Unstable manifolds of recurrent flows in pipe flow

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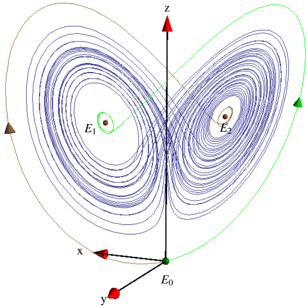
Motivation



Lorenz system:

$$
\begin{aligned}
\dot{x} &= \sigma(y - x)\,, \\
\dot{y} &= \rho x - y - xz\,, \\
\dot{z} &= xy - bz\,,
\end{aligned}
$$

Unstable manifolds *shape* the attractor.

Motivation

Plane Couette flow (Gibson, Halcrow, Cvitanović (2008))

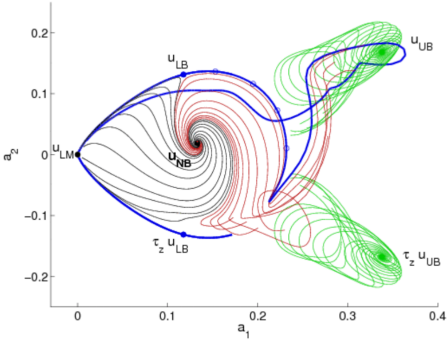
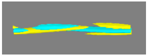


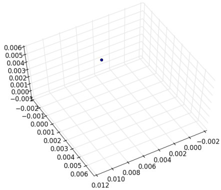
FIGURE 5. A state space portrait of plane Couette flow for $Re = 400$ and $[L_x, L_y, L_z] = [2\pi/1.14, 2, 4\pi/5]$, projected from 61 506 dimensions to 2. The labeled points are exact equilibrium (steady-state) solutions of the Navier-Stokes equation (see §3); the curved trajectories are fully-resolved time-dependent numerical integrations of Navier-Stokes projected onto the $(\mathbf{e}_1, \mathbf{e}_2)$ plane defined by (4.3). W^u_{LB}, the 1d unstable manifold of the 'lower-branch' equilibrium $\mathbf{u}_{\mathrm{LB}}$, and $\tau_z W^u_{\mathrm{LB}}$, its half-cell translation in z, are shown with thick blue lines. $W^{u(1,2)}_{\mathrm{NB}}$, a 2d portion of the unstable manifold of the 'newbie' equilibrium $\mathbf{u}_{\mathrm{NB}}$, is shown with thin black and red spirals emanating from $\mathbf{u}_{\mathrm{NB}}$. Similarly, the thin green lines spirally out of $\mathbf{u}_{\mathrm{UB}}$ and $\tau_z \mathbf{u}_{\mathrm{UB}}$ indicate $W^{u,S}_{\mathrm{UB}}$ and $\tau_z W^{u,S}_{\mathrm{UB}}$, the 2$d$ unstable manifolds of $\mathbf{u}_{\mathrm{UB}}$ and its half-cell translation $\tau_z \mathbf{u}_{\mathrm{UB}}$ within the S-invariant subspace $\mathbb{U}_S$. Open dots along W^u_{LB} show initial conditions for Newton-GMRES searches used to find $\mathbf{u}_{\mathrm{NB}}$. The plane of the projection is defined in terms of the equilibrium solutions; it is *dynamically invariant* and independent of the numerical representation. See §4.2 and §4.3 for discussions of the projection and the dynamics.

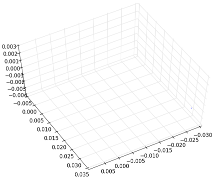
The problem

- Continuous symmetries $\rightarrow$ *relative* invariant solutions
- Can be avoided in invariant subspaces of plane Couette flow
- *Cannot* be avoided in pressure-driven flows (pipes, channels)

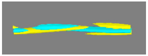
Relative equilibria and relative periodic orbits

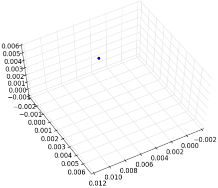


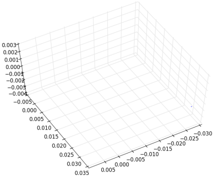




Symmetry reduction







Pipe flow

- Navier-Stokes equations for fluctuations around base flow:

$$\mathbf{u}_\tau + \mathbf{u}_{HP} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_{HP} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + 32\frac{\beta}{Re}\hat{\mathbf{z}} + \frac{1}{Re}\nabla^2 \mathbf{u}$$

- $\beta = \beta(\tau)$ ensures constant-flux
- Incompressibility: $\nabla \cdot \mathbf{u} = 0$
- No-slip on the wall: $\mathbf{u}(z, \theta, r = D/2) = 0$
- Periodic in axial and azimuthal directions:
  $\mathbf{u}(z, \theta, r) = \mathbf{u}(z + L, \theta, r)$ and $\mathbf{u}(z, \theta, r) = \mathbf{u}(z, \theta + 2\pi, r)$

Navier-Stokes equations along with incompressibility and boundary conditions induces a finite-time flow

$$x(\tau) = f^\tau(x(0)),$$

where x is the state space vector corresponding to a point in the space of allowable velocity fields.

Symmetries

- Translation: $g_z(l)\mathbf{u}(z, r, \theta) = \mathbf{u}(z - l, r, \theta)$
- Rotation: $g_\theta(\phi)\mathbf{u}(z, r, \theta) = \mathbf{u}(z, r, \theta - \phi)$
- Reflection: $\sigma[u, v, w](z, r, \theta) = [u, v, -w](z, r, -\theta)$

- Equivariance under $\mathrm{SO}(2)_z \times \mathrm{O}(2)_\theta$:

$$G = \{g_z(l), g_\theta(\phi), \sigma\}$$

- In state space:

$$g f^\tau(x) = f^\tau(gx)\,, \text{ where } g \in G$$

Symmetry reduction

- Each solution x has infinitely many "symmetry copies" on its "group orbit":

$$\mathcal{M}_{gx} = \{g\,x \,|\, \forall g \in G\}$$

- Symmetry reduction is a coordinate transformation $x \to \hat{x}$ such that each group orbit $\mathcal{M}_{gx}$ is represented by a single point $\hat{x}$.

SO(2) symmetry reduction: First Fourier mode slice

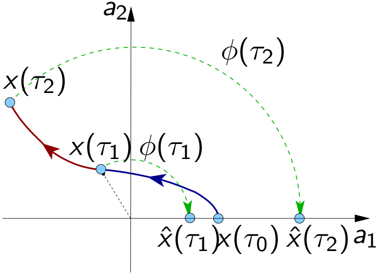
Project the solution $x(\tau)$ onto a subspace where group orbits are circular, then fix the polar angle on this projection:

Let $\hat{x}'$ be a state space vector corresponding to the velocity field

$$[\hat{u}', \hat{v}', \hat{w}'](z, \theta, r) = J_0(\alpha r) \cos(2\pi z / L)$$

then

$$a_1 = \langle x, \hat{x}' \rangle , \quad a_2 = \langle x, g_z(-L/4)\hat{x}' \rangle$$
$$\hat{x} = g_z(-l)x , \quad l = \frac{L}{2\pi} \arg a_1 + i a_2$$

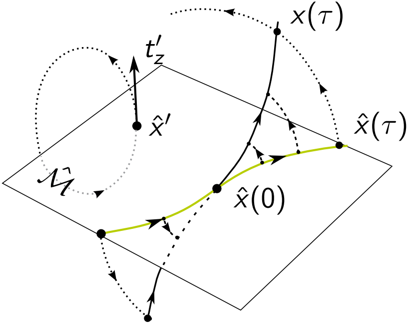


Budanur et al. (2015) PRL, Willis, Short, Cvitanović (2016) PRE

Why call it a slice?

"Polar" coordinate transformation defines a codimension-1 "slice hyperplane" in state space:

$$\langle \hat{x}(\tau) - \hat{x}',\, t'_z \rangle = 0\,, \quad \langle t_z(\hat{x}),\, t'_z \rangle > 0$$

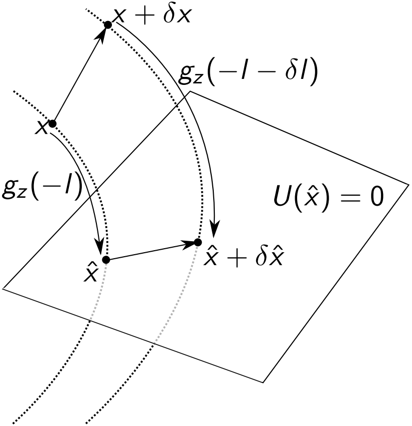


- $t_z(\hat{x}) = T_z \hat{x}$: Group tangent
- T_z: Generator of infinitesimal translations i.e. $g_z(l) = e^{l T_z}$
- $t'_z = t_z(\hat{x}')$: Slice tangent

Projecting perturbations

Small perturbations δx to x can be brought to the slice by the transformation

$$\delta\hat{x} = \left(1 - \frac{t_z(\hat{x}) \otimes t'_z}{\langle t_z(\hat{x}),\ t'_z \rangle}\right) g_z(-l)\delta x$$



Projection operator

$$H(x) = 1 - \frac{t_z(\hat{x}) \otimes t_z'}{\langle t_z(\hat{x}),\, t_z' \rangle}$$

allows us to project tangent space onto the slice.

Example: If $\dot{x} = v(x) = \lim_{\delta\tau \to 0} f^{\delta\tau}(x)/\delta\tau$ then

$$\dot{\hat{x}} = \hat{v}(\hat{x}) = v(\hat{x}) - \frac{\langle t_z',\, v(\hat{x}) \rangle}{\langle t_z(\hat{x}),\, t_z' \rangle} t_z(\hat{x})$$

is the symmetry reduced state space velocity.

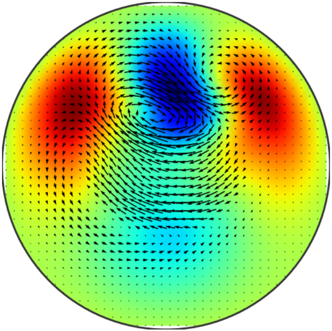
By acting with $H(x)$, we are able to bring stability eigenvectors of traveling waves and Floquet vectors of relative periodic orbits to the slice.

Unstable manifolds of traveling waves

- "Asymmetric" traveling wave, $S1$
- Invariant under "shift-and-reflect"

$$x_{S1} = \sigma g_z(L/2) x_{S1}$$

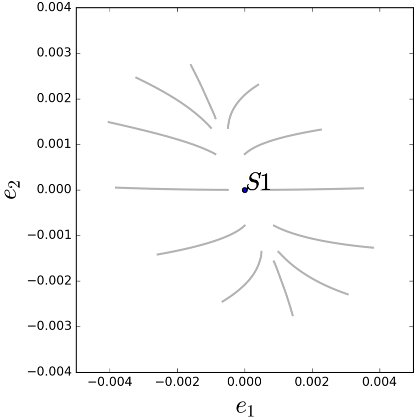
- Lies on laminar-turbulent boundary
- Has two real unstable ($\mu > 0$) stability eigenvalues



Here, computed at $Re = 3000$ for a pipe of length $L = 5D$.

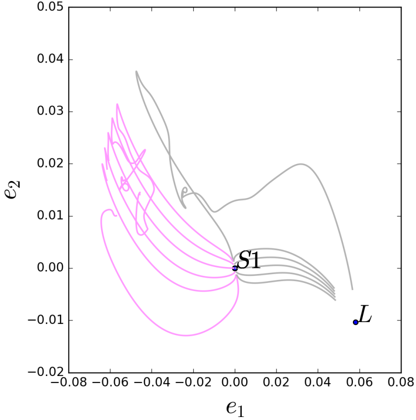
(Pringle & Kerswell 2007, Mellibovsky & Meseguer 2007)

Unstable manifold of $S1$



$$\hat{x}_\phi(\tau = 0) = \hat{x}_{S1} + \epsilon \left(\frac{\hat{V}_1}{\mu_1} \cos\phi + \frac{\hat{V}_2}{\mu_2} \sin\phi \right) , \quad e_{1,2} = \langle \hat{x}_\phi(\tau),\ \hat{V}_{1,2} \rangle$$

Unstable manifold of $S1$



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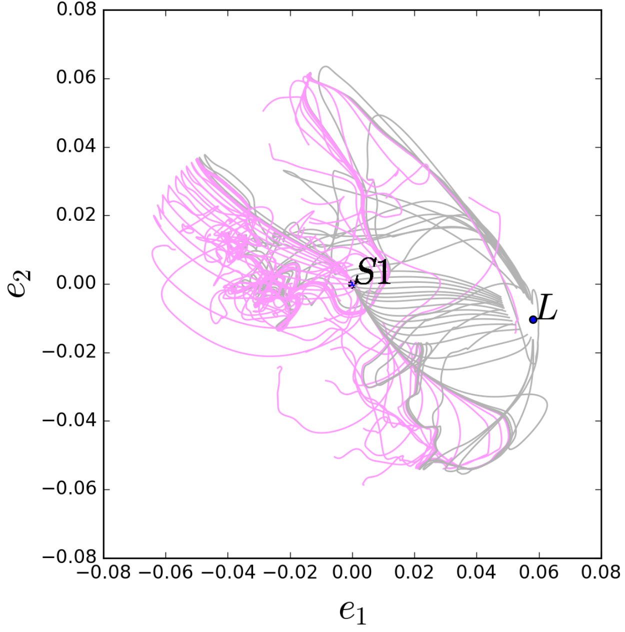
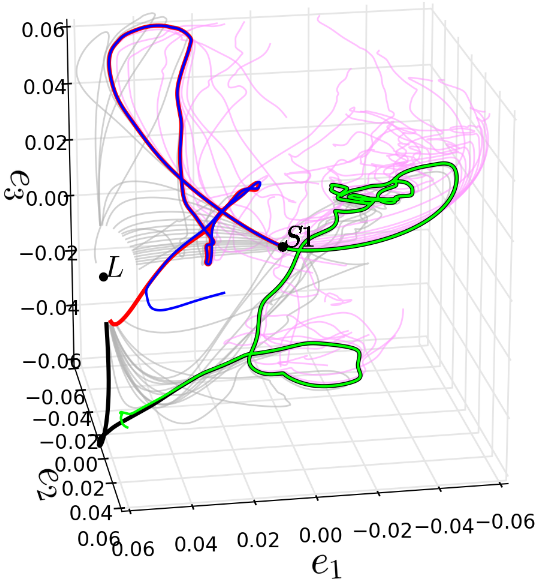
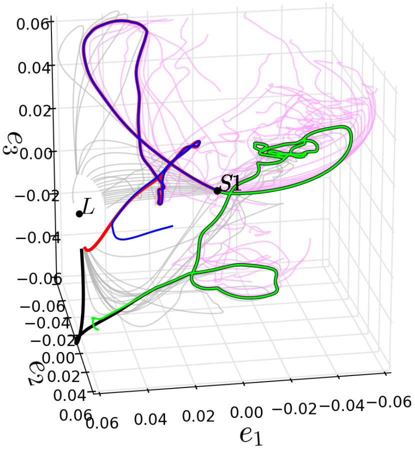
Unstable manifold of $S1$



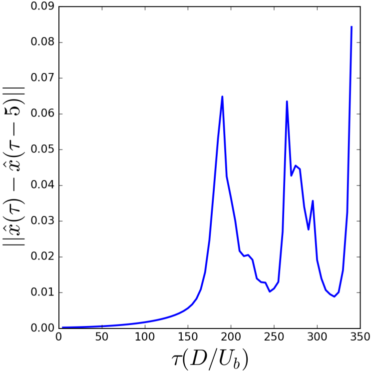
Unstable manifold of $S1$

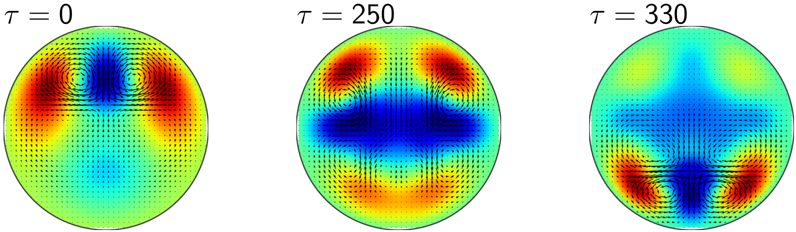


Unstable manifold of $S1$



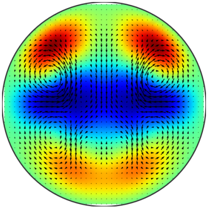
Recurrences on the unstable manifold



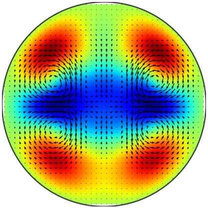


A new traveling wave on the "edge"

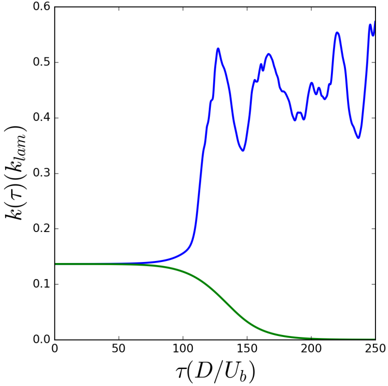
Initial condition:



Converged solution:

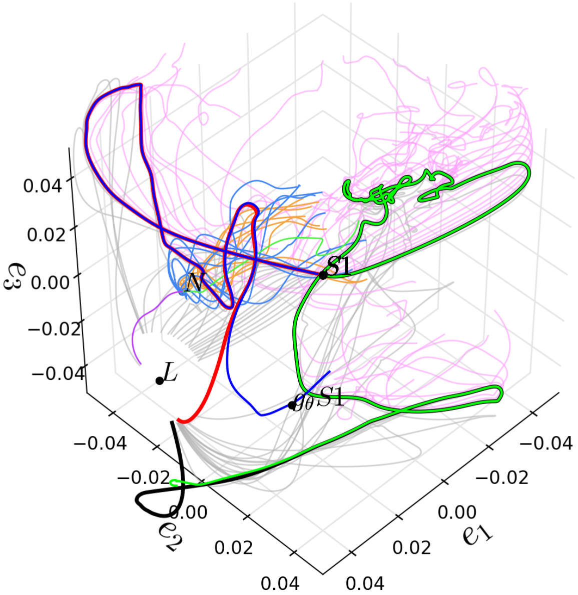


$$\lambda_{1,2} = 0.131 \pm 0.178i$$
$$\lambda_{3,4} = 0.095 \pm 0.066i$$
$$\lambda_5 = 0.073$$

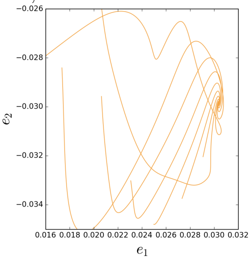


$$x(0) = x_{New} \pm \epsilon V_5$$

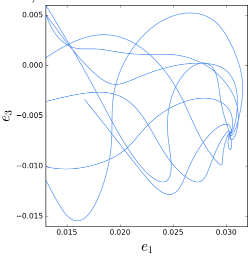
“Almost” complete picture



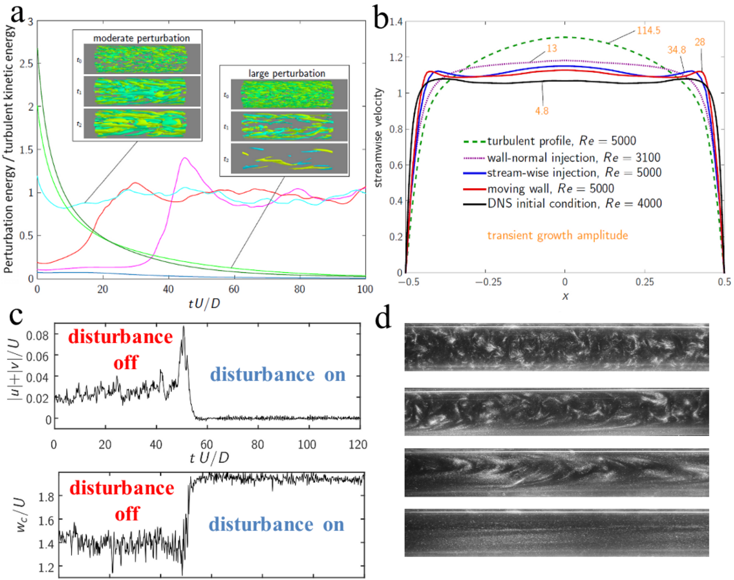
$\lambda_{1,2} = 0.131 \pm 0.178i$



$\lambda_{3,4} = 0.095 \pm 0.066i$



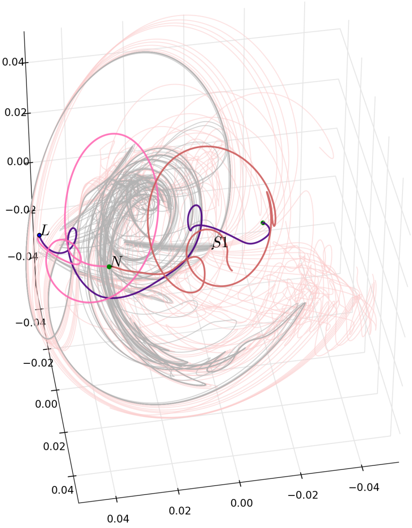
"Destabilizing" turbulence

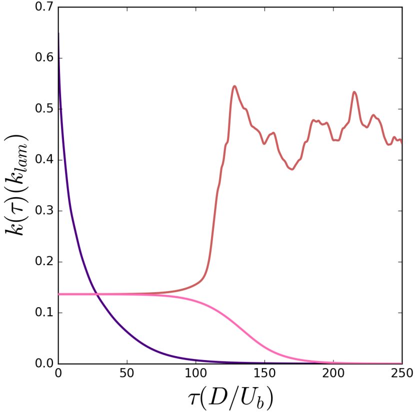


Kühnen et al. (2017) *to appear*

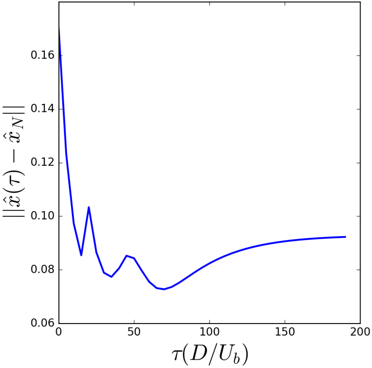
Too much turbulence kills turbulence

$\mathrm{SO}(2)_z \times \mathrm{SO}(2)_\theta$-reduced manifold:

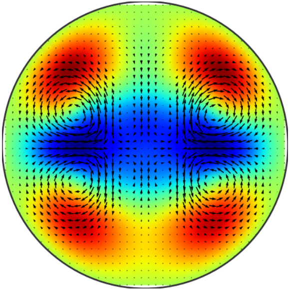




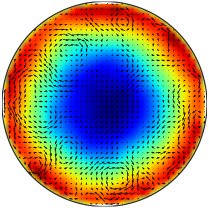
Too much turbulence kills turbulence



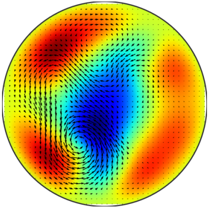
N:



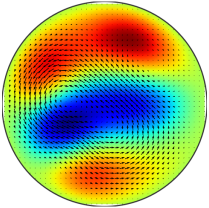
$\tau = 0$



$\tau = 35$



$\tau = 70$



Localized relative periodic orbits

PRL **110,** 224502 (2013) PHYSICAL REVIEW LETTERS week ending 31 MAY 2013

Streamwise-Localized Solutions at the Onset of Turbulence in Pipe Flow

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[2]*Castelldefels School of Telecom and Aerospace Engineering (EETAC), Universitat Politècnica de Catalunya, 08860 Barcelona, Spain*
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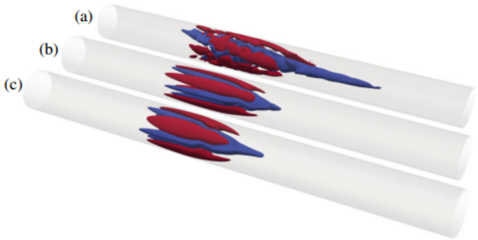
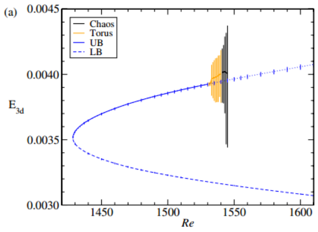
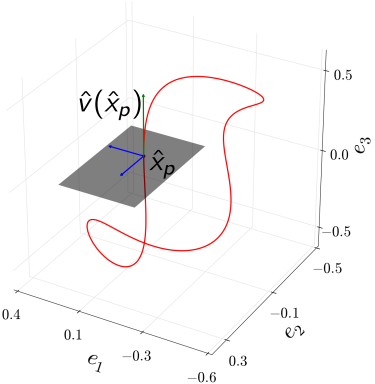
(Received 30 January 2013; published 29 May 2013)

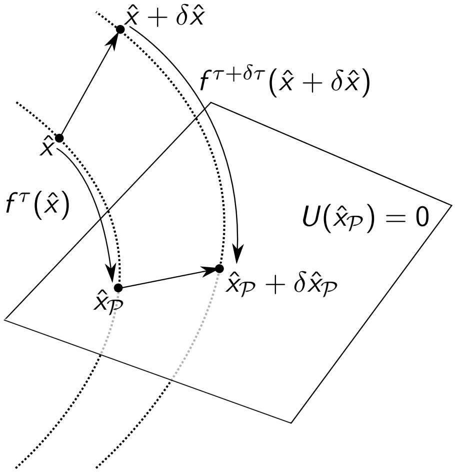
Although the equations governing fluid flow are well known, there are no analytical expressions that describe the complexity of turbulent motion. A recent proposition is that in analogy to low dimensional chaotic systems, turbulence is organized around unstable solutions of the governing equations which provide the building blocks of the disordered dynamics. We report the discovery of periodic solutions which just like intermittent turbulence are spatially localized and show that turbulent transients arise from one such solution branch.





Poincaré section

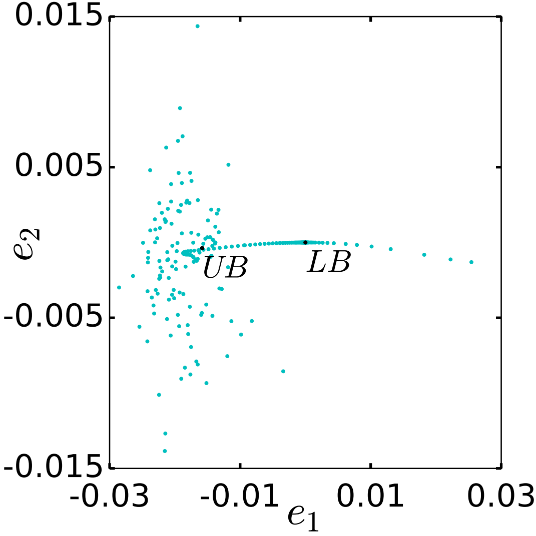




$$\begin{aligned}\langle \hat{x}_{\mathcal{P}} - \hat{x}_p,\, \hat{v}(\hat{x}_p)\rangle &= 0\\ \langle \hat{v}(\hat{x}_{\mathcal{P}}),\, \hat{v}(\hat{x}_p)\rangle &> 0\end{aligned}$$

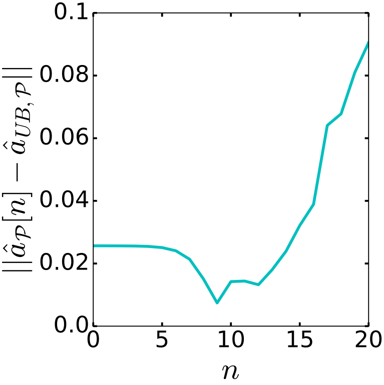
$$\delta\hat{x}_{\mathcal{P}} = \left(1 - \frac{\hat{v}(\hat{x}_{\mathcal{P}}) \otimes \hat{v}(\hat{x}_p)}{\langle \hat{v}(\hat{x}_{\mathcal{P}}),\, \hat{v}(\hat{x}_p)\rangle}\right) \frac{df^{\tau}(\hat{x})}{d\hat{x}} \delta\hat{x}$$

Unstable manifold of the lower branch at $Re = 1700$

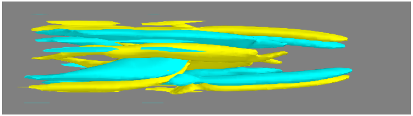


$$\hat{x}_{\mathcal{P}}(\delta) = \hat{x}_{p,\mathcal{P}} \pm \epsilon \Lambda_1^{\delta} \hat{V}_{1,\mathcal{P}}\,, \quad \delta \in (0,1]$$

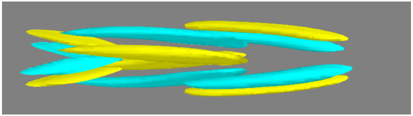
Heteroclinic connection



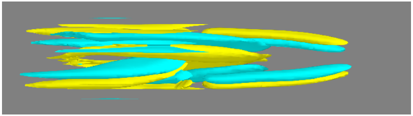
UB:



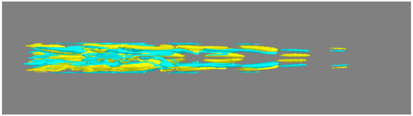
$n = 0$ (LB):



$n = 9$ (closest approach):

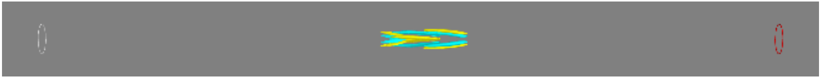


$n = 20$:

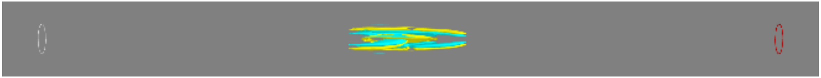


On the same scale

$n = 0$ (LB):

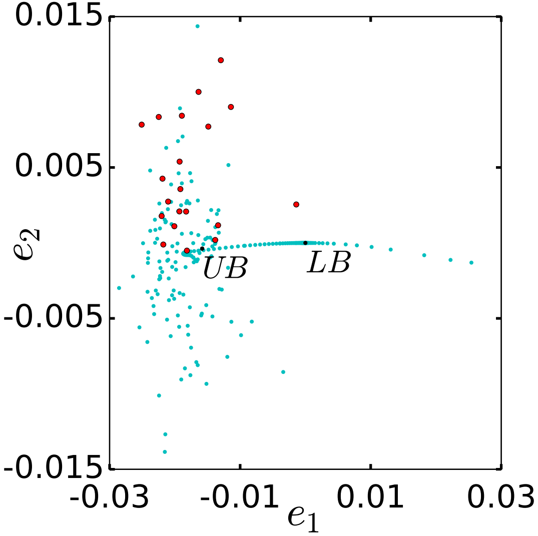


$n = 9$ (closest approach to UB):

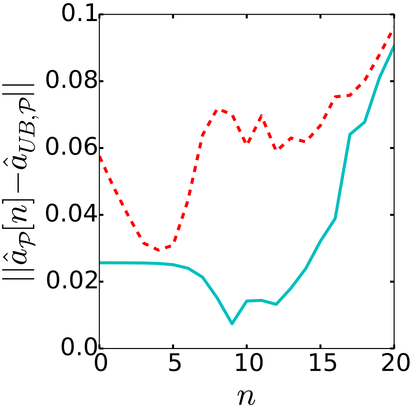


$n = 20$:

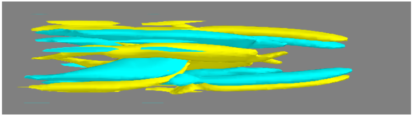
Puff formation from a random initial perturbation



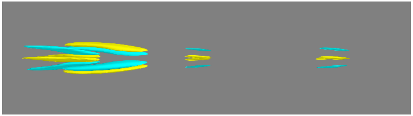
Flow structures



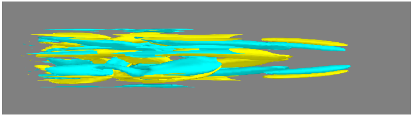
UB:



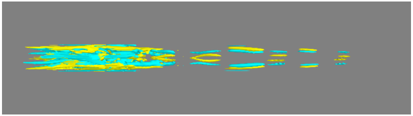
$n = 0$:



$n = 4$ (closest approach):



$n = 20$:



Conclusions

- We computed and visualized unstable manifolds of traveling waves and relative periodic orbits
- We found a new traveling wave on the laminar-turbulent boundary of a small pipe
- We found strong numerical evidence of a heteroclinic connection between localized relative periodic solutions of pipe flow

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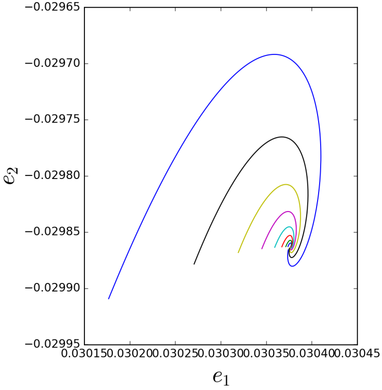
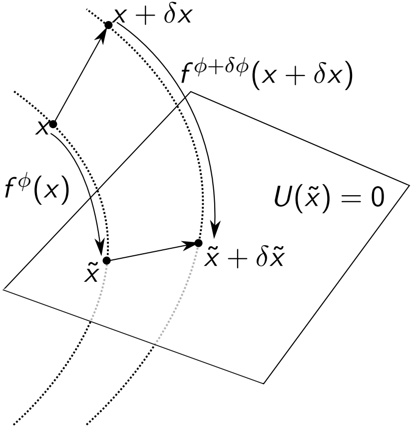
Complex conjugate eigenvalues

$$\lambda_{1,2} = \mu \pm i\omega$$



$$\hat{x}(0) = \hat{x}_{TW} + \epsilon e^{\frac{2\pi\mu}{\omega}\delta} \mathrm{Re}\, \hat{V}_1 \,, \quad \delta = (0, 1]$$

Projection



$$\delta\tilde{x} = \left(1 - \frac{\partial_\theta f^\theta(x)|_{\theta=\phi} \otimes \nabla U(\tilde{x})}{\langle \partial_\theta f^\theta(x)|_{\theta=\phi}, \nabla U(\tilde{x}) \rangle}\right) \nabla f^\phi(x) \delta x$$