariant PEPS

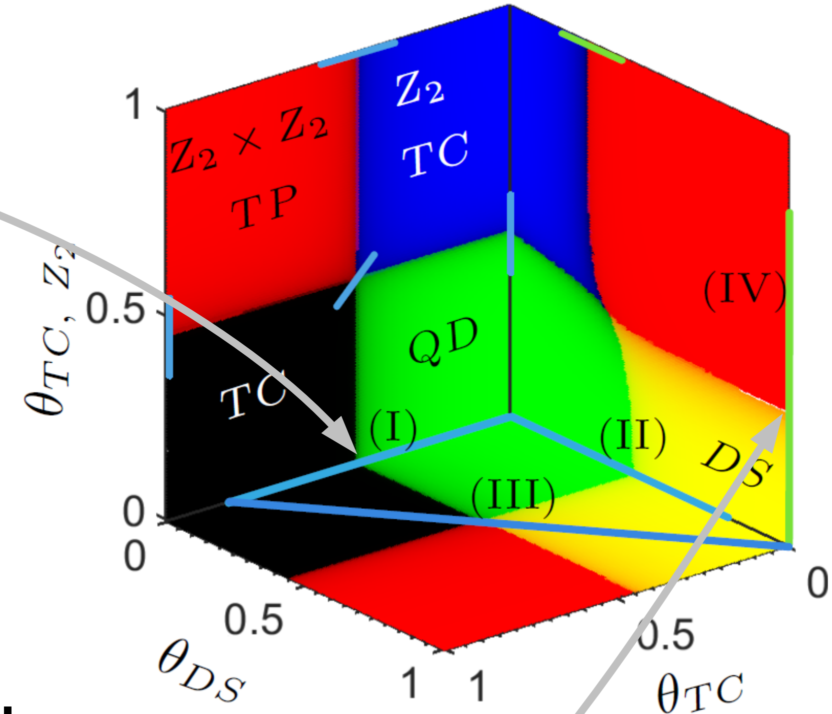


[Iqbal *et al.*, in preparation]

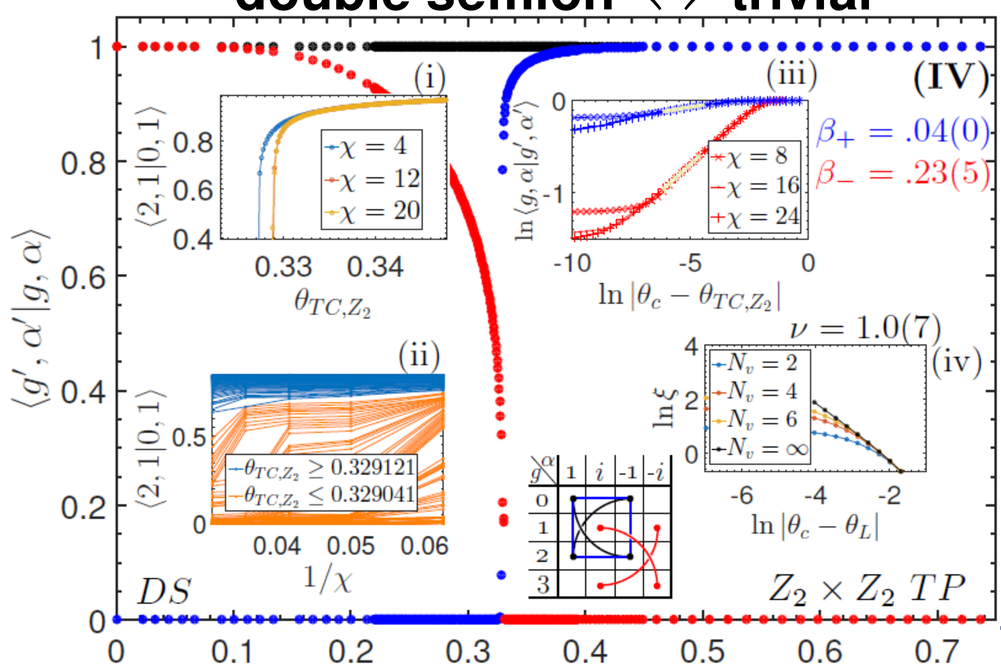
Topological phase transitions

$\mathbb{Z}_4$ double model $\leftrightarrow$ toric code



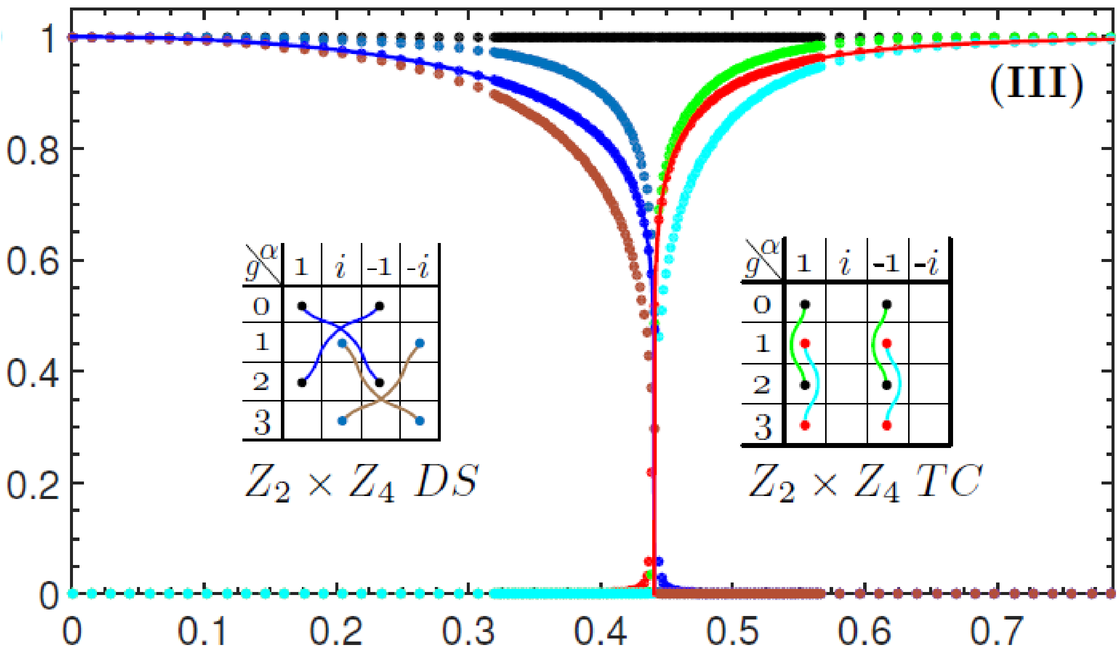


double semion $\leftrightarrow$ trivial



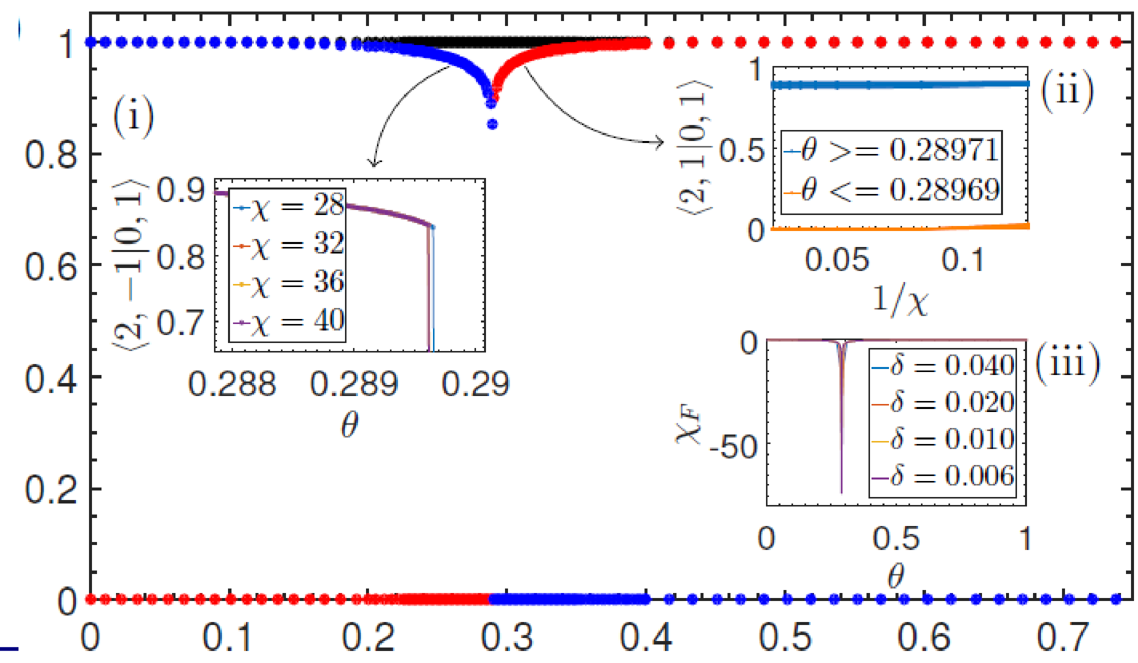
[Iqbal *et al.*, in preparation]

Toric Code – Doubled Semion transition



2[nd] order

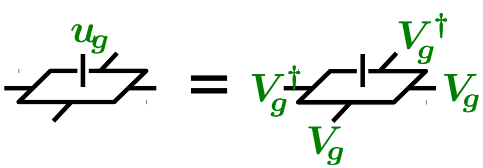
1[st] order

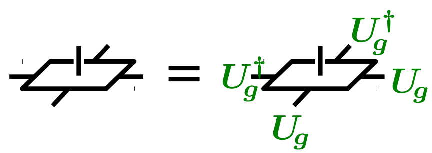


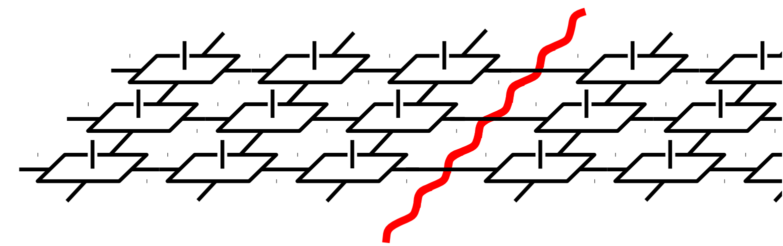
[Iqbal *et al.*, in preparation]

Summary

- PEPS: **entanglement-based local description** of many-body systems
- PEPS encode **physical structure locally**
- PEPS provide an explicit **“entanglement space” at the boundary**
- **topological order** in PEPS $\leftrightarrow$ **local symmetry** in entanglement







- Application 1: **Spin-liquid nature** of RVB state
- Application 2: study of **topological phases** “holographically” through **1D phases at the boundary**