The Generalised Quasilinear Approximation: Application to Jets, HMRI & Rotating Couette Flows

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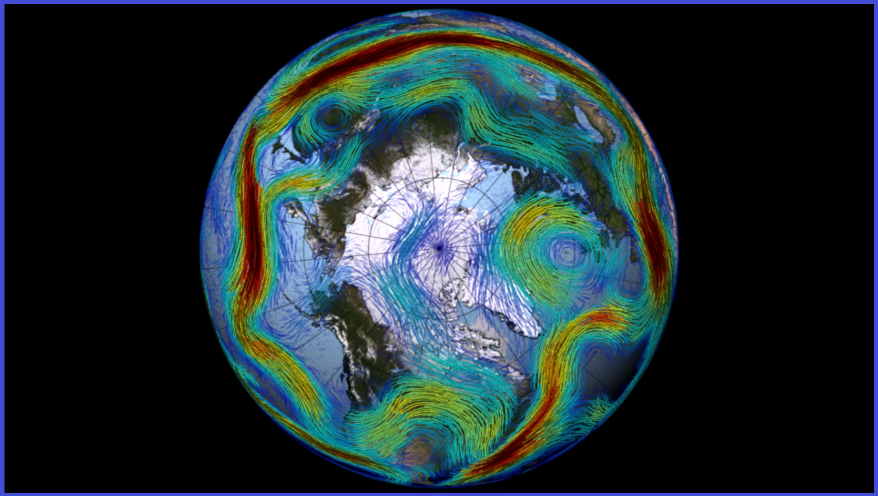
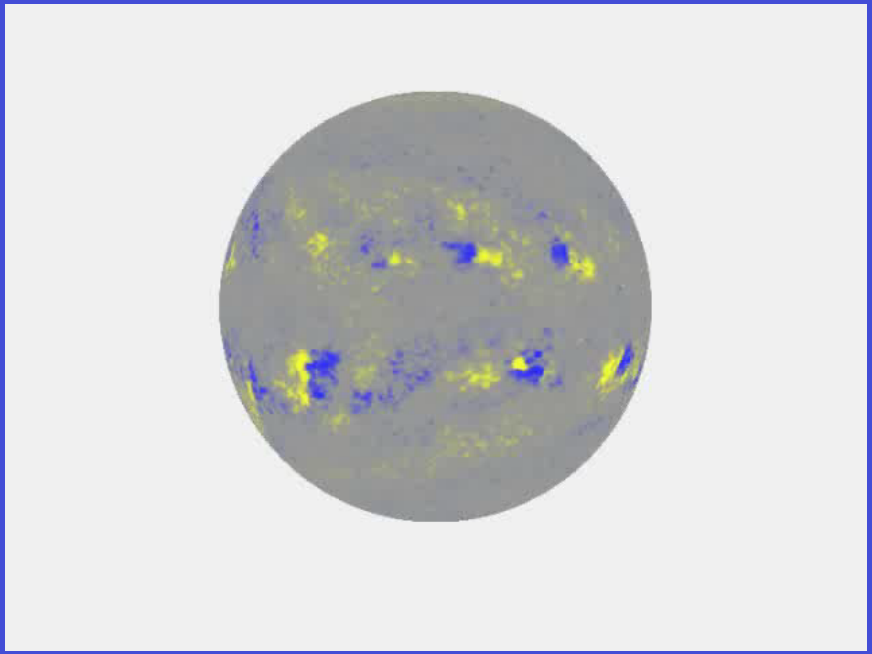
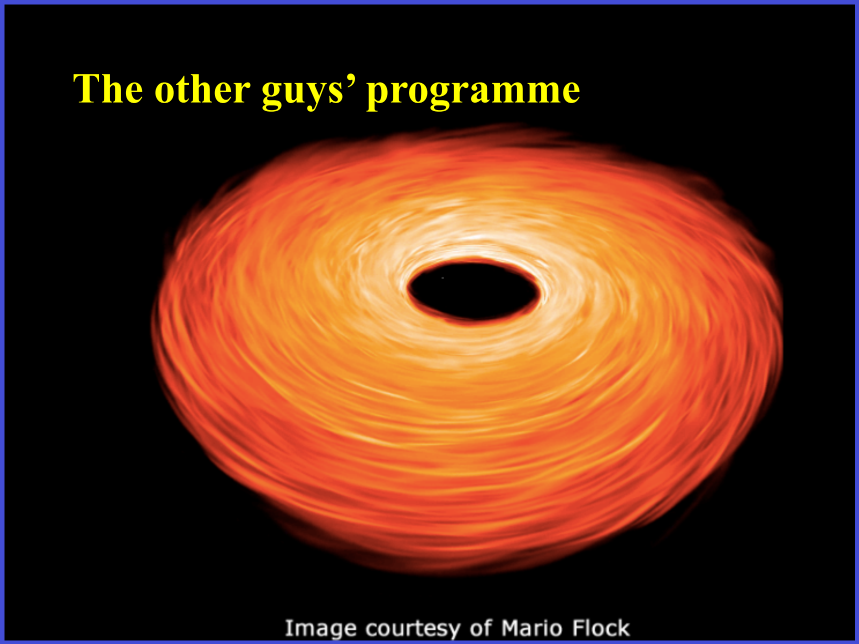
Talk Outline

- Motivation
  - Some geophysical and astrophysical flows.
  - The need to go beyond DNS – but where?
    - Reduced Models
- The Quasilinear approximation
  - When does linearising about the adjusted (to the fluctuation interactions) mean state work?
- The Generalised Quasilinear Approximation: Some model problems…
  - Jet formation
  - Magnetised Taylor-Couette (HMRI)
  - Rotating Couette flow
- [Direct Statistical Simulation]
- Conclusions

The “Big Bad” flows of AGFD

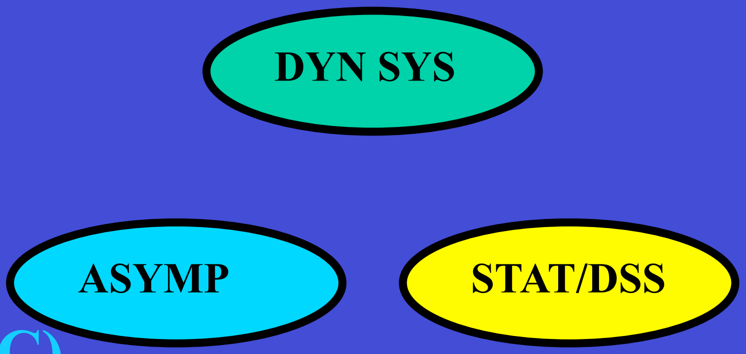






Reduced Models

- Often important to derive "reduced" models of turbulence that yield insight into the dynamics/statistics of the full problems in certain parameter regimes



- Dynamical Systems theory
  - Find building blocks of turbulence
  - Fully nonlinear solutions
- Asymptotic theories (talks by KAJ, GPC)
  - Models valid in some asymptotic wedge in parameter space.
  - Often validity of models extends beyond the formal limit of the theory.
- Statistical Models
  - Derived to yield information about the statistical properties of a system
  - Or the statistical properties of certain unresolved scales.
  - Rely on truncations/approximations
    - Often unwarranted (e.g. homogeneity, isotropy)

Herring (1963), Kraichnan (1980), Farrell & Ioannou (2007, 2009), Tobias et al (2011), Squire & Bhattacharjee (2014, 2015), Marston et al (2014), Bakas & Ioannou (2011, 2013), Srinivasan & Young (2012), Parker & Krommes (2013,2014), Tobias & Marston (2013), Constantinou et al (2016)

Two types of turbulent interaction with means

- **Energy input at small/ moderate scales**
  - large scales/mean flows emerge owing to correlations in turbulence (rotn/strat)
- **Convective driving of mean flows in planets/stars**
- **Driving of jets on giant planets**
- **Large-scale Dynamos**
  - **e.g. solar cycle**

- **Energy input via mean flows/fields at large scales**
  - small scales emerge owing to instability of large scales.
  - these act back to modify large scales flows
- **KH instability**
- **Taylor-Couette**
- **Magnetorotational Instability**
- **Joint tachocline instabilities**
- **Pipe Flow**
- **Rotating Couette Flow**

An aside: Direct Statistical Simulation: Why *simulate* the statistics?

Low-order statistics: smoother in space than instantaneous flow.

Statistics evolve slowly in time, or not at all, and hence may be described by a fixed point, or at least a slow manifold.

Statistics usually describe mean/average behaviour and variations about that mean (e.g. 2pt correlations)

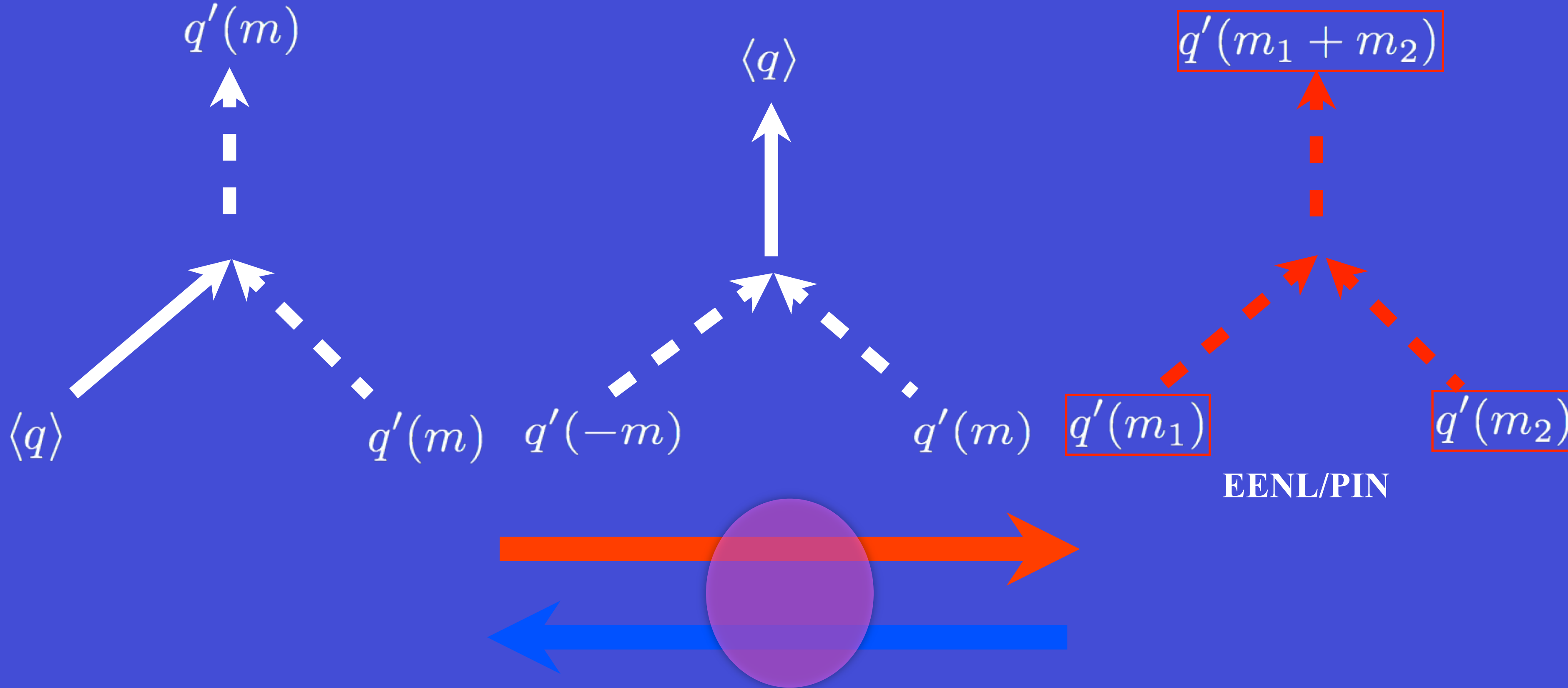
Statistics are less sensitive to changes in underlying parameters than detailed dynamics.

In geophysics/astrophysics correlations are *non-local* and highly anisotropic and inhomogeneous. Statistical formulations *must* respect this. They should also respect conservation laws

Solution of Statistical Equations is an old idea: Boussinesq, Reynolds, Lorenz, Herring, Kraichnan, Frisch, Farrell, Ioannou

Degrees of Approximation

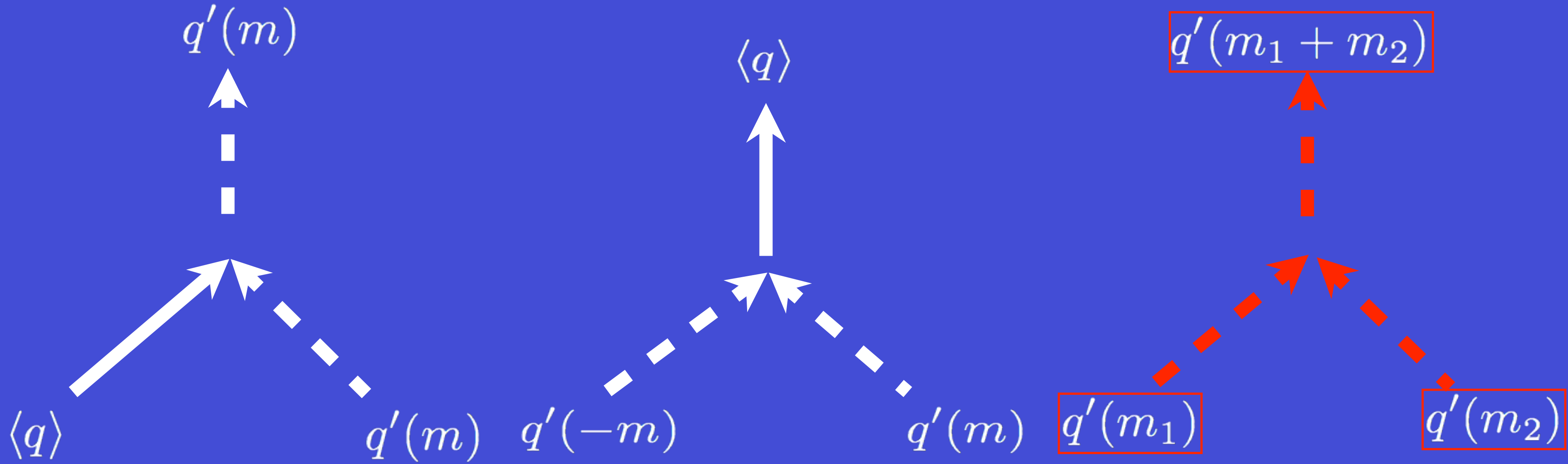
Quasilinear Approximation (QL) $q = \langle q \rangle + q'$



- Eddies sheared apart by mean flows; exact in certain asymptotic limits
- Conservation of global linear and quadratic invariants
  - Constrained triad decimation in pairs (Kraichnan 1985)
- No local cascade or inverse cascade (all non-local)
- The (inhomogenous/anisotropic) statistical formulation of this approach is CE2/SSST

Degrees of Approximation

Quasilinear Approximation (QL) $q = \langle q \rangle + q'$



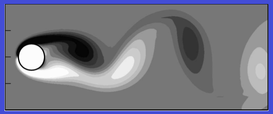
The effectiveness of this approximation in fluids is often measured by the Kubo number (see e.g. Diamond et al 2005, Tobias & Marston 2016)

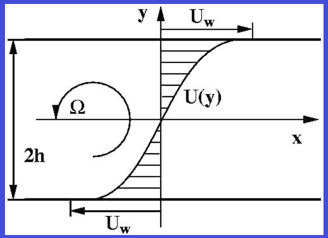
$$R = \frac{u_{rms}\tau_c}{l_c}$$

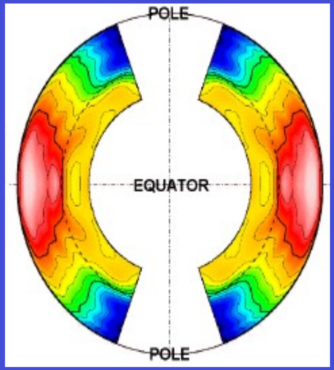
If this approximation works the mean adjusts owing to perturbation/perturbation interactions only. The perturbations only respond to the mean

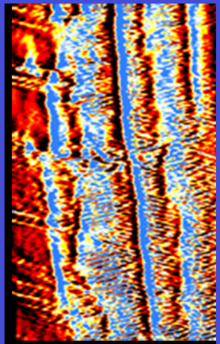
The statistical formulation of this approximation can be shown to break down as one moves away from statistical equilibrium (Tobias & Marston 2013)

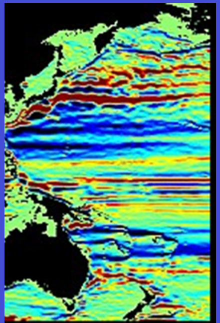
When does QL work for turbulence?

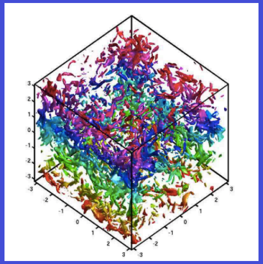






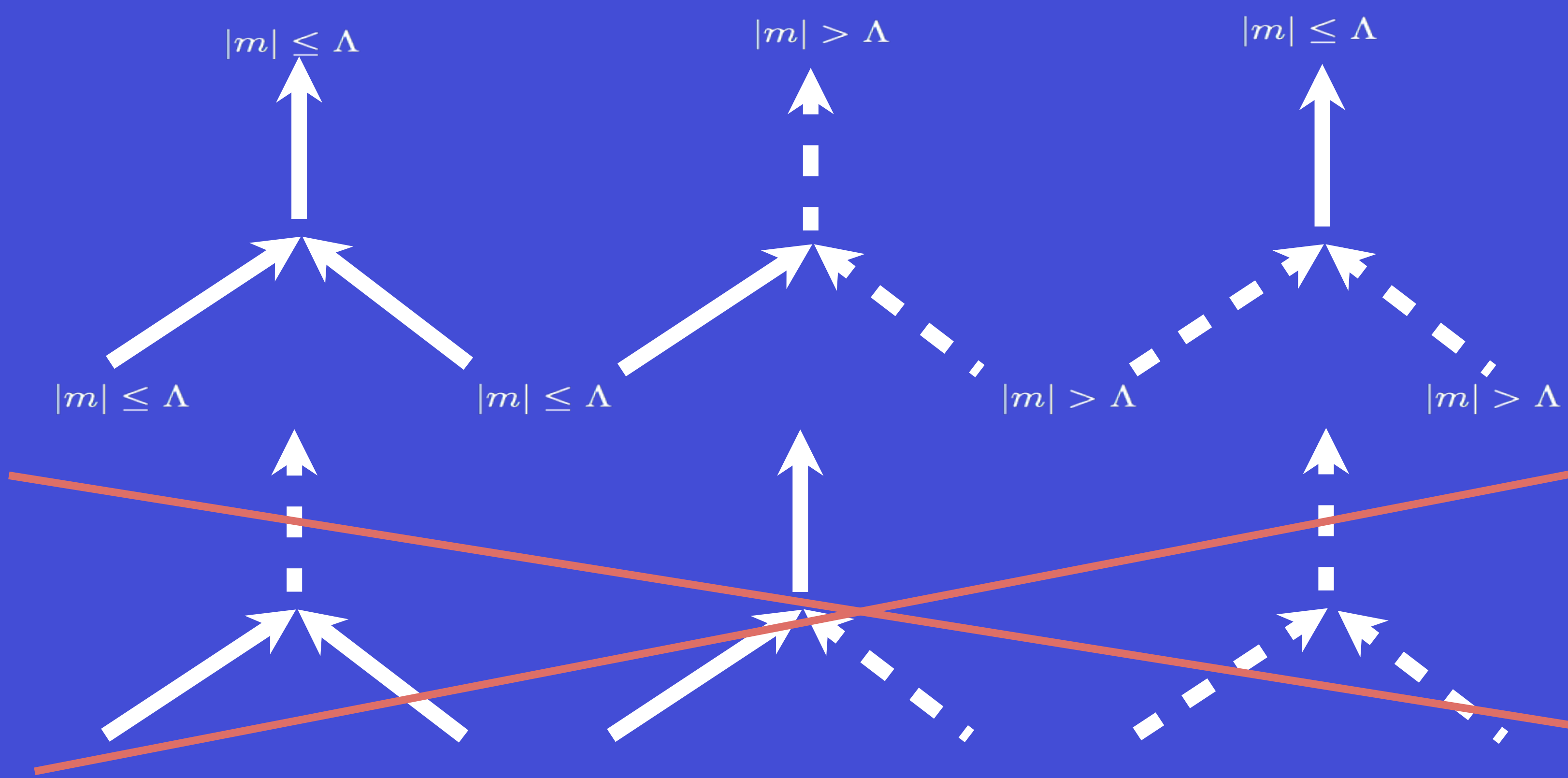

Magn. plasma
[Dif-Pradalier 2010]


Earth ocean
[Richards, 2006]



- Kubo number small
- Zonostrophy parameter large
- Eddy/eddy → eddy nonlinearity subdominant

Generalised Quasilinear Approximation (GQL): Separate Triads Into Long and Short Scale interactions

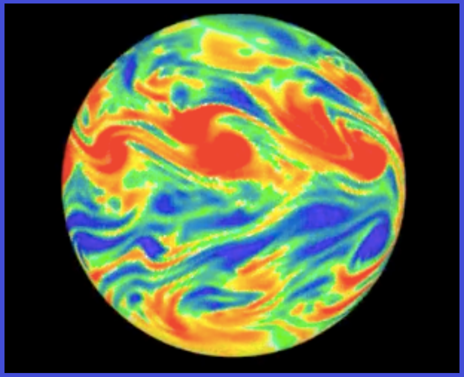


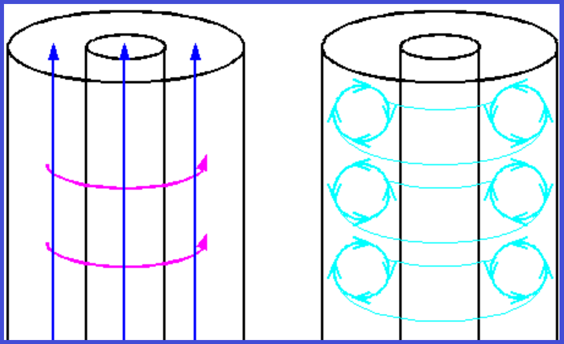
Marston, Chini & Tobias, PRL (2016)
Marston online KITP talk (2014)

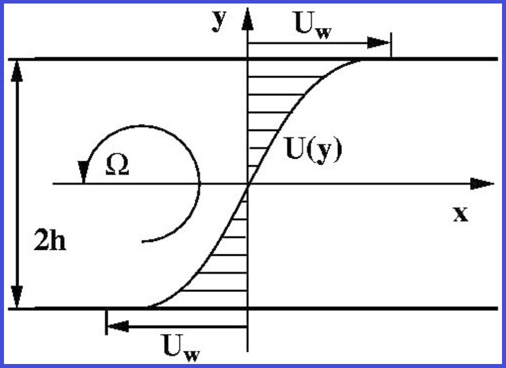
- **Energy can be scattered between short-scale modes via interactions with the large-scale flows.**
- **Energy can be scattered *into* "long" via interactions of short at different wavenumbers.**
- **Quadratic Conservation laws are maintained.**
- **Here we give examples for spatial scale interactions (wavenumbers)**

QL vs GQL: Beyond Quasilinearity

- QL/GQL have been tested *via direct numerical simulation* on 3 paradigm problems







- Driving of Barotropic jets on a sphere and beta-plane
  - Marston, Chini & Tobias (PRL, 2016)
- Inverse anisotropic cascade
- Reynolds stresses
- Energy to large scales.
- See also Constantinou et al (2016)

- Helical Magnetorotational instability (2D)
  - Magnetised Taylor-Couette (Child, Hollerbach, Marston & Tobias, Journal of Plasma Physics, 2016)
- Large-scale instability
- Energy to small scales

- Rotating Couette flow (3D)
  - Tobias & Marston (2016)
- Instability for anticyclonic shear
- Transport of absolute vorticity
- Energy to small scales

Paradigm Problems

$$\partial_t q = \mathcal{L}[q] + \mathcal{N}[q\ q] + f(t)$$

1. Barotropic turbulence. 2D flow on a rotating sphere or beta-plane

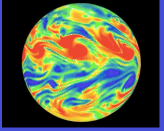
$$\partial_t \zeta = -\kappa\zeta - \nu_4 \nabla^4 \zeta - \mathbf{v}.\nabla(\zeta + 2\Omega \sin\theta) + \eta(t)$$

$$\zeta = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{r}}$$

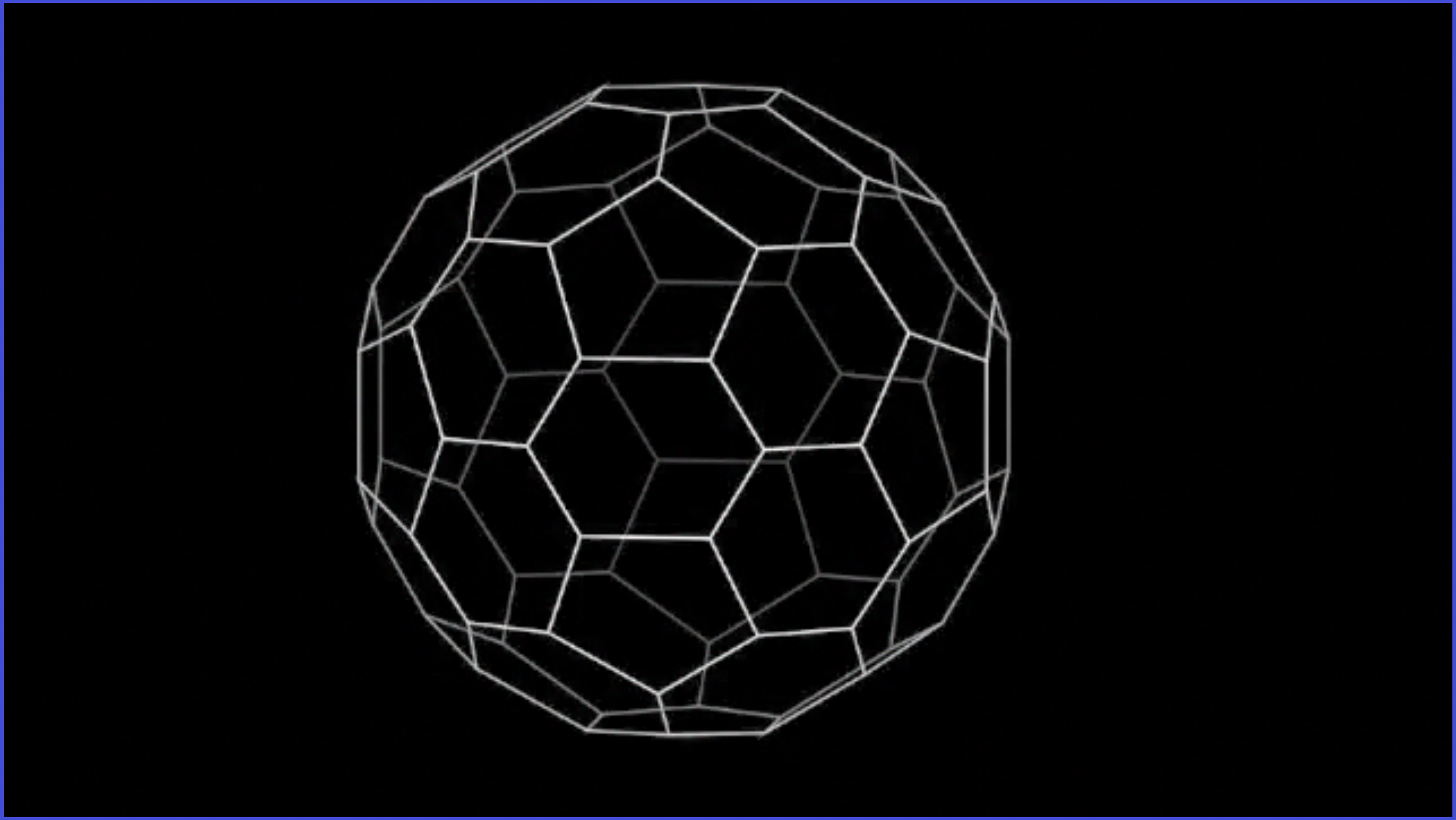
$$\zeta(\theta, \phi) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \zeta_{\ell,m} Y_\ell^m(\theta, \phi)$$

Radial component of the vorticity

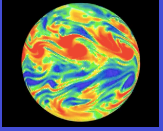
Rhines (1975), Vallis & Maltrud (1993), Galperin et al (2006, 2010), Farrell & Ioannou (2007), Tobias, Dagon & Marston (2011), Scott & Dritschel (2012), Srinivasan & Young (2012), Tobias & Marston (2013), Bakas & Ioannou (2014), Constantinou et al (2016)



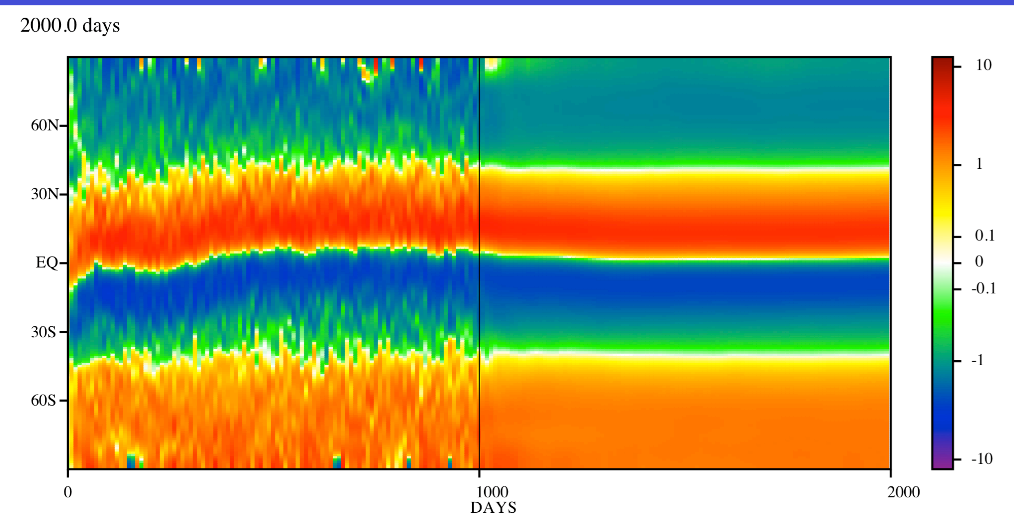
DNS Movie of hydro jet formation



Relative Vorticity



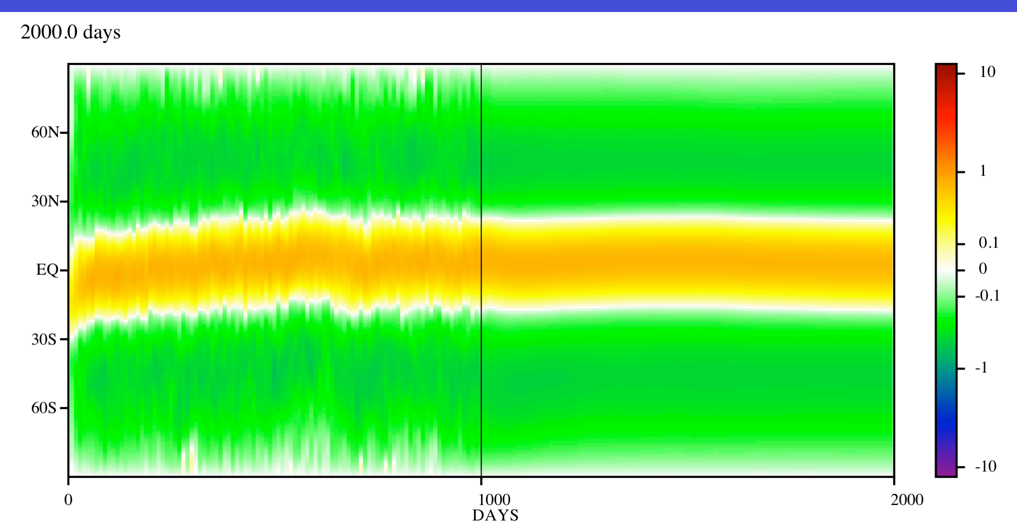
Jet (mean flow) formation



Latitude

Time

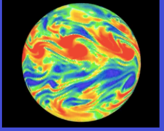
Zonal Mean Zonal Vorticity



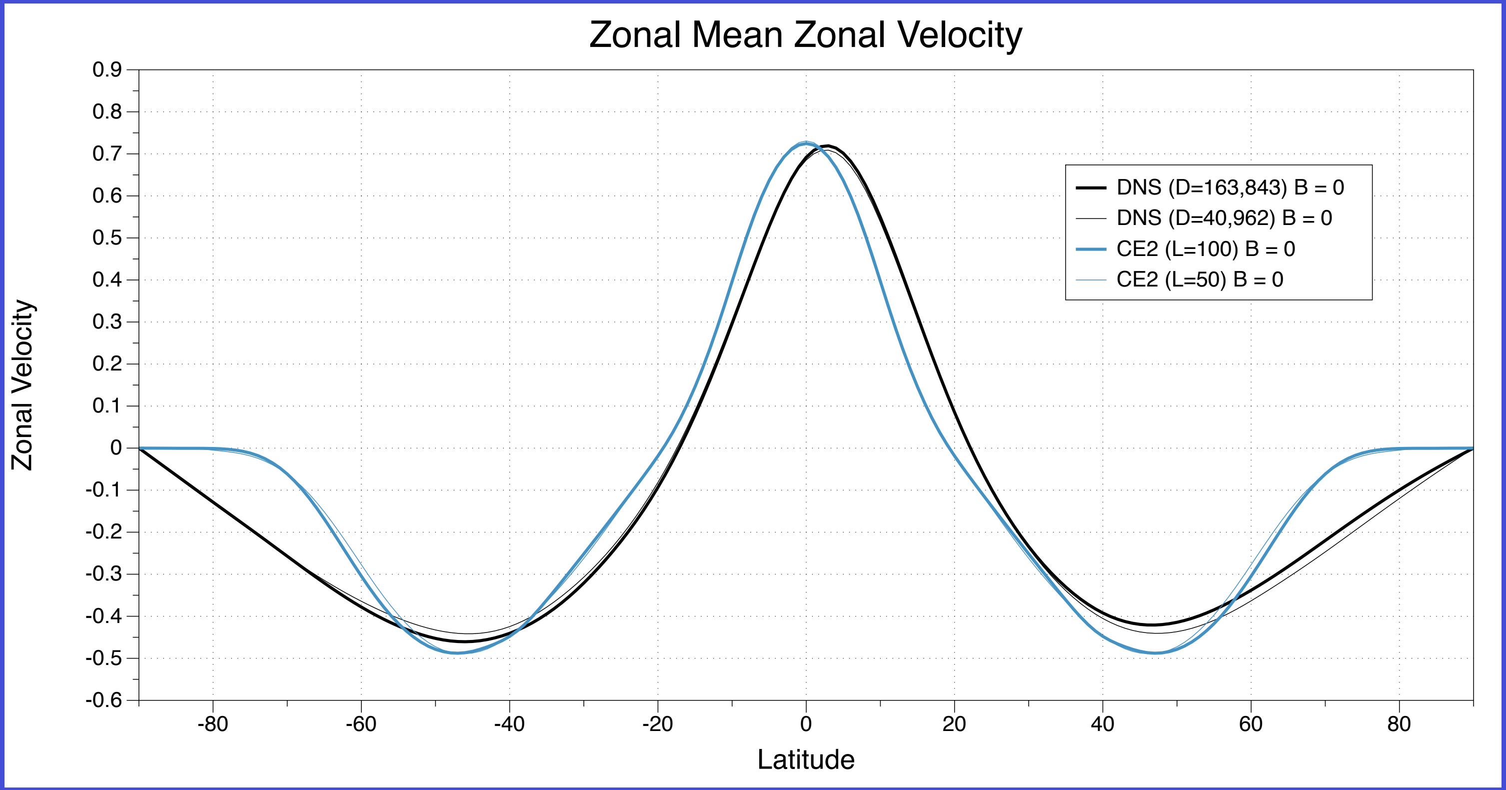
Latitude

Time

Zonal Mean Zonal Relative Velocity

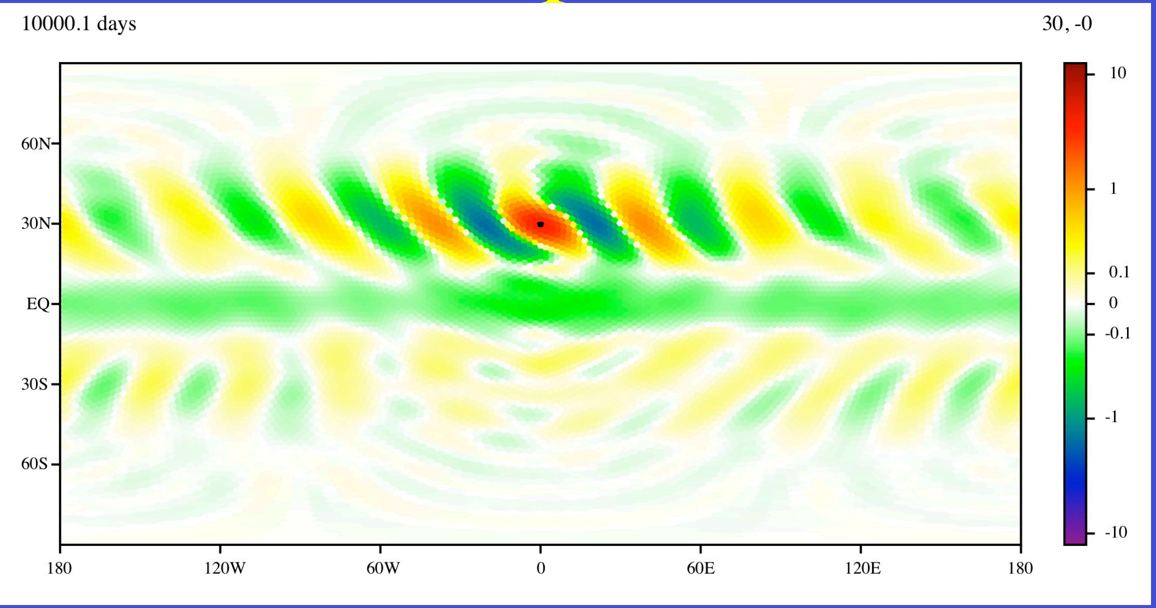


Hydrodynamic: QL DSS

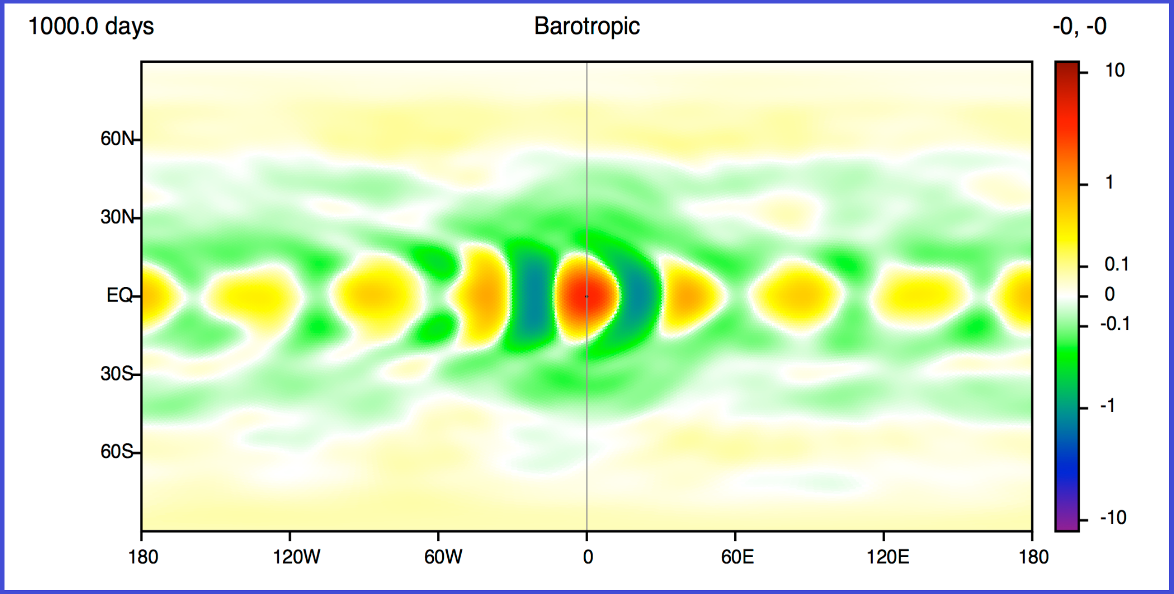


- Detailed comparison: QL reproduces mean shape of jet fairly well (except at poles where mean is small)
- No eddy-eddy scattering and no forcing at the poles)

Covariance/2pt correlation?

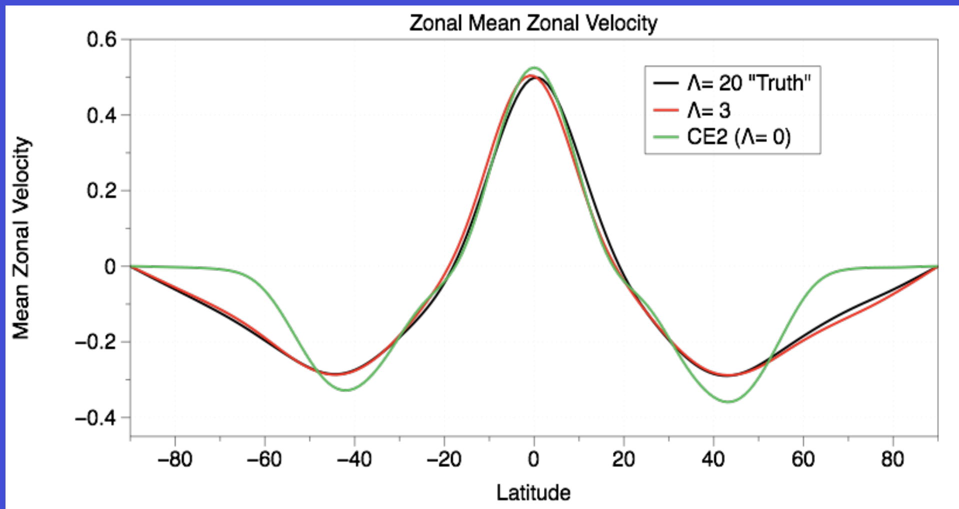


DNS



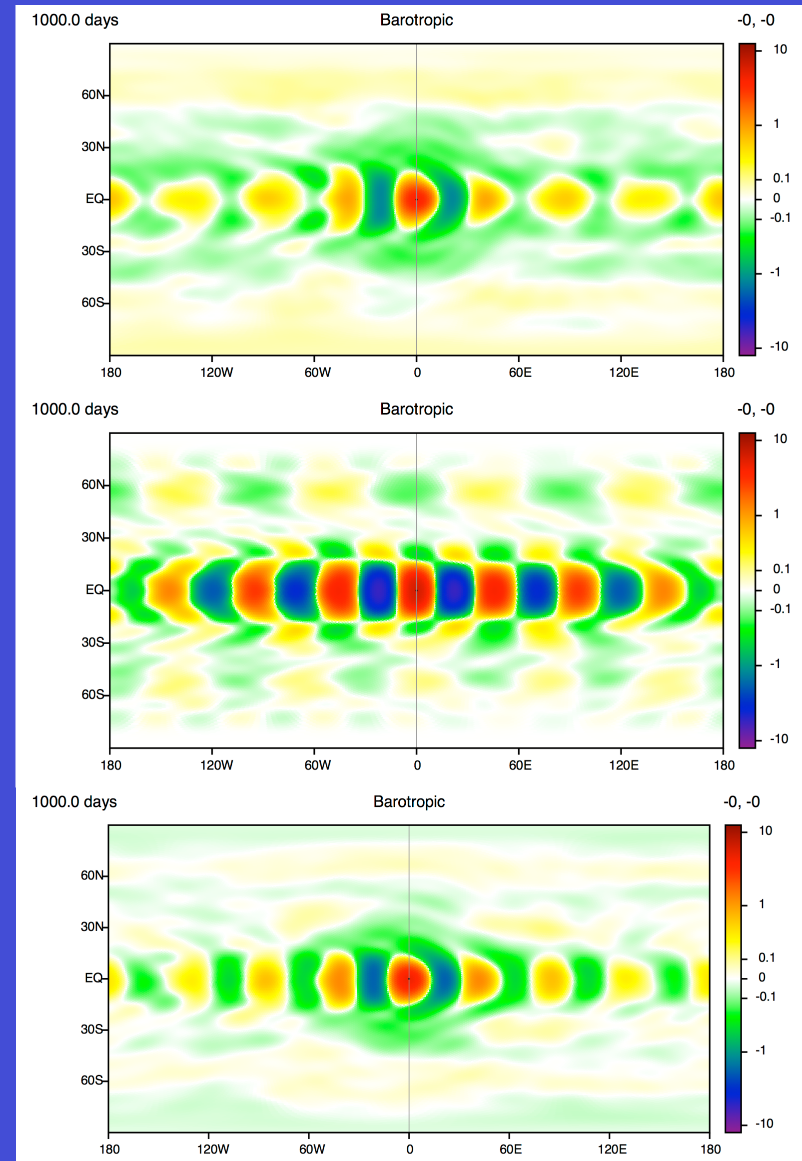
DNS

Barotropic Jet. QL vs GQL (sphere)



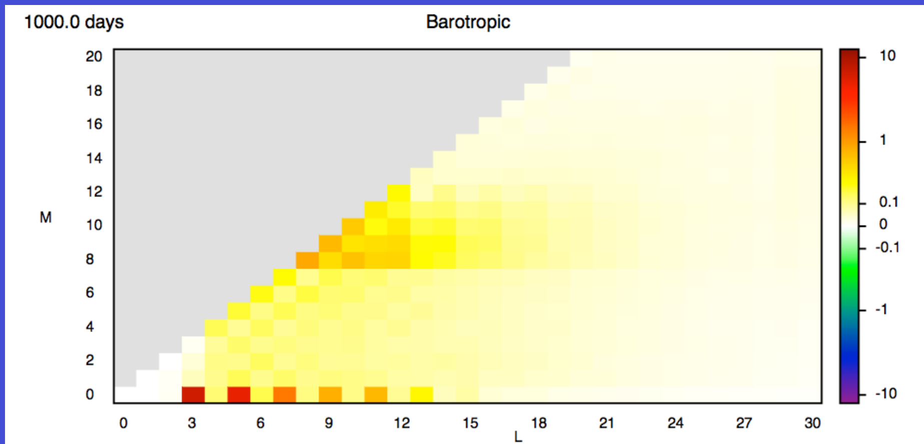
Marston, Chini & Tobias (PRL, 2016)

- QL (and therefore CE2) overemphasises the role of waves.
- Overemphasises long range correlations
- Can not scatter energy to where there is no direct forcing (i.e. the poles)

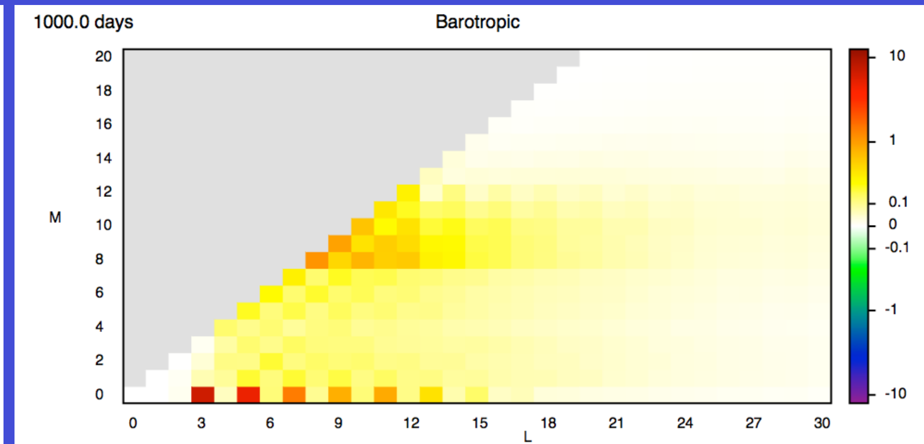


$\Lambda = 3$

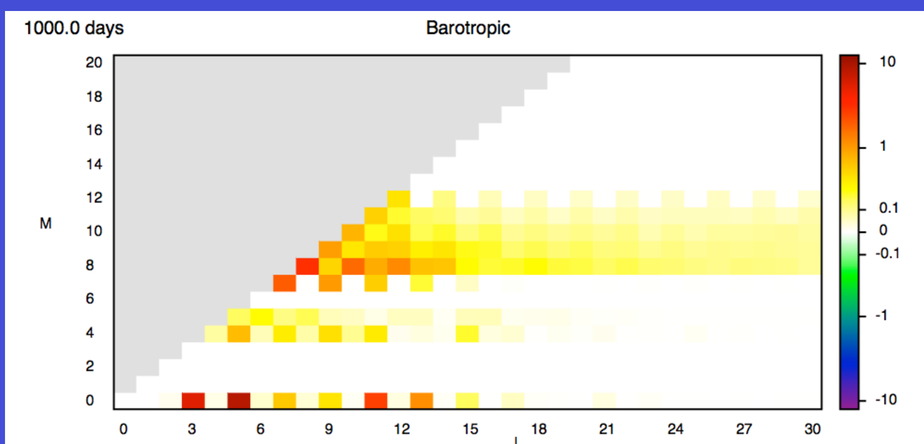
Vorticity Power Spectra (GQL)



$\Lambda = 20$ **"Truth"**

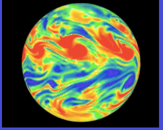


$\Lambda = 3$

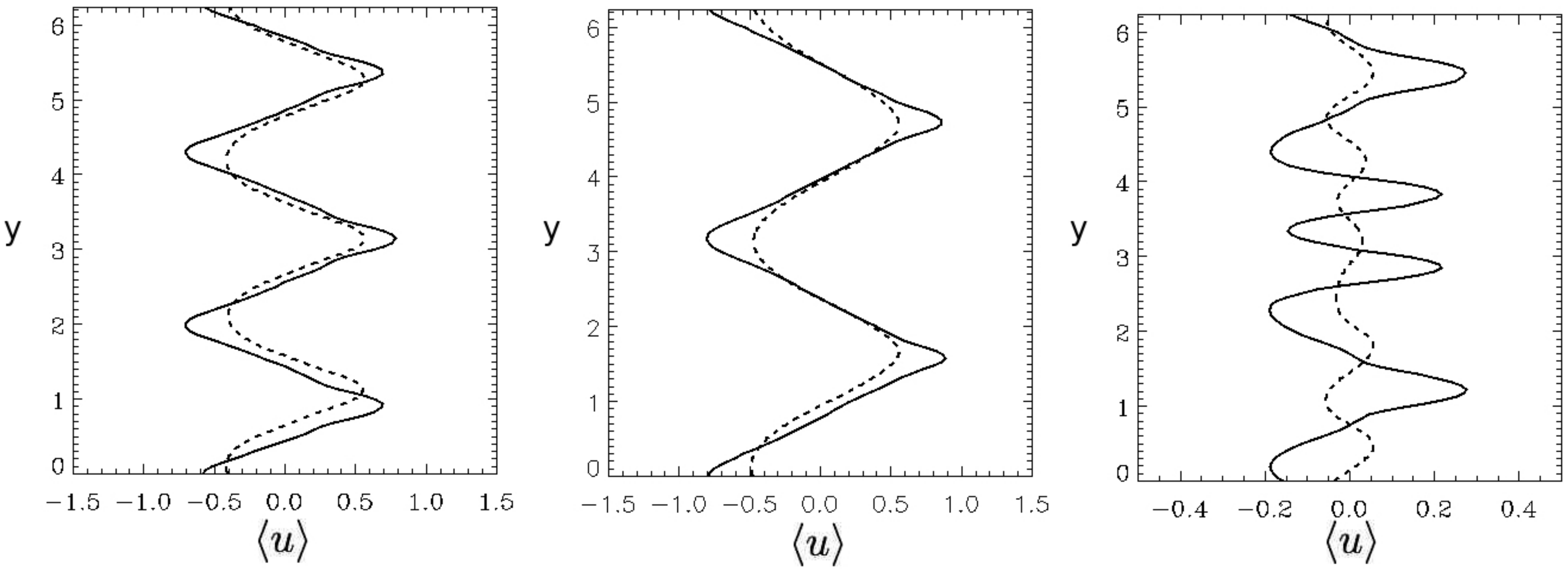


$\Lambda = 0$

- **GQL can scatter energy between small scales off the large scales**
- **This is impossible in standard quasilinear closure…**
- **Hence QL overemphasises the importance of the Rossby wave instability of the emerging mean flow.**



Quasilinear approx may break down as one moves away from statistical equilibrium



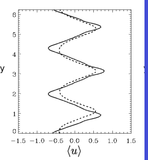
Tobias & Marston 2013 PRL

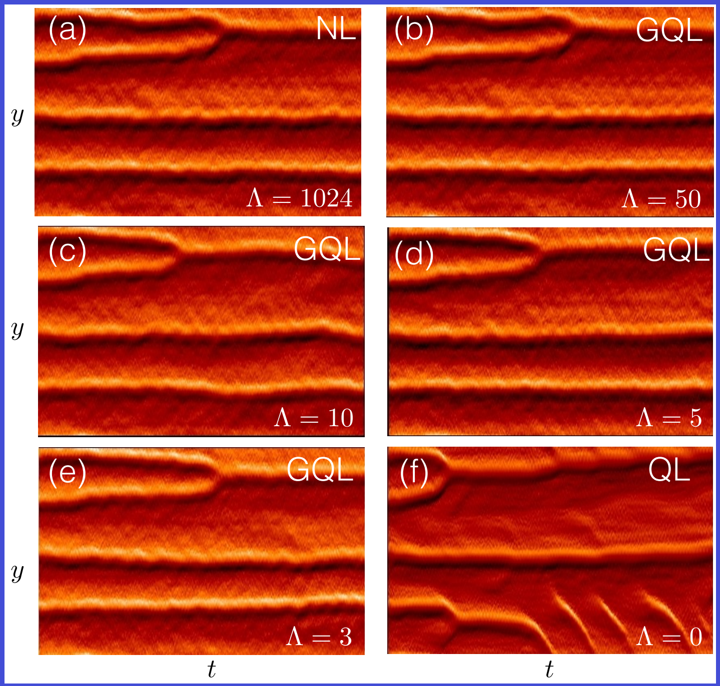
Jets on a β-plane: decreasing Zonostrophy Parameter: Jupiter jets → Ocean jets

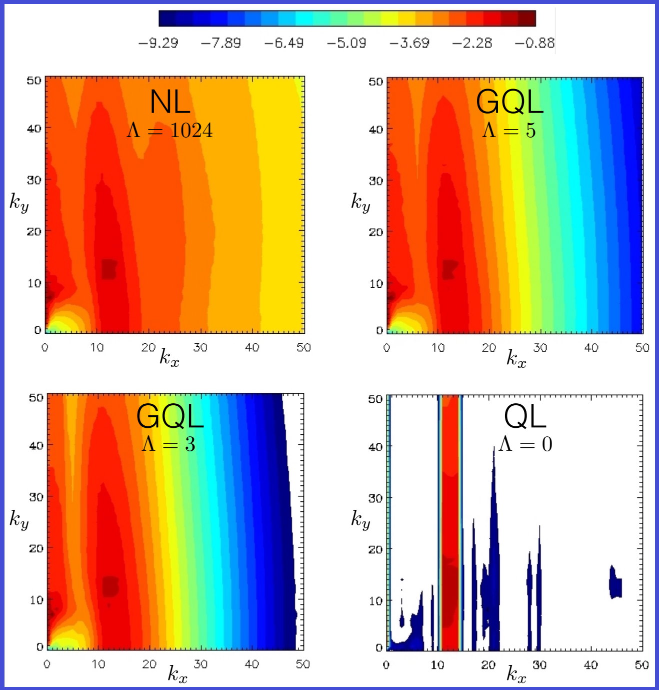
Move further away from equilibrium. More energy in eddies…

Need to include more sophisticated interactions

Barotropic Jet. QL vs GQL (Beta-plane)







- On a beta-plane even QL with a small cut-off Λ=3 can reproduce merging of jets.
- QL struggles to get number of jets correct.
- Formal derivation of GQL can be achieved via asymptotic expansion
  - Small parameter is related to degree of lack of statistical equilibrium
  - In this case, the small parameter can be related to ratio of dissipation to forcing.

Marston, Chini & Tobias, PRL (2016)

Low modes

$$\partial_T \bar{\zeta} + J(\bar{\psi}, \bar{\zeta}) + \bar{\beta} \partial_X \bar{\psi} = -\partial_y (\overline{\zeta' \partial_\chi \psi'}) - \bar{\kappa} \bar{\zeta} + \bar{D},$$

$$\partial_y^2 \bar{\psi} = \bar{\zeta},$$

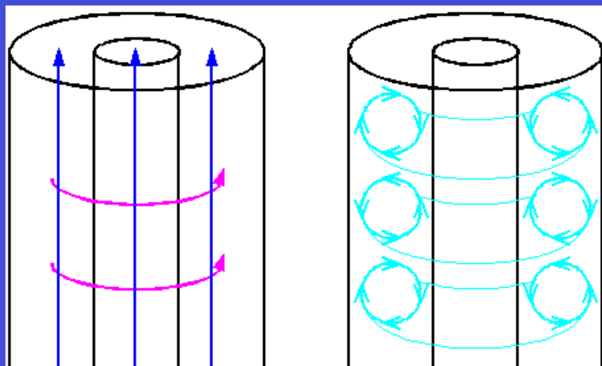
High modes

$$\partial_\tau \zeta' - \partial_y \bar{\psi} \partial_\chi \zeta' + \partial_\chi \psi' \partial_y \bar{\zeta} + \bar{\beta} \partial_\chi \psi' = \eta'(\tau),$$

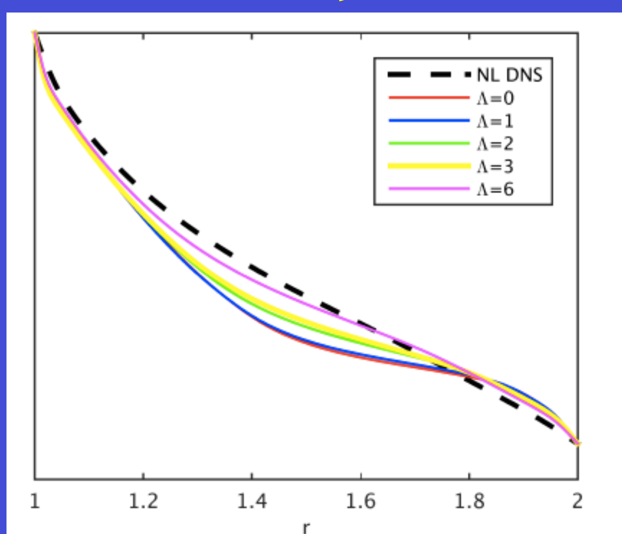
$$(\partial_\chi^2 + \partial_y^2) \psi' = \zeta',$$

Axisymmetric Helical MRI. QL vs GQL

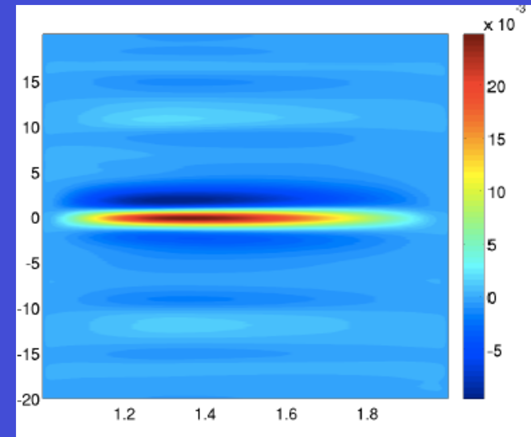
- **Helical MRI**



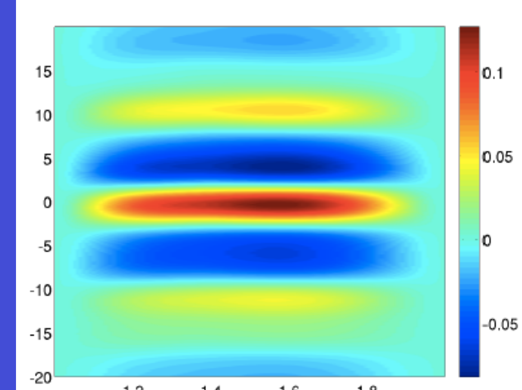
  - Helical Magnetorotational instability (2D, axisymmetric)
    - Forward cascade
    - Magnetised Taylor-Couette (Child, Hollerbach, Marston & Tobias, Journal Plasma Physics, 2016)



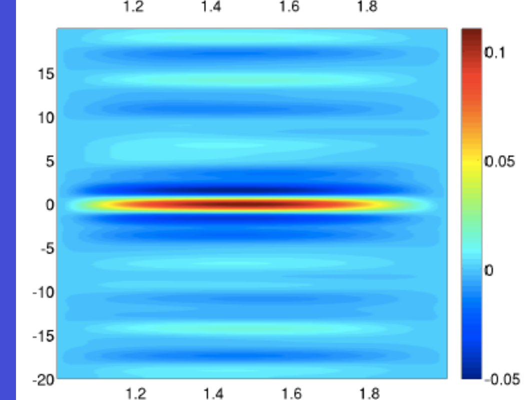
Two Point Correlation Functions



Full DNS

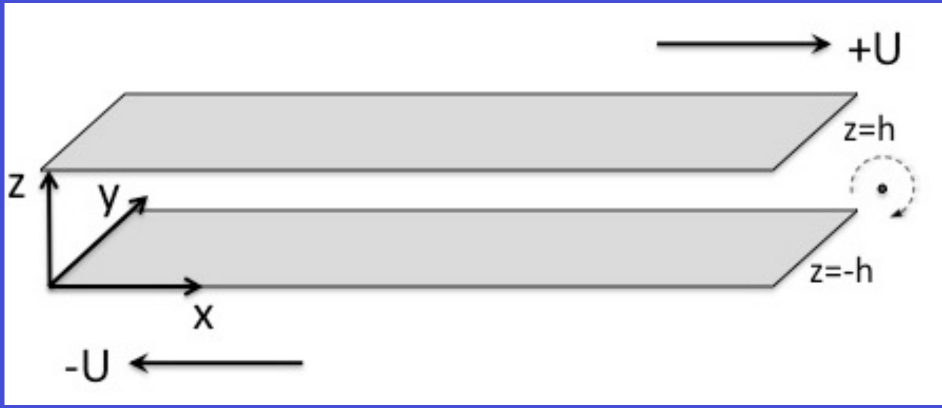


Λ=0



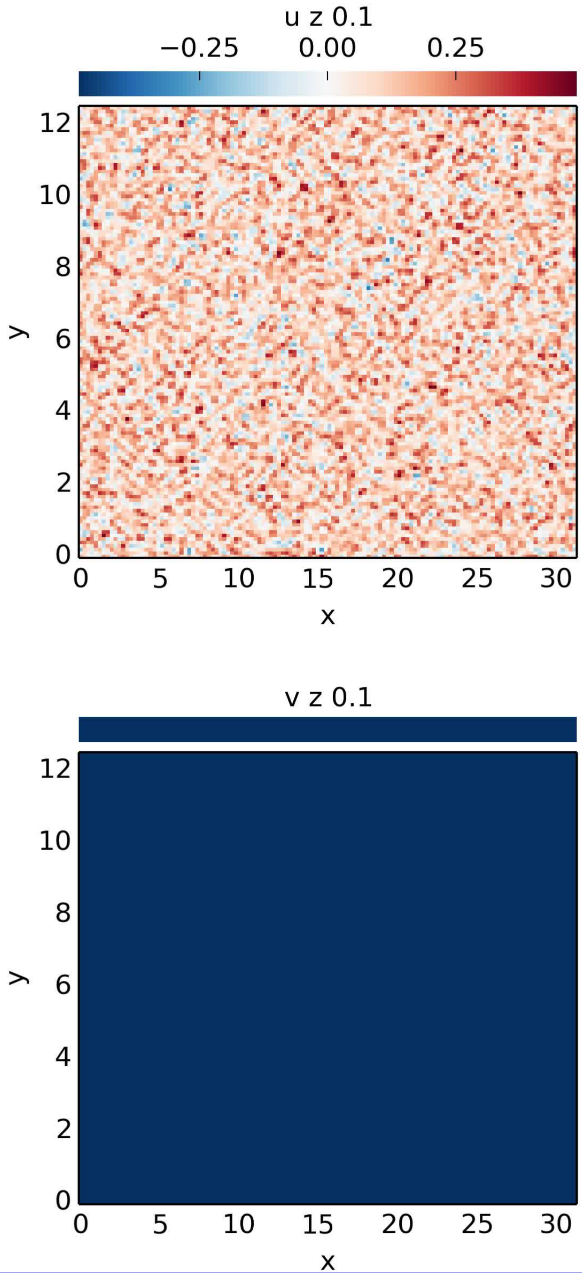
Λ=3

Rotating Couette Flow

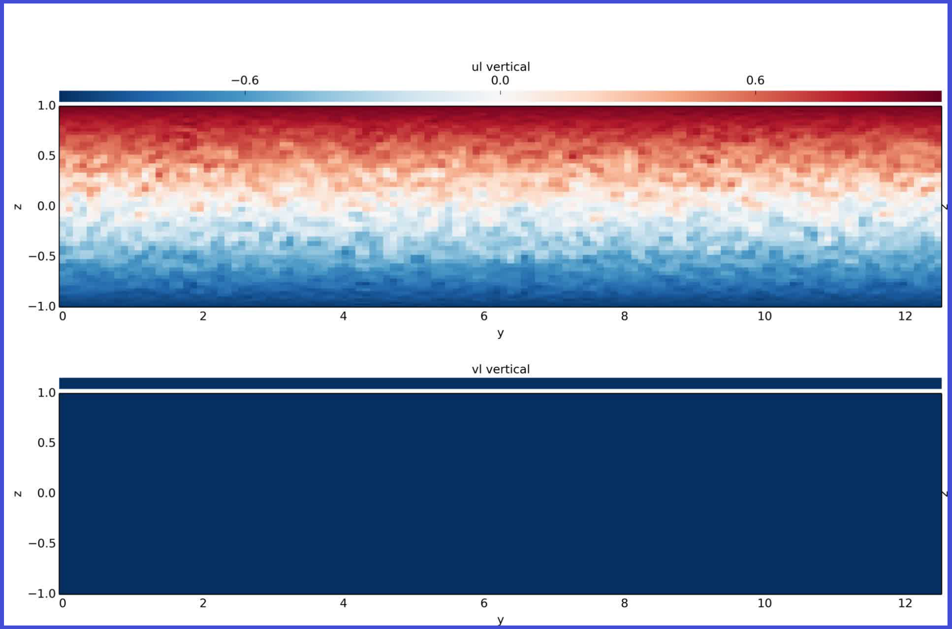


- Narrow gap limit of Taylor-Couette flow (Faisst & Eckhardt (2000))
- Experiments: Tilmark & Alfredsson (1992), Hiwatashi et al (2007), Tsukahara et al (2010)
- Some very complicated dynamics at moderate Re
  - co-existence of coherent structures and turbulence Bech & Andersson (1996, 1997)
  - Wall shear stress non-monotonic function of Rotation. Salewski & Eckhardt (2015)
- Two important non-dimensional parameters
  - Reynolds number: $Re = Uh/\nu$
  - Rossby number: $Ro = U/(2\,\Omega\,h)$
  - (Note in literature some use Rotation number $Ro = (2\,\Omega\,h)/U$)
  - → $Ro = 1/Ro$!

Tobias & Marston, JFM (2016)

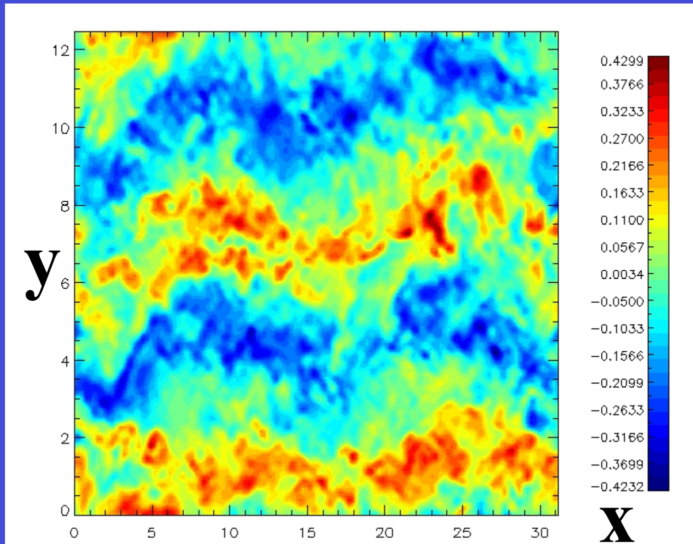


- Re=1200, Ro=100 (weakly rotating).
- Evolves to a state with 2 time-dependent wavy rolls.
  - Has transport both by these coherent structures and by turbulence (Bech & Andersson 1996,1997)
  - Kubo number R ~ 0.5
- (Evolution is via a state with 3 straight rolls.)

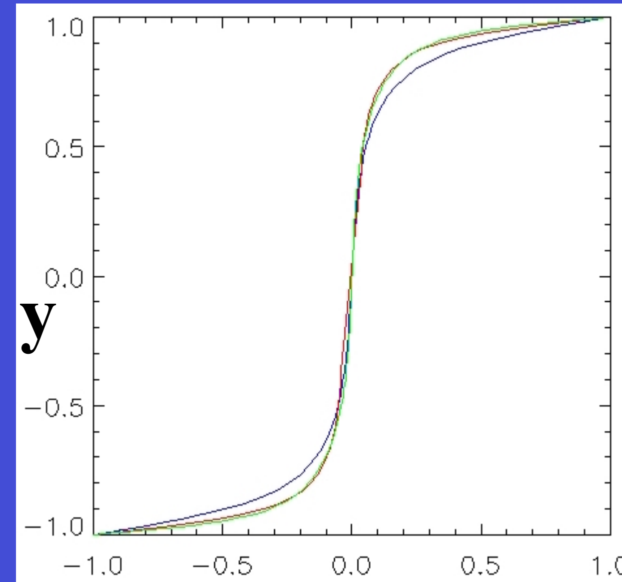


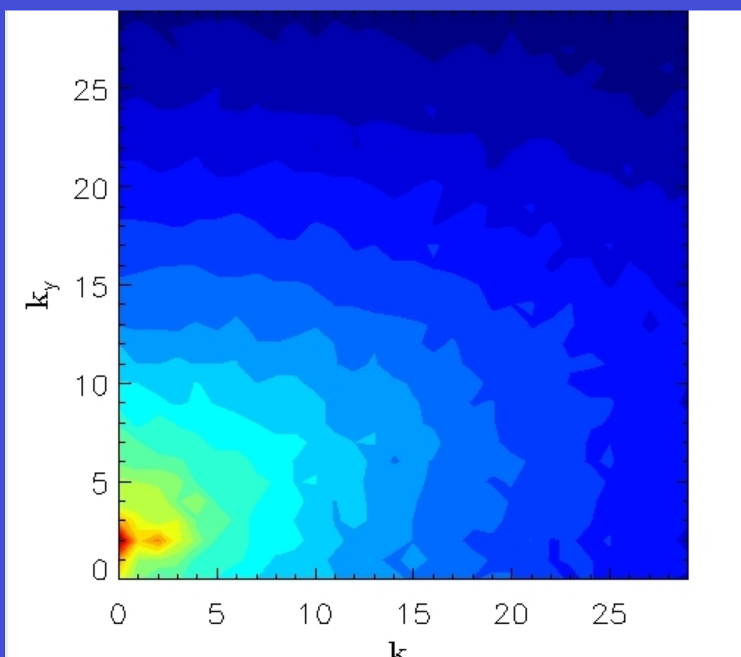
- Solved using pseudo-spectral *Dedalus* framework dedalus-project.org

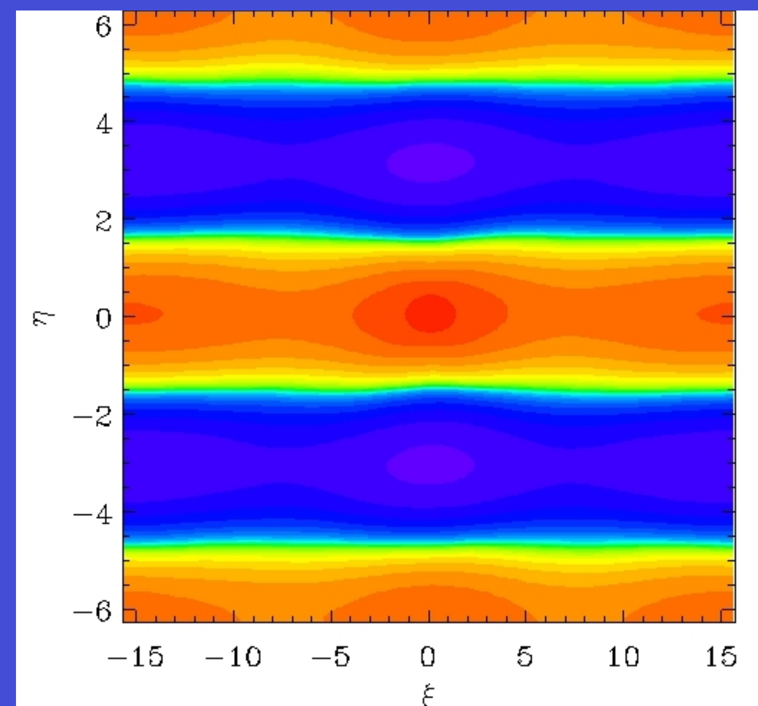
Snapshot of flow in plane z=0.1

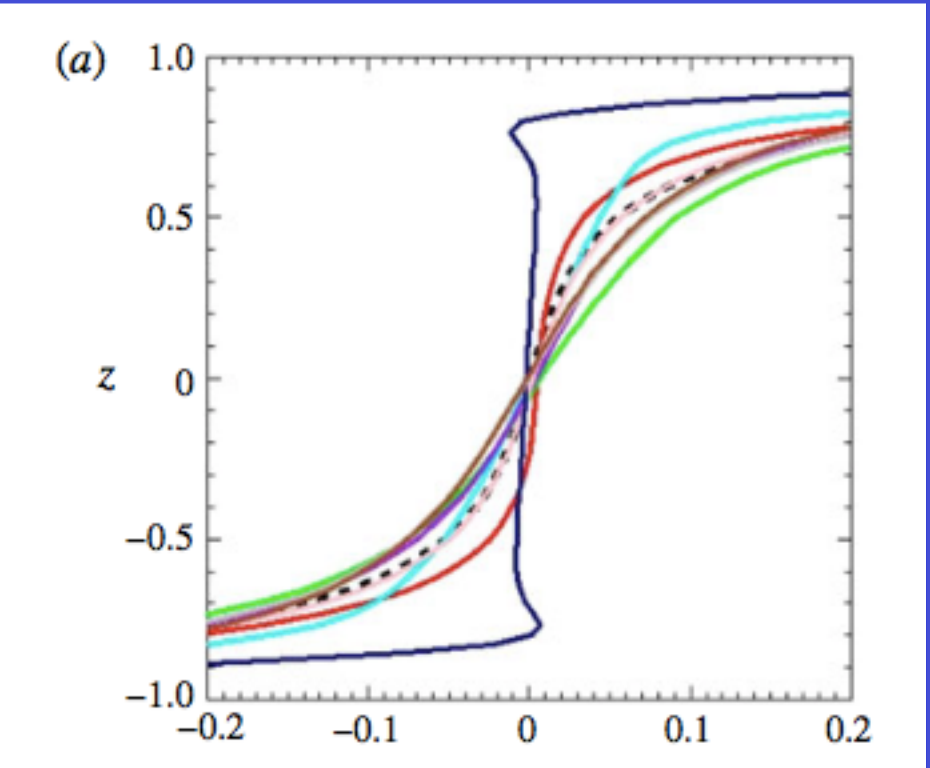


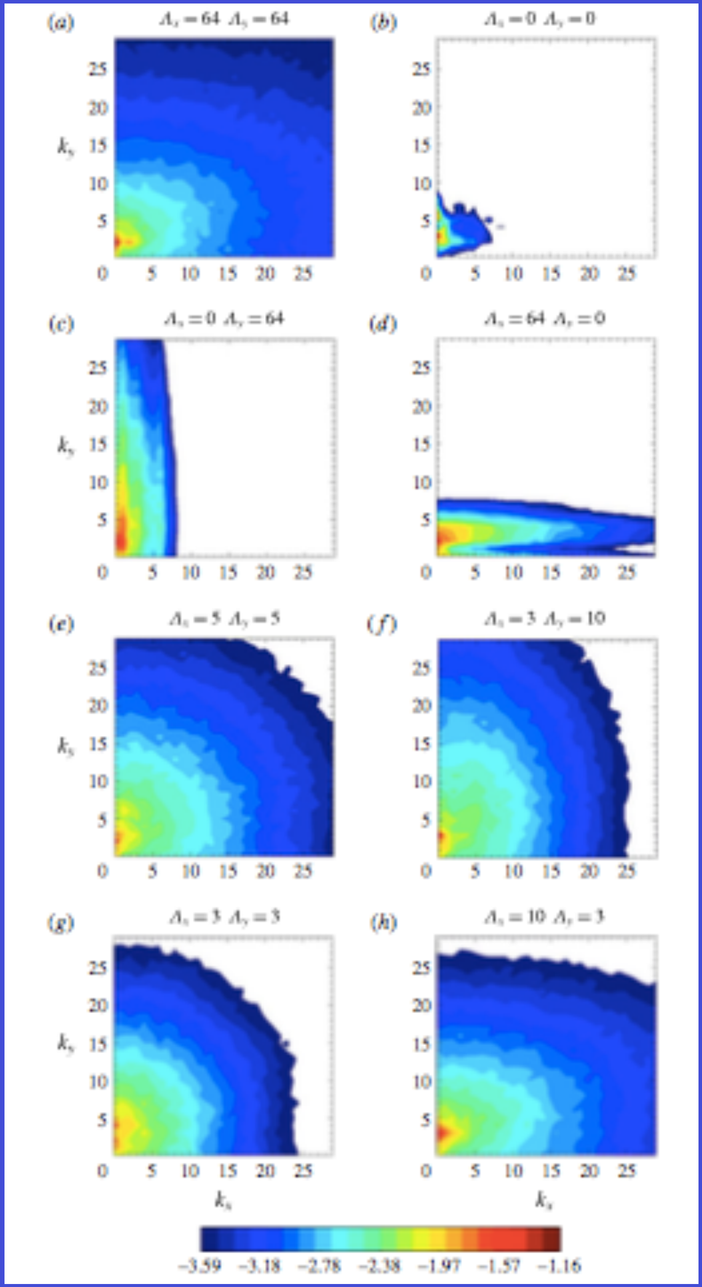
Mean Flow





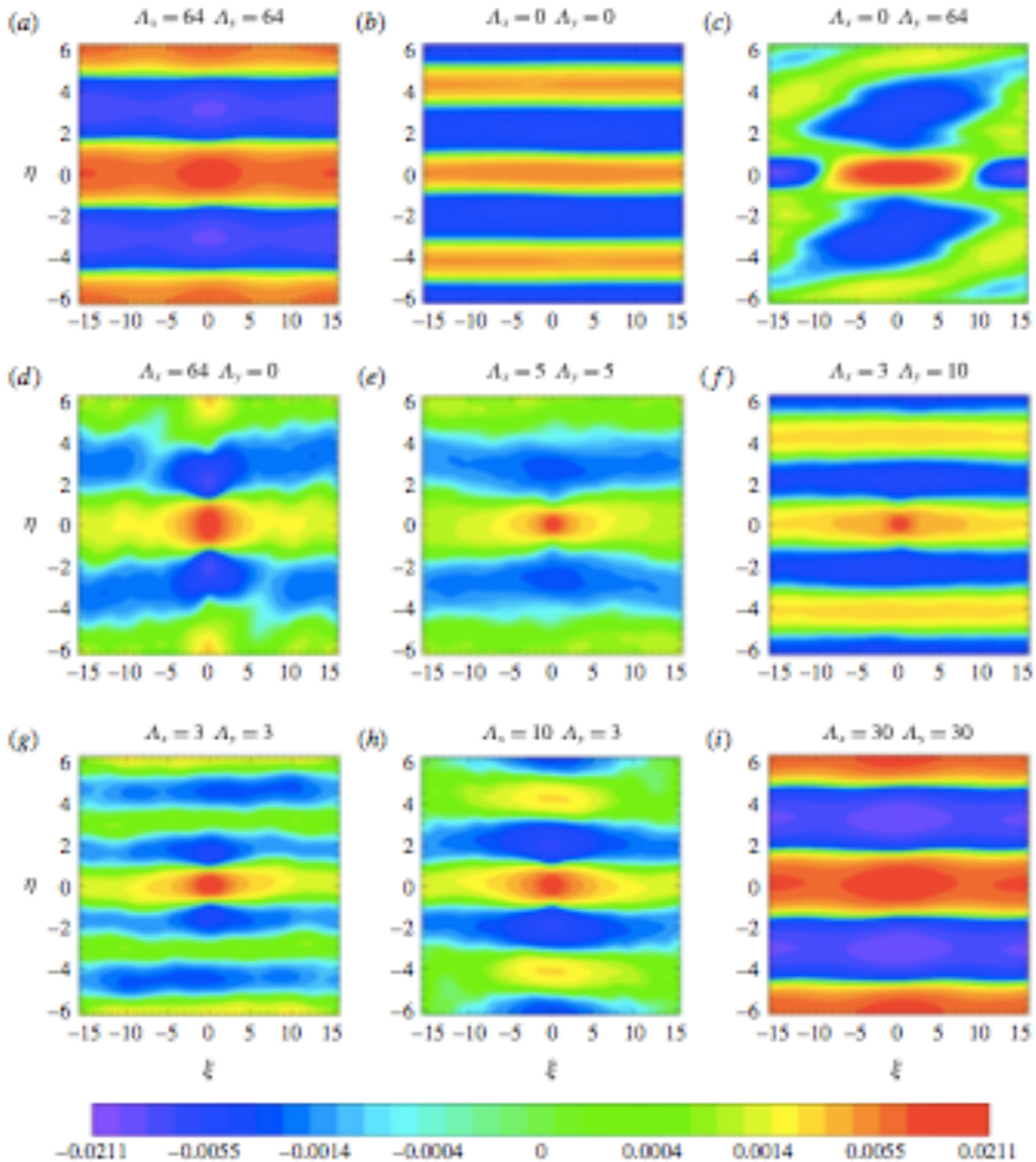






Mean flows, dissipation rate, and spectra can be reproduced even if the cut-off is O(5) in each direction.

Note the QL approximation performs more poorly. even if quasilinear in one direction and fully nonlinear in the other (cf Thomas et al 2014, 2015)



Second cumulant can be reproduced even if The cut-off is O(5) in each Direction!

GQL can lead to a closure for DSS (GCE2)

$$\frac{\partial}{\partial t} q = L[q] + Q[q,\ q]$$

$$q = \ell + h$$

$$\frac{\partial}{\partial t} \ell = Q[\ell,\ \ell] + Q[(h,\ h)]$$

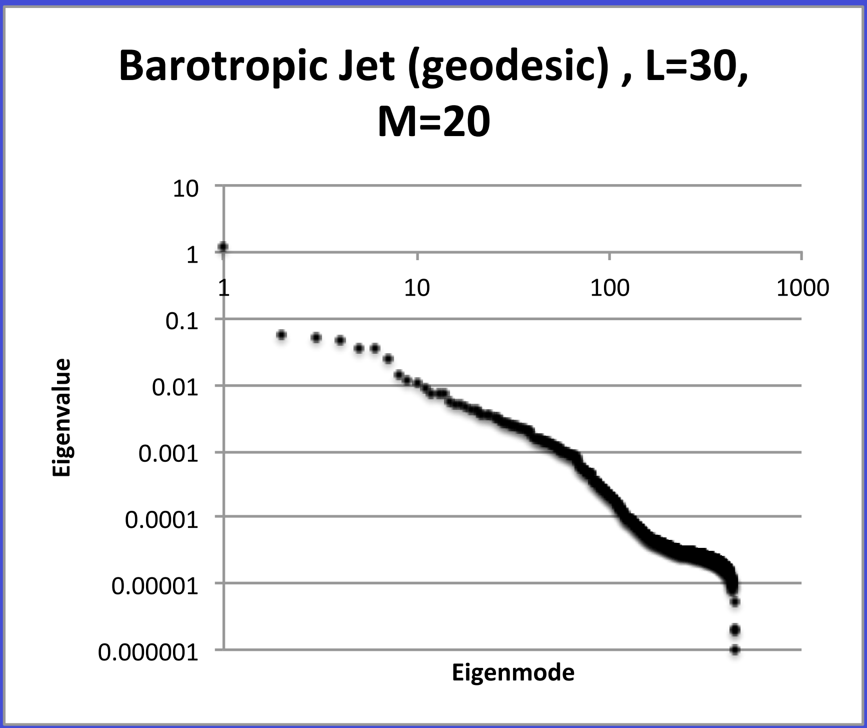
$$\frac{\partial}{\partial t} h = Q[\ell,\ h]$$

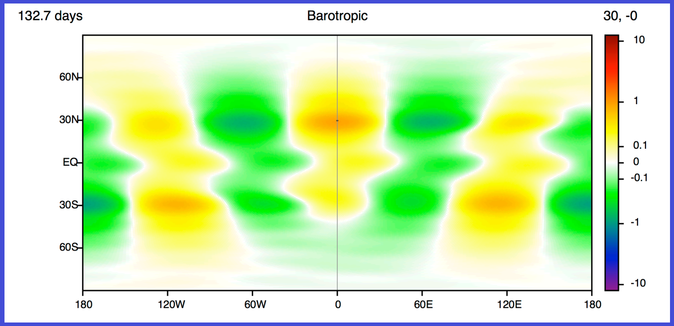
$$\frac{\partial}{\partial t} (h\ h) = 2Q[\ell,\ (h]\ h)$$

Closure

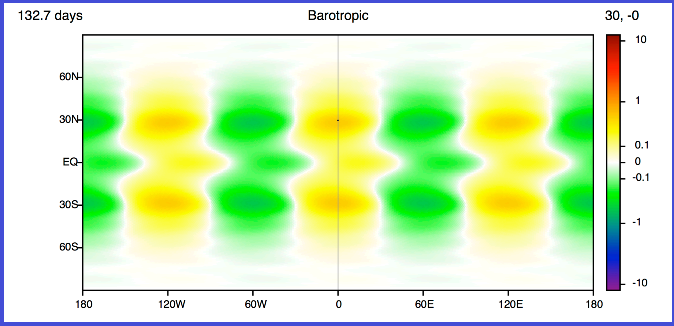
Model Reduction via POD

- POD usually used as a tool to analyse modes of greatest importance in flows (eg atmospheric flows, experiments)

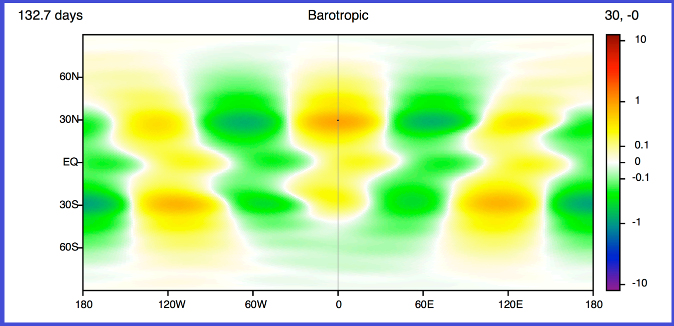




600



1



3

But can be used as a *tool for calculation* to project *statistics of DSS* onto the most important modes: massive speed up

Allawala et al 2017

Approximation

Statistical Theory

QUASILINEAR DNS
(QL-DNS)

CE2 (SSST)
eCE2

GENERALISED
QUASILINEAR DNS
(GQL-DNS)

GCE2

FULL DNS

GCE2
(cutoff → large)

Some eddy/eddy→ eddy interactions

CE2.5

Inhom/
anisotropic
EDQNM

CE3

Marston, Qi & Tobias (2017), arXiv:1412.0381 (2014)

Conclusions

- The Quasilinear Approximation can be valid in cases where the projection for the third cumulant onto the second cumulant is small (low Kubo number)
- A better approximation is GQL, which generalises the definition of "mean" to large-scales
  - Allows for scattering of energy from eddies at one scale to another via the large scales (non-local transfer)
  - Conserves quadratic invariants
  - Can form the basis of a statistical closure (GCE2)
  - GQL is better at reproducing the low-order statistics of flows than QL.
    - Driving of mean flows
    - Magnetised Taylor Couette
    - Rotating Couette
  - Can be generalised to timescales, rather than spatial scales.
  - Could form the basis of a subgrid statistical model
- Statistical models (DSS) can be made more efficient using POD

Some References

Tobias et al (2011) ApJ, 727, 127
Tobias & Marston (2013) PRL 110, 104502.
Marston et al (2015) arXiv:1412.0381 (review)
Marston et al (2016) PRL, 7116, 214501
Child et al (2016) Journal of Plasma Physics, 82, 905820302
Tobias & Marston (2016) JFM
Allawala et al (2017)

Direct Statistical Simulation via Cumulants

Take PDE or set of PDEs

e.g. Momentum, induction, energy equations

$$\partial_t q = \mathcal{L}[q] + \mathcal{N}[q\ q] + f(t)$$

Define Cumulants

1st cumulants: means

2nd cumulants: two-point correlation functions

cross correlations

Split into means and fluctuations
Reynolds averaging

$$q = \langle q \rangle + q'$$

$$c_1 = \langle q\ (\mathbf{x}) \rangle, \quad c_2 = \langle q'(\mathbf{x_1}) q'(\mathbf{x_2}) \rangle$$

$$c_3 = \langle q'(\mathbf{x_1}) q'(\mathbf{x_2}) q'(\mathbf{x_3}) \rangle$$

Derive Evolution Eqns for Cumulants

Use Hopf functional technique or brute force

$$\dot{c}_1 = \mathcal{L}[c_1] + \mathcal{N}[c_1 c_1 + c_2]$$

$$\dot{c}_2 = \mathcal{L}[c_2] + \mathcal{N}[c_1 c_2 + c_3] + \Gamma$$

$$\dot{c}_3 = \mathcal{L}[c_3] + \mathcal{N}[c_1 c_3 + c_2 c_2 + c_4] \quad \ldots$$

Truncate cumulant hierarchy

Conservation Laws
Realizability

Truncation at second order is a Quasilinear, self-consistent mean-field theory (CE2)

Formally analogous to Farrell & Ioannou S3T

Truncation at third order is an anisotropic, inhomogeneous EDQNM (CE3)