The range of scale coupling in turbulent cascades

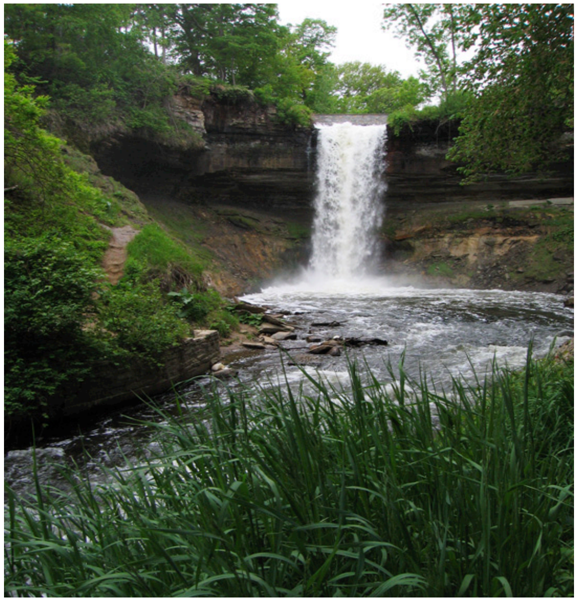
Hussein Aluie
Los Alamos National Lab

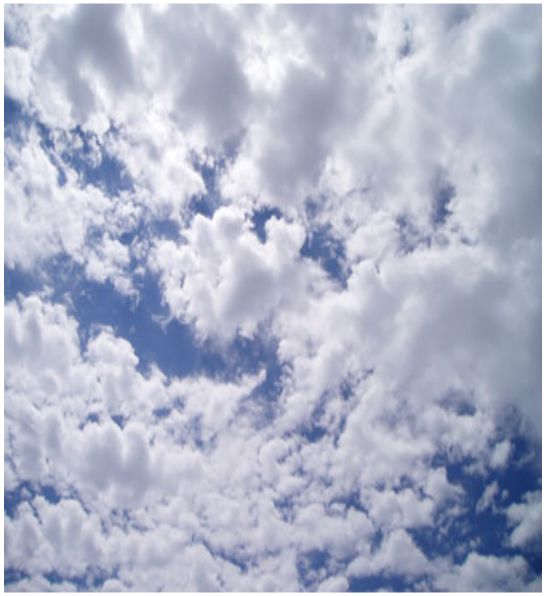
Gregory L. Eyink
Johns Hopkins University

Aluie H., Eyink, G. L., *Phys. Rev. Lett.* (2010).
Eyink, G. L., Aluie H., *Phys. Fluids* (2009).
Aluie H., Eyink, G. L., *Phys. Fluids* (2009).

Hydrodynamics

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p \quad + \nu \nabla^2 \mathbf{u} \qquad\qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

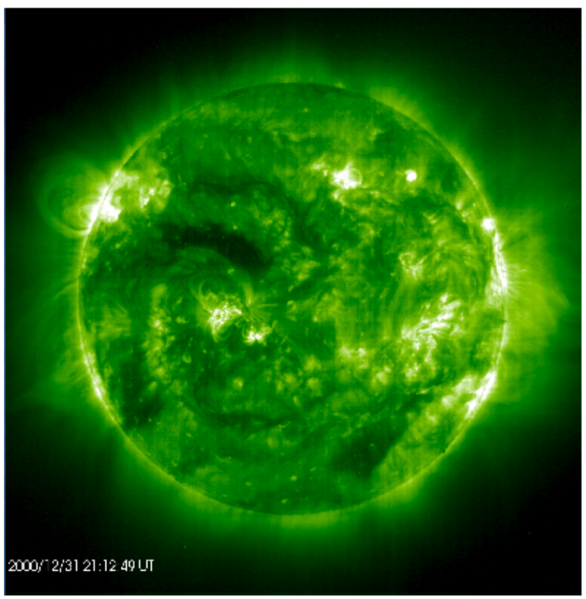


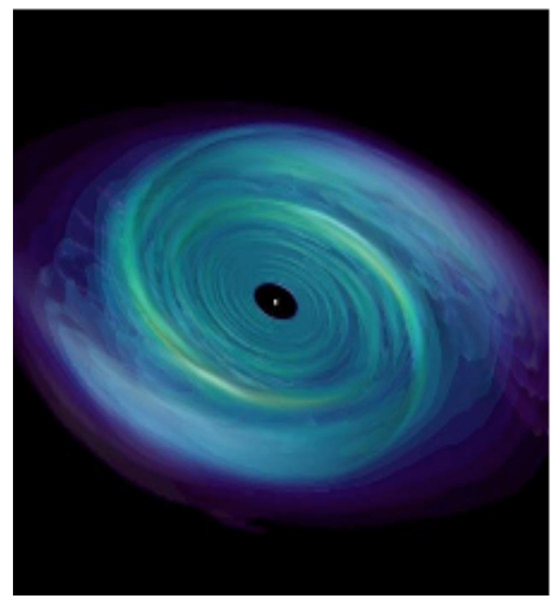


Magnetohydrodynamics (MHD)

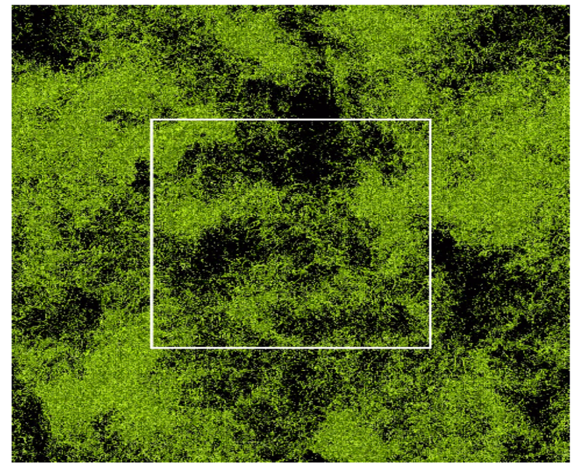
$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{u} \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$



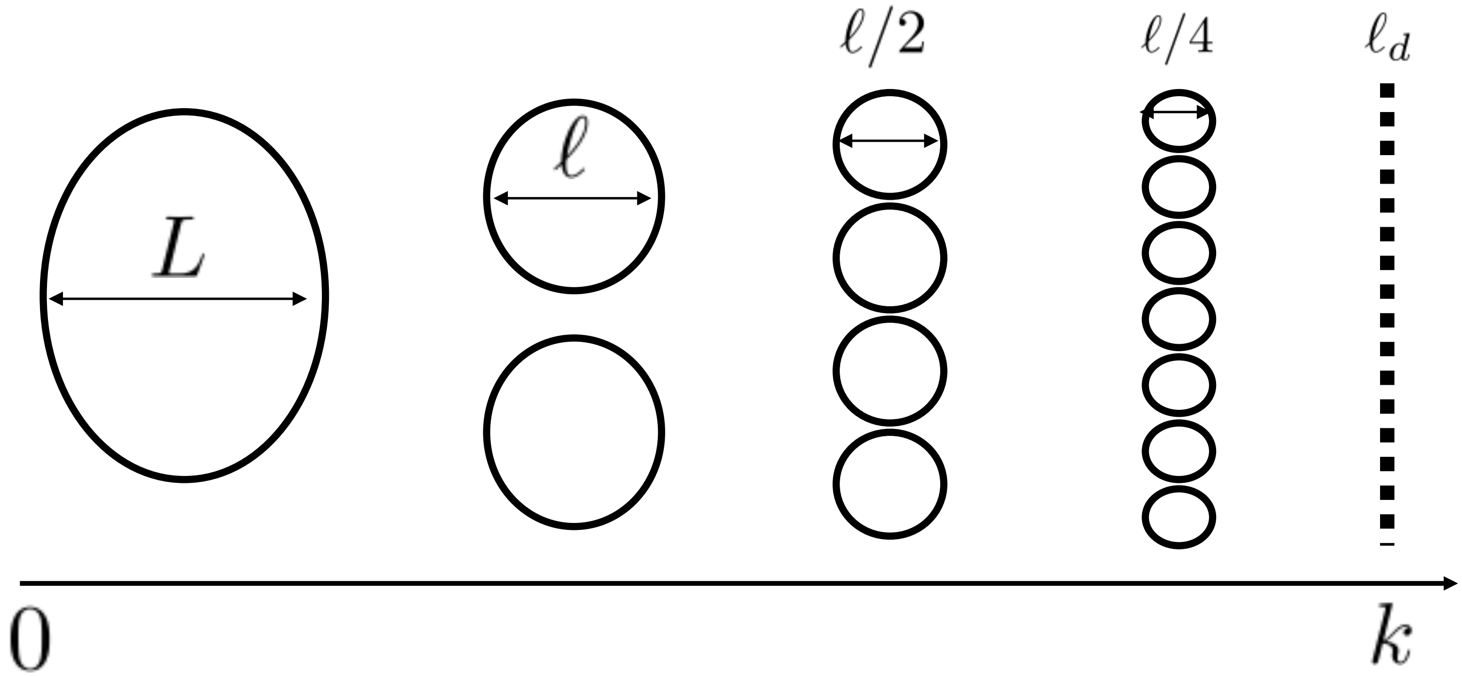


Multi-scale interactions & Turbulence



Hierarchy of scales: $L \gg \ell \gg \ell_d$

Fourier wavenumber: $k = 1/\ell$

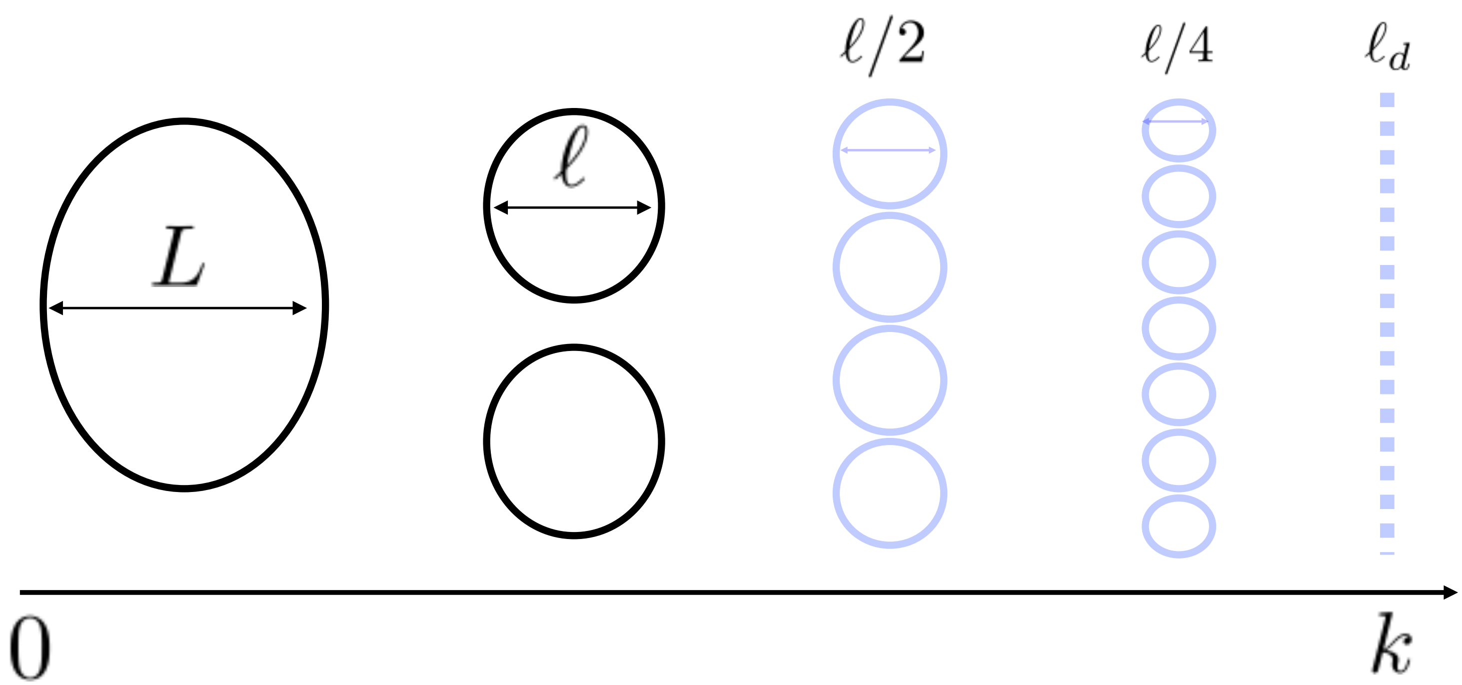


Our Approach

Coarse-graining (Smoothing)

$$\overline{\mathbf{u}}_\ell(\mathbf{x}) = \int d\mathbf{r}\ G_\ell(\mathbf{r})\mathbf{u}(\mathbf{x}+\mathbf{r})$$

$$\overline{\mathbf{B}}_\ell(\mathbf{x}) = \int d\mathbf{r}\ G_\ell(\mathbf{r})\mathbf{B}(\mathbf{x}+\mathbf{r})$$

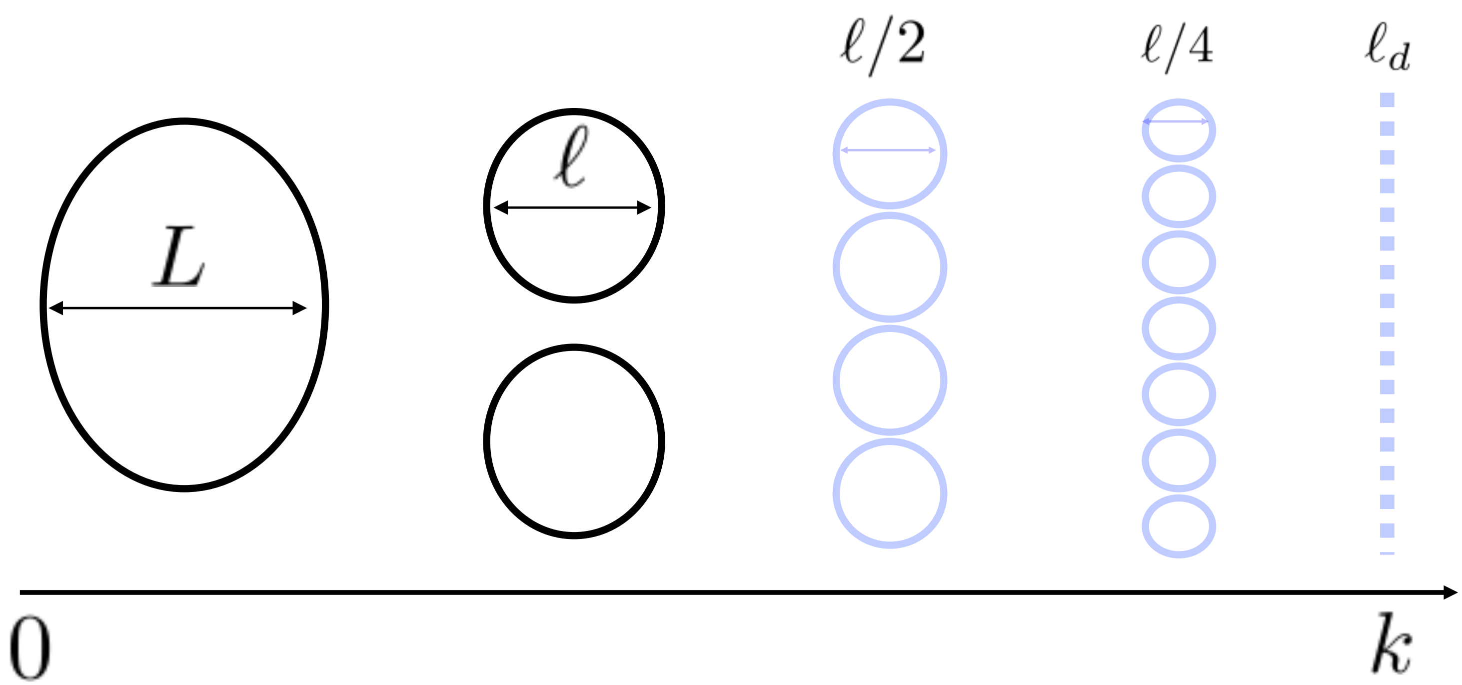


Our Approach

Coarse-grained dynamics

$$\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \boldsymbol{\nabla})\overline{\mathbf{u}} = -\boldsymbol{\nabla}\overline{p} + \overline{\mathbf{J}} \times \overline{\mathbf{B}} - \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell + \nu \nabla^2 \overline{\mathbf{u}} \qquad \boldsymbol{\nabla} \cdot \overline{\mathbf{u}} = 0$$

$$\partial_t \overline{\mathbf{B}} = \boldsymbol{\nabla} \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \boldsymbol{\nabla} \times \boldsymbol{\varepsilon}_\ell + \eta \nabla^2 \overline{\mathbf{B}} \qquad \boldsymbol{\nabla} \cdot \overline{\mathbf{B}} = 0$$



Our Approach

Coarse-grained dynamics

$$\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \boldsymbol{\nabla})\overline{\mathbf{u}} = -\boldsymbol{\nabla}\overline{p} + \overline{\mathbf{J}} \times \overline{\mathbf{B}} - \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell + \nu \nabla^2 \overline{\mathbf{u}} \qquad \boldsymbol{\nabla} \cdot \overline{\mathbf{u}} = 0$$

$$\partial_t \overline{\mathbf{B}} = \boldsymbol{\nabla} \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \boldsymbol{\nabla} \times \boldsymbol{\varepsilon}_\ell + \eta \nabla^2 \overline{\mathbf{B}} \qquad \boldsymbol{\nabla} \cdot \overline{\mathbf{B}} = 0$$

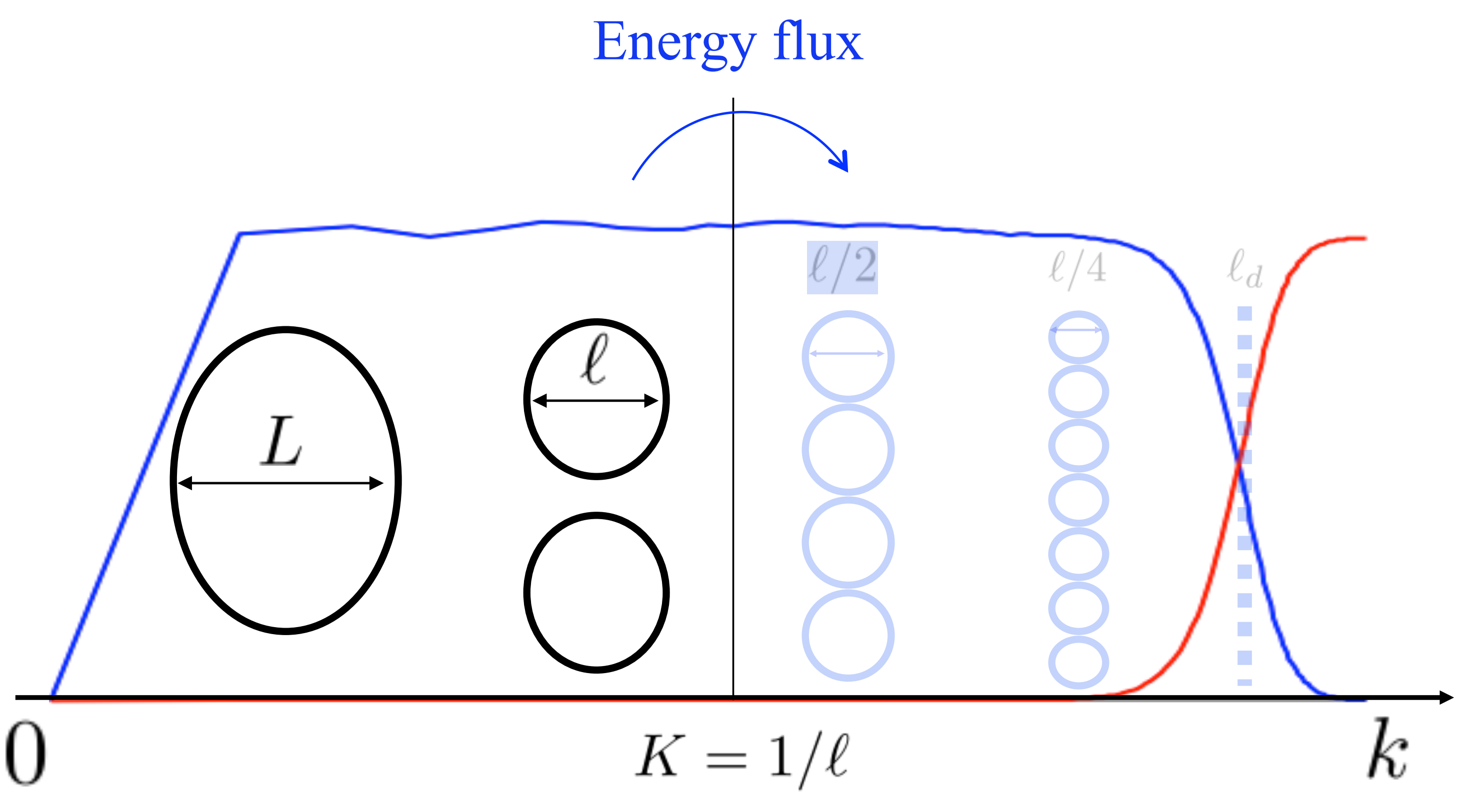
The stress: $\boldsymbol{\tau}_\ell = [\overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\ \overline{\mathbf{u}}] - [\overline{\mathbf{B}\mathbf{B}} - \overline{\mathbf{B}}\ \overline{\mathbf{B}}]$

The EMF: $\boldsymbol{\varepsilon}_\ell = \overline{\mathbf{u} \times \mathbf{B}} - \overline{\mathbf{u}} \times \overline{\mathbf{B}}$

Cascade of Energy

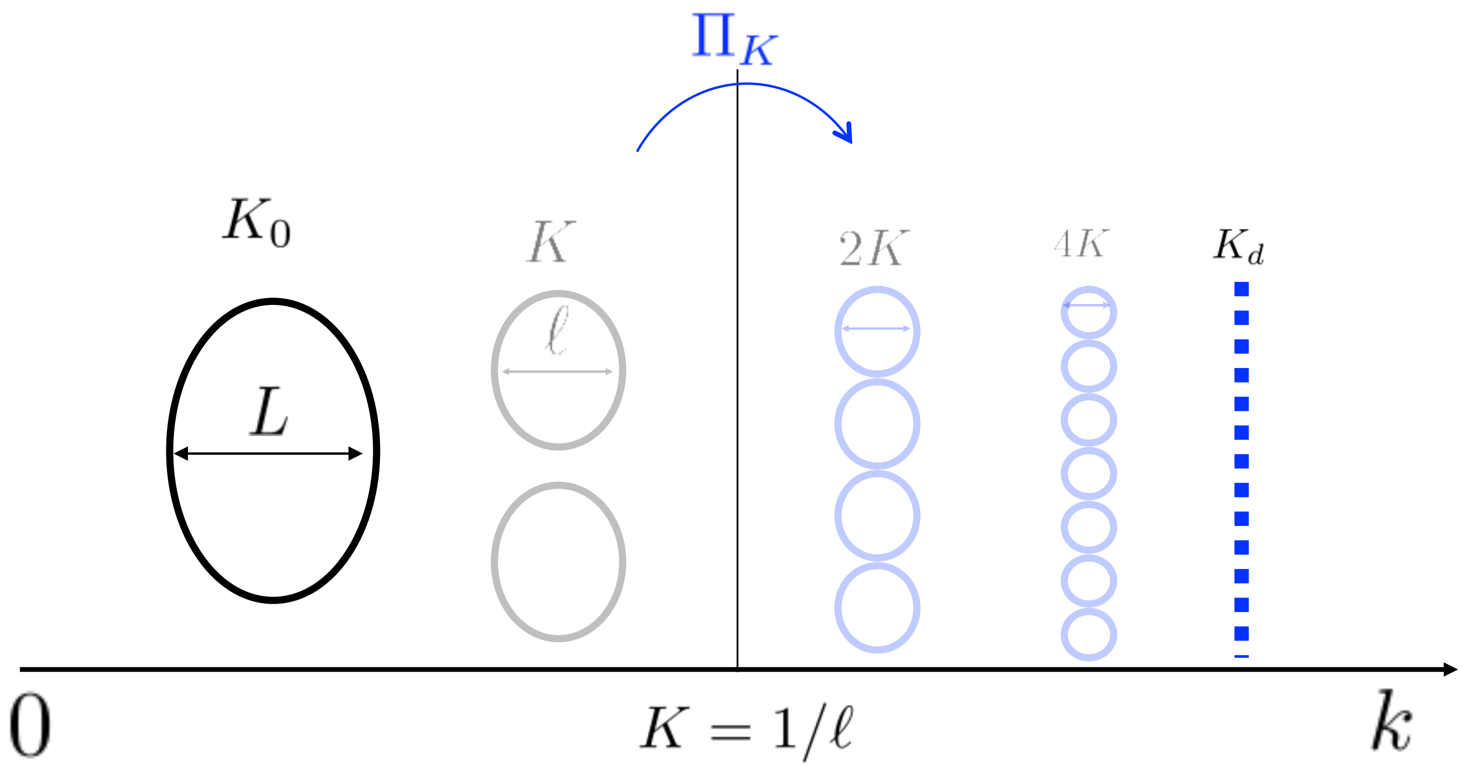
Large-scale energy budget

$$\partial_t\left(\frac{|\overline{\mathbf{u}}|^2}{2} + \frac{|\overline{\mathbf{B}}|^2}{2}\right) + \nabla \cdot [\cdots] = -\Pi_\ell^E \; -\nu|\nabla\overline{\mathrm{u}}|^2 - \eta|\nabla\overline{\mathbf{B}}|^2$$



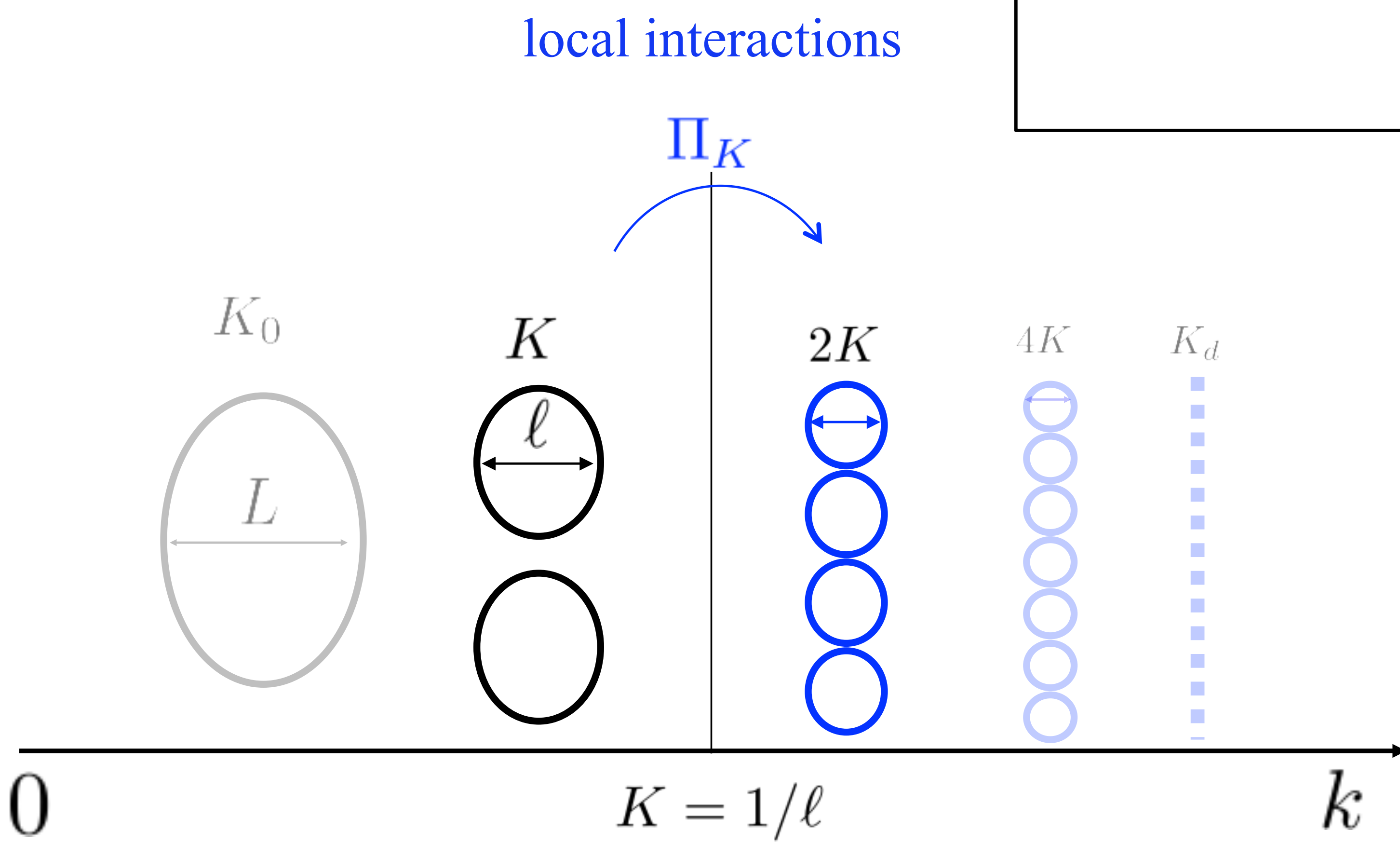
Scale-Locality

Non-local interactions



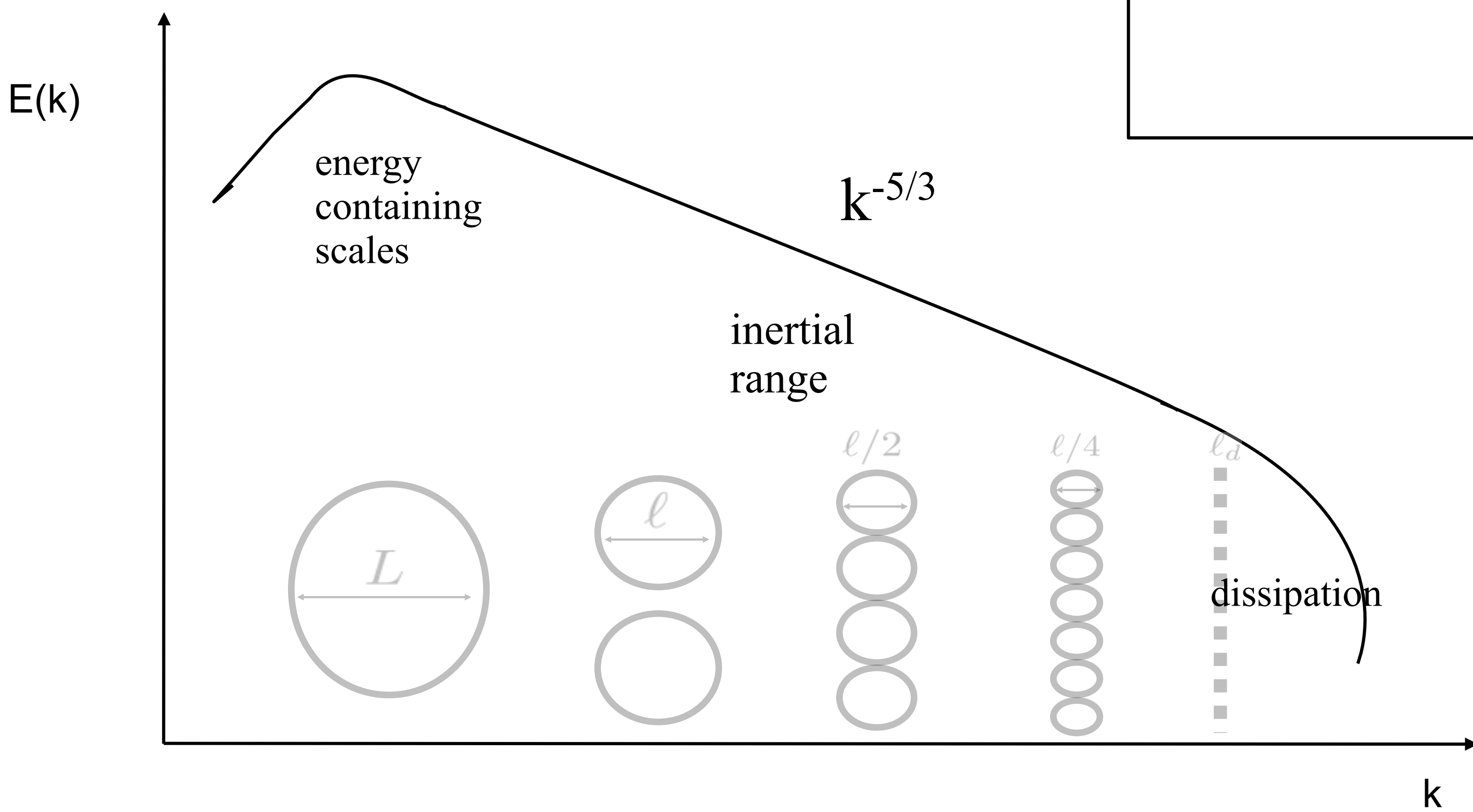
Scale-Locality

- Richardson 1922
- Kolmogorov 1941
- Onsager 1945



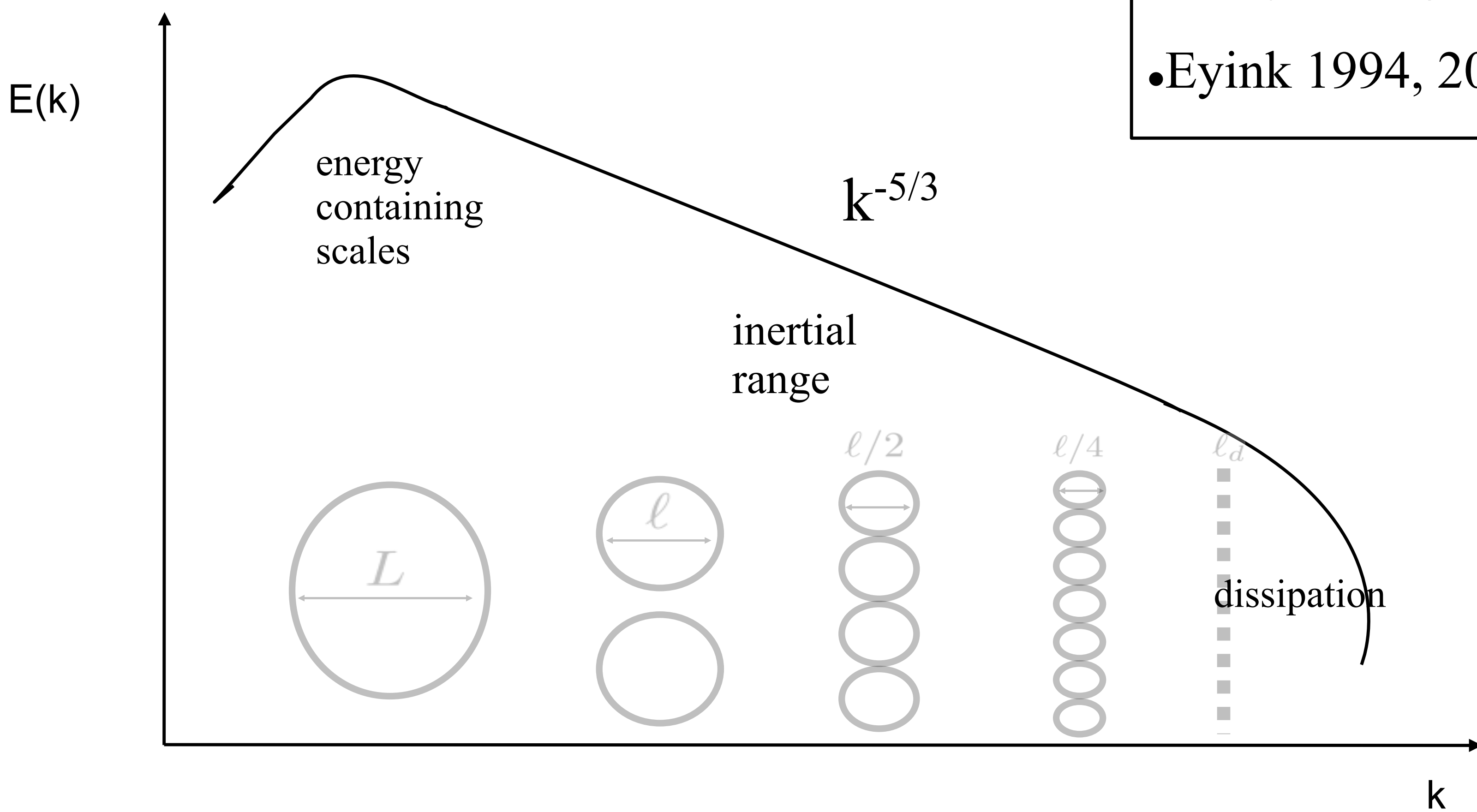
Scale-Locality

- Richardson 1922
- Kolmogorov 1941
- Onsager 1945



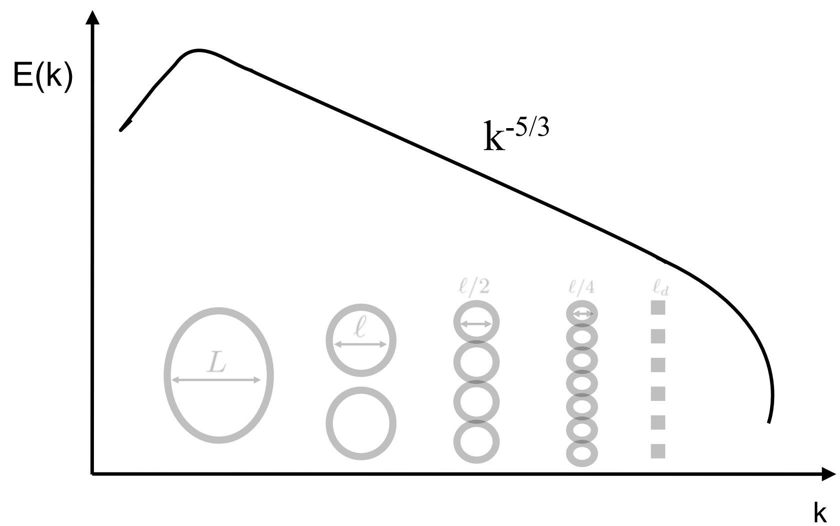
Scale-Locality

- Richardson 1922
- Kolmogorov 1941
- Onsager 1945
- Kraichnan 1959
- Eyink 1994, 2005

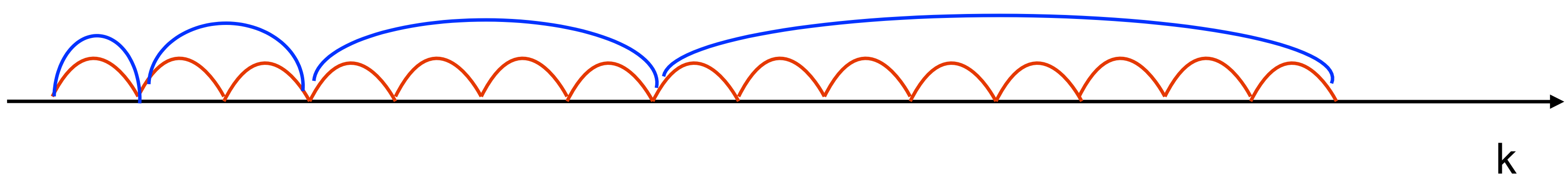


Doubts cast on locality

- Brasseur & Corrsin (1987)
- Domaradzki & Rogallo (1990)
- Yeung and Brasseur (1991)
- Ohkitani and Kida (1992)
- Zhou, Yeung, and Brasseur (1996)
- Alexakis, Mininni, and Pouquet (2005)
- Mininni, Alexakis, and Pouquet (2006)
- Mininni, Alexakis, and Pouquet (2008)



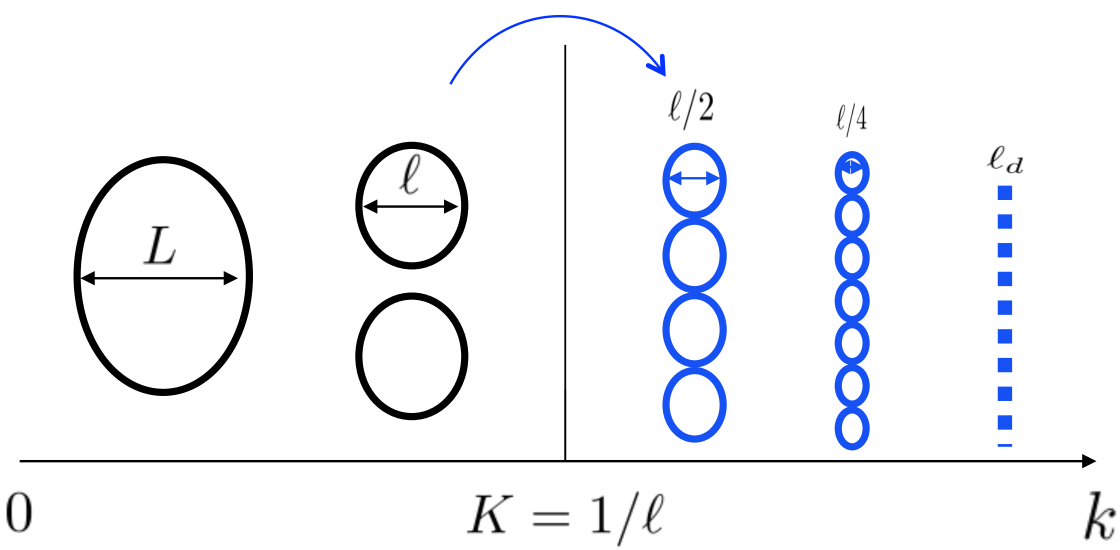
energy cascade: fixed steps vs. multiplicative steps



Nonlinear Interactions

$$\Pi_K(u_i, u_j, u_i) \equiv -\partial_j u_i{}^{<K}[(u_j u_i)^{<K} - u_j{}^{<K} u_i{}^{<K}]$$

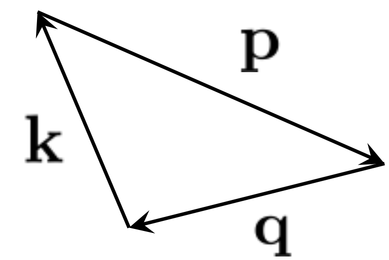
$$\mathbf{u}^{<K}(\mathbf{x}) \equiv \sum_{|\mathbf{k}|<K} \widehat{\mathbf{u}}(\mathbf{k})\ e^{i\mathbf{k}\cdot\mathbf{x}}$$

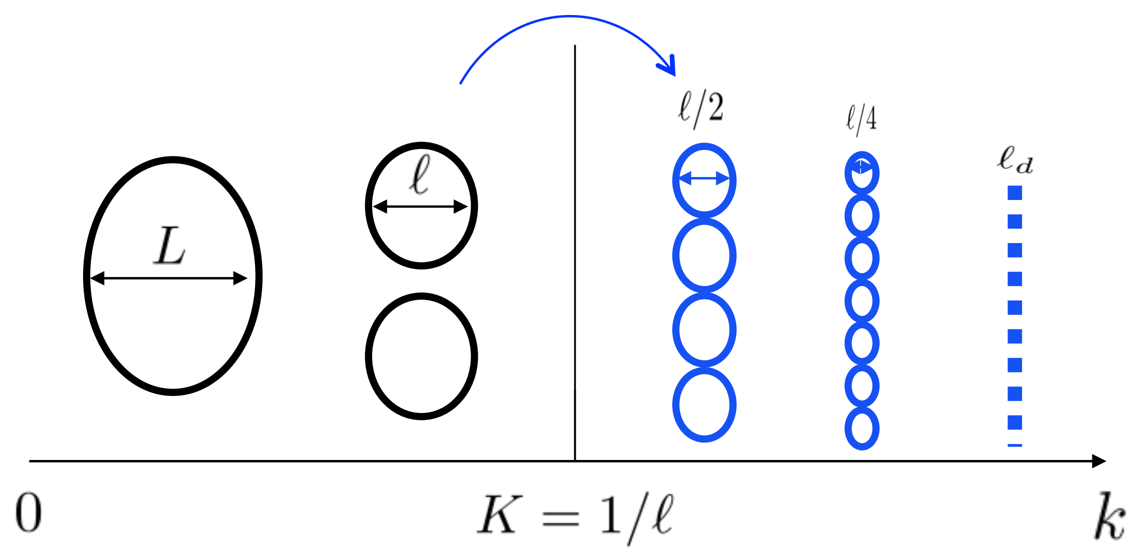


Nonlinear Interactions

$$\Pi_K(u_i, u_j, u_i) \equiv -\partial_j u_i{}^{<K}[(u_j u_i)^{<K} - u_j{}^{<K} u_i{}^{<K}]$$

cubic quantity ⟶ 3 Fourier modes interacting $k_j \hat{u}_i(\mathbf{k})\hat{u}_j(\mathbf{p})\hat{u}_i(\mathbf{q})$

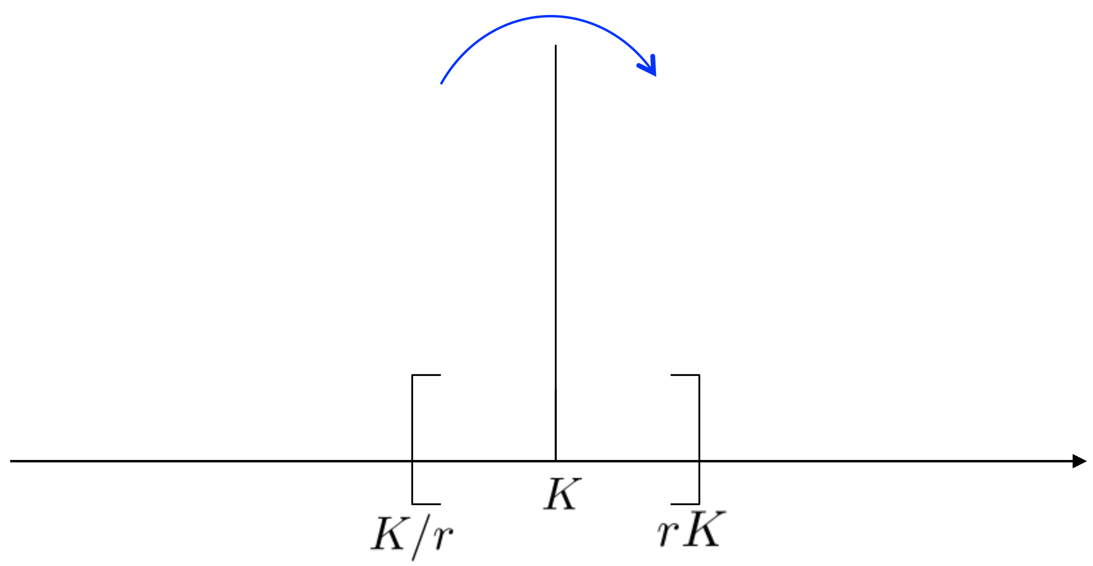




(Non) Locality of Interactions

$$\Pi_K(u_i, u_j, u_i) \equiv -\partial_j u_i{}^{<K}[(u_j u_i)^{<K} - u_j{}^{<K} u_i{}^{<K}]$$

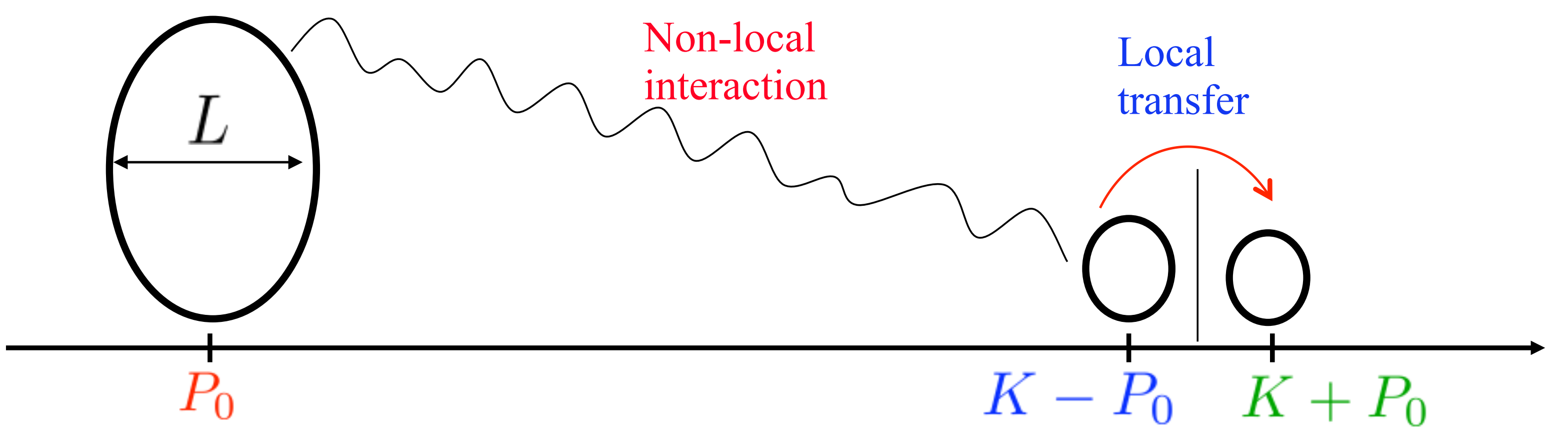
$$\Pi_K(u_i, u_j, u_i) \approx \Pi_K(u_i{}^{[\frac{K}{r}, rK]}, u_j{}^{[\frac{K}{r}, rK]}, u_i{}^{[\frac{K}{r}, rK]})$$



(Non) Locality of Interactions

$$\Pi_K(u_i, u_j, u_i) \equiv -\partial_j u_i{}^{<K}[(u_j u_i)^{<K} - u_j{}^{<K} u_i{}^{<K}]$$

$$\Pi_K(u_i, u_j, u_i) \approx \Pi_K(u_i{}^{[\frac{K}{r}, rK]}, u_j{}^{[\frac{K}{r}, rK]}, u_i{}^{[\frac{K}{r}, rK]})$$



(Non) Locality of Interactions

$$\Pi_K(u_i, {u_j}^{[P]}, u_i) = -\partial_j {u_i}^{<K}[({u_j}^{[P]} u_i)^{<K} - {u_j}^{[P],<K} {u_i}^{<K}]$$

$$\mathbf{u}^{[P]} = \sum_{\frac{P}{2}<|\mathbf{k}|<P} \hat{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

Locality of Interactions

$$\Pi_K(u_i, {u_j}^{[P]}, u_i) = -\partial_j {u_i}^{<K}[({u_j}^{[P]} u_i)^{<K} - {u_j}^{[P],<K} {u_i}^{<K}]$$

for $P < \frac{K}{2}$, $$({u_j}^{[P]} u_i)^{<K} - {u_j}^{[P]} {u_i}^{<K}$$

$$= ({u_j}^{[P]} u_i^{<K+P})^{<K} - {u_j}^{[P]} {u_i}^{<K}$$

$$= ({u_j}^{[P]}(u_i^{<K-P} + u_i^{[K-P,K+P]}))^{<K} - {u_j}^{[P]}(u_i^{<K-P} + u_i^{[K-P,K]})$$

$$= ({u_j}^{[P]} u_i^{[K-P,K+P]})^{<K} - {u_j}^{[P]} u_i^{[K-P,K]}$$

Locality of Interactions

$$\Pi_K(u_i, {u_j}^{[P]}, u_i) = -\partial_j {u_i}^{<K} [({u_j}^{[P]} u_i)^{<K} - {u_j}^{[P],<K} {u_i}^{<K}]$$

for $P < \frac{K}{2}$, $\quad = -\partial_j {u_i}^{<K} [({u_j}^{[P]} u_i^{[K-P,K+P]})^{<K} - {u_j}^{[P]} u_i^{[K-P,K]}]$

Advecting modes get stronger for smaller P

Thinner Fourier bands get weaker

Rigorous result for $P \ll K \qquad \leq (\text{const.}) P^{-\sigma_4} P^{1/2}$

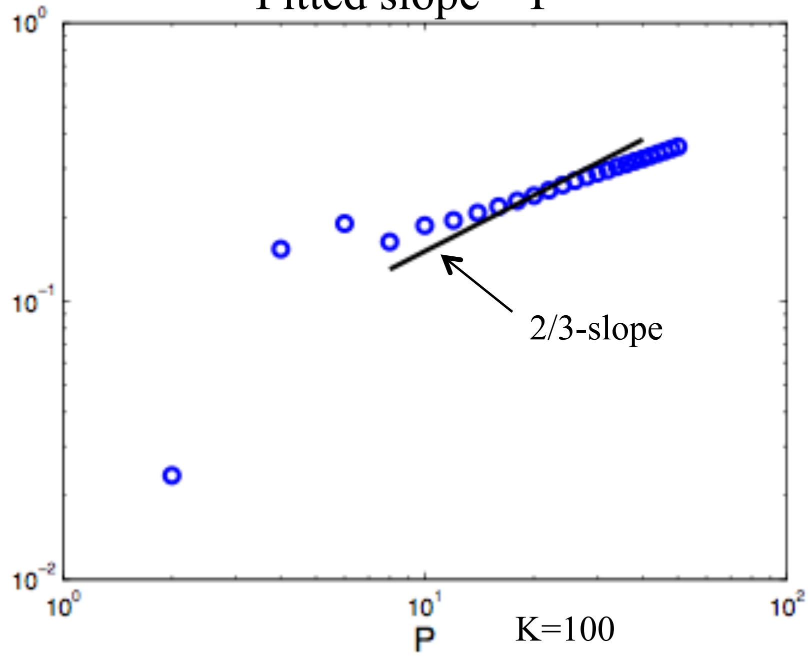
$\sigma_4 \doteq 1/3 \qquad \sim P^{1/6}$

where $\langle |\delta \mathbf{u}(\ell)|^s \rangle^{1/s} \sim \ell^{\sigma_s}$

Locality of Interactions

$$\Pi_K(u_i, u_j{}^{[P]}, u_i) = -\partial_j u_i{}^{<K}[(u_j{}^{[P]}u_i)^{<K} - u_j{}^{[P],<K}u_i{}^{<K}]$$

Fitted slope ~ $P^{0.45}$



K=100

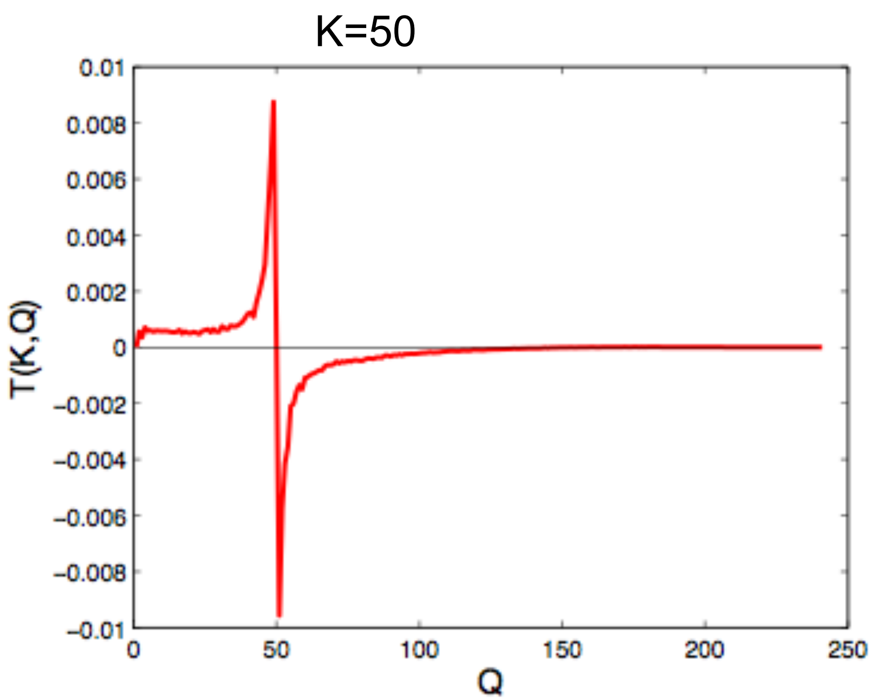
Rigorous result for $P \ll K$ $\leq (\text{const.})P^{-\sigma_4}P^{1/2}$

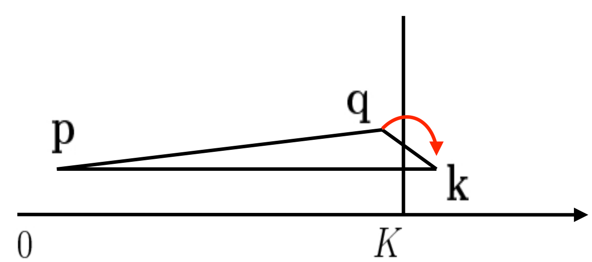
$\sigma_4 \doteq 1/3$ $\sim P^{1/6}$ where $\langle|\delta\mathbf{u}(\ell)|^s\rangle^{1/s} \sim \ell^{\sigma_s}$

Comparing with Previous Work

$$T(K,Q) = \langle \partial_j u_i^{[K-1,K]} \underbrace{u_j}\, u_i^{[Q-1,Q]} \rangle$$

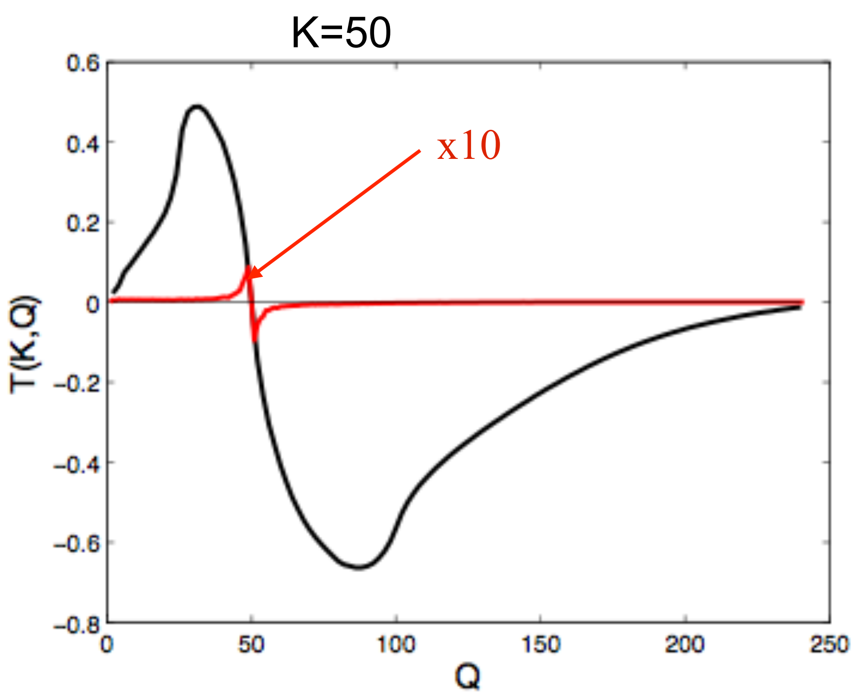
Advecting modes get
stronger for smaller P

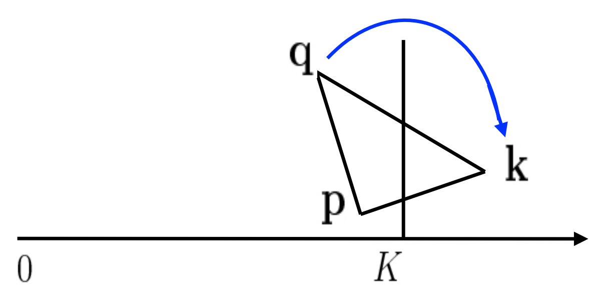




Comparing with Previous Work

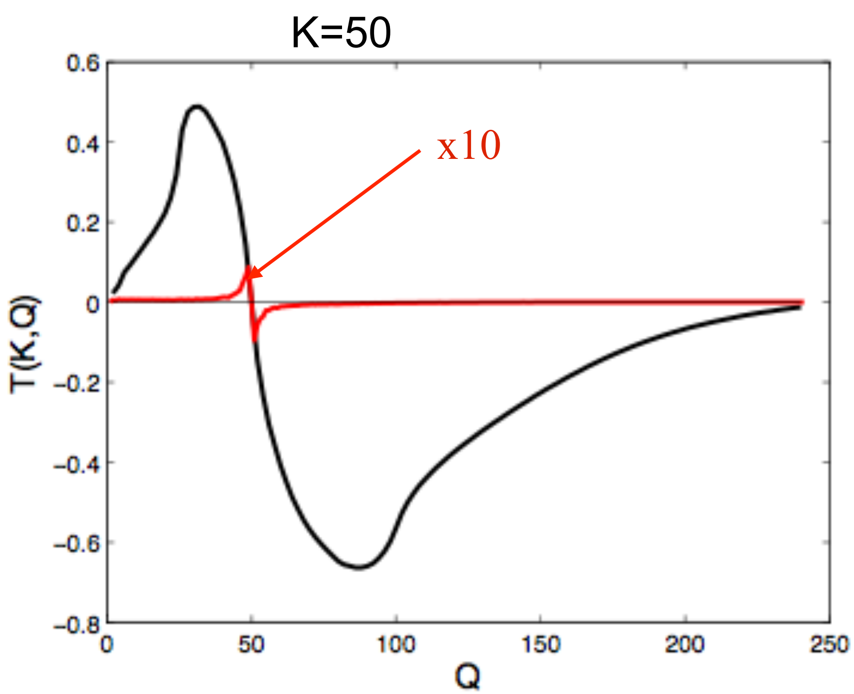
$$T(K,Q) = \langle \partial_j u_i^{[K/2,K]} u_j \, u_i^{[Q/2,Q]} \rangle$$

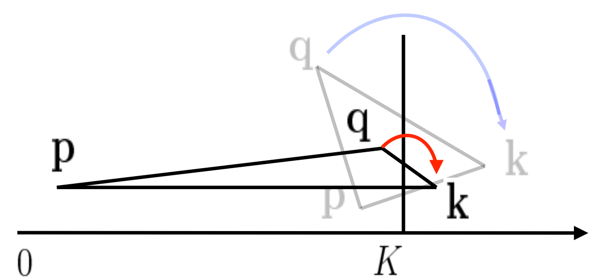




Comparing with Previous Work

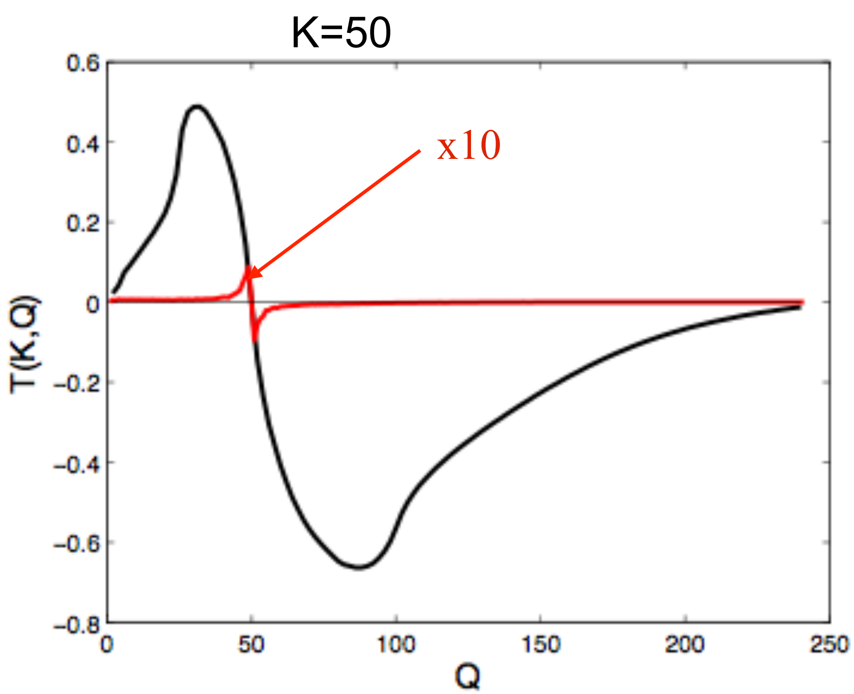
$$T(K,Q) = \langle \partial_j u_i^{[K/2,K]} u_j \, u_i^{[Q/2,Q]} \rangle$$

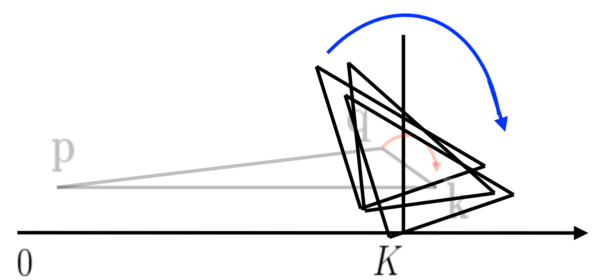




Comparing with Previous Work

$$T(K,Q) = \langle \partial_j u_i^{[K/2,K]} u_j \, u_i^{[Q/2,Q]} \rangle$$





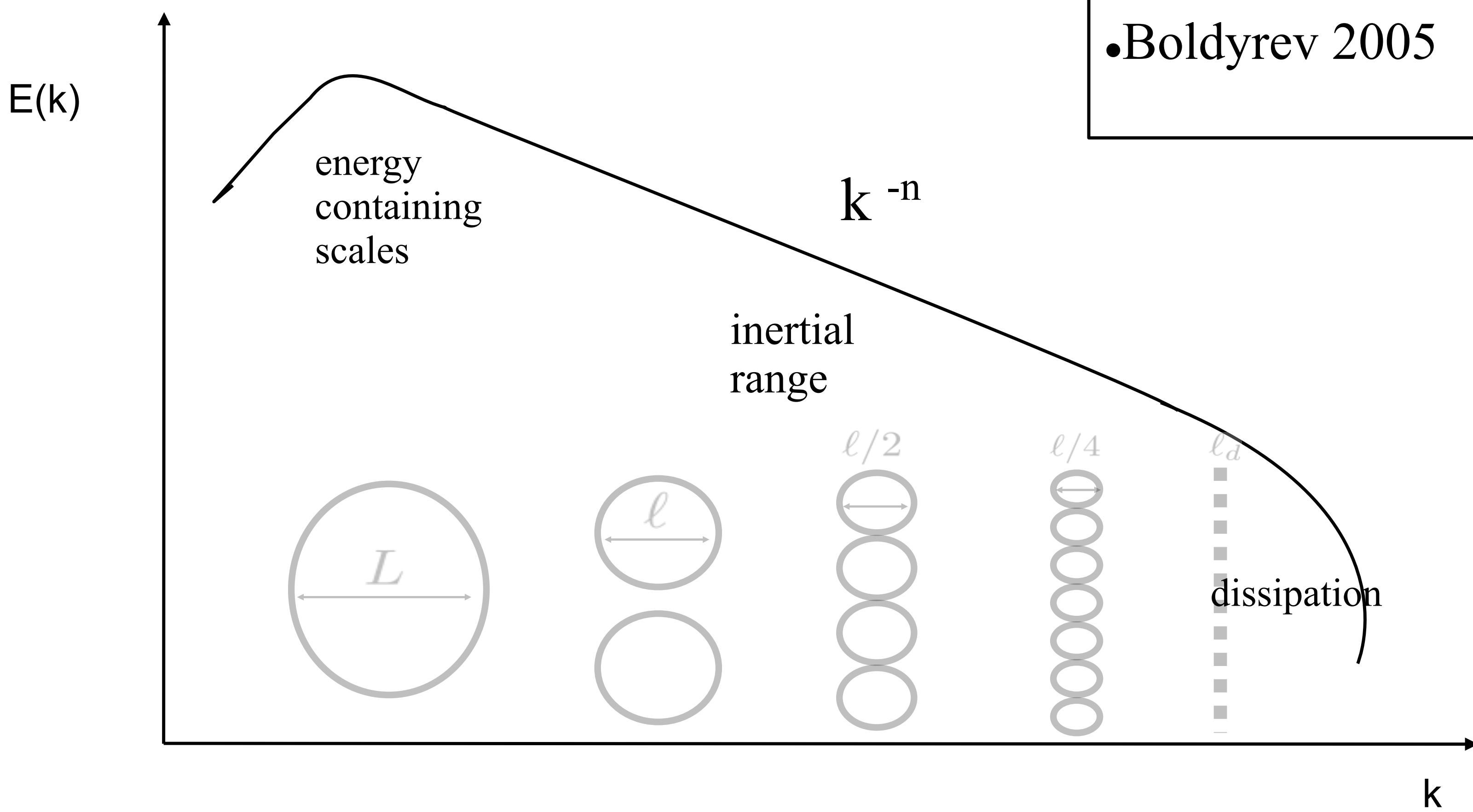
Conclusion

Proved locality of the energy cascade in Fourier space

1. The SGS definition of the flux, with sweeping effects subtracted, is the proper measure of the cascading energy.
2. Sharp spectral filter has a firm theoretical basis for use in LES modeling.
3. It is inadequate to associate <u>single</u> Fourier modes with length-scales in turbulent flows.

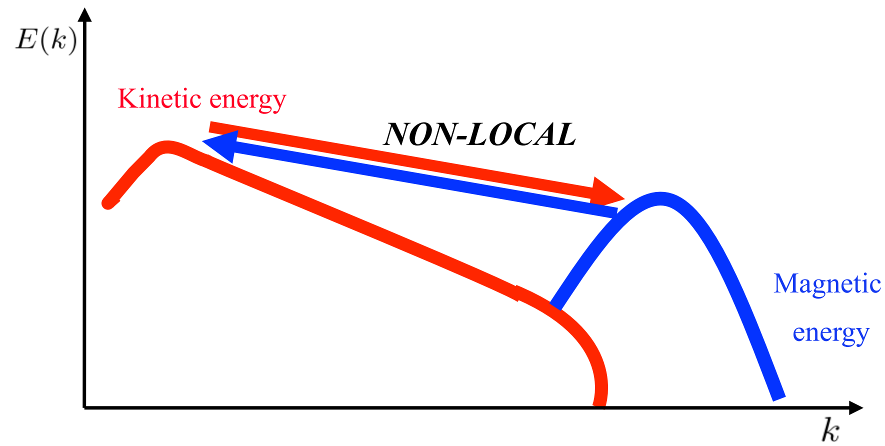
Scale-Locality: MHD Turbulence

- Iroshnikov1964
- Kraichnan 1965
- Goldreich-Sridhar 1995
- Boldyrev 2005

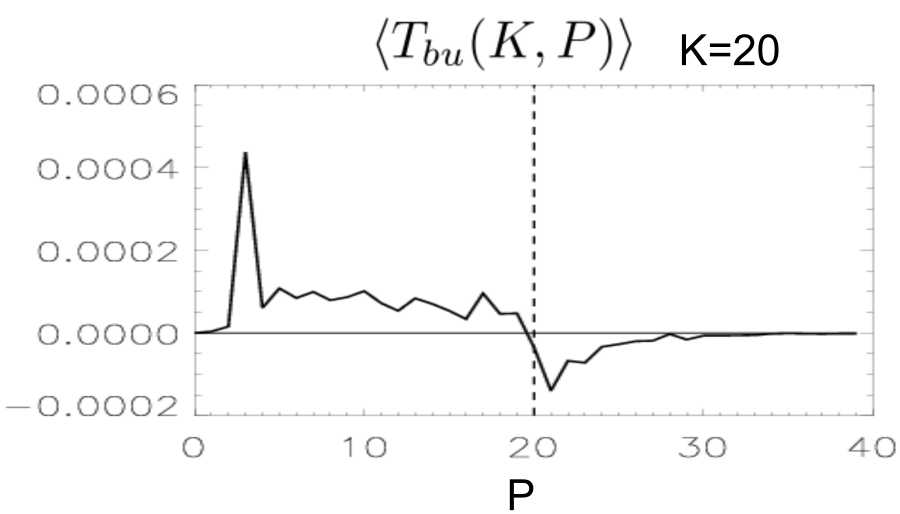


Claims of non-local transfer

- Yousef, Rincon, and Schekochihin 2007
- Schekochihin & Cowley 2008
- Alexakis, Mininni, and Pouquet 2005
- Carati, et al. 2006



Adapted from
Schekochihin & Cowley 2008

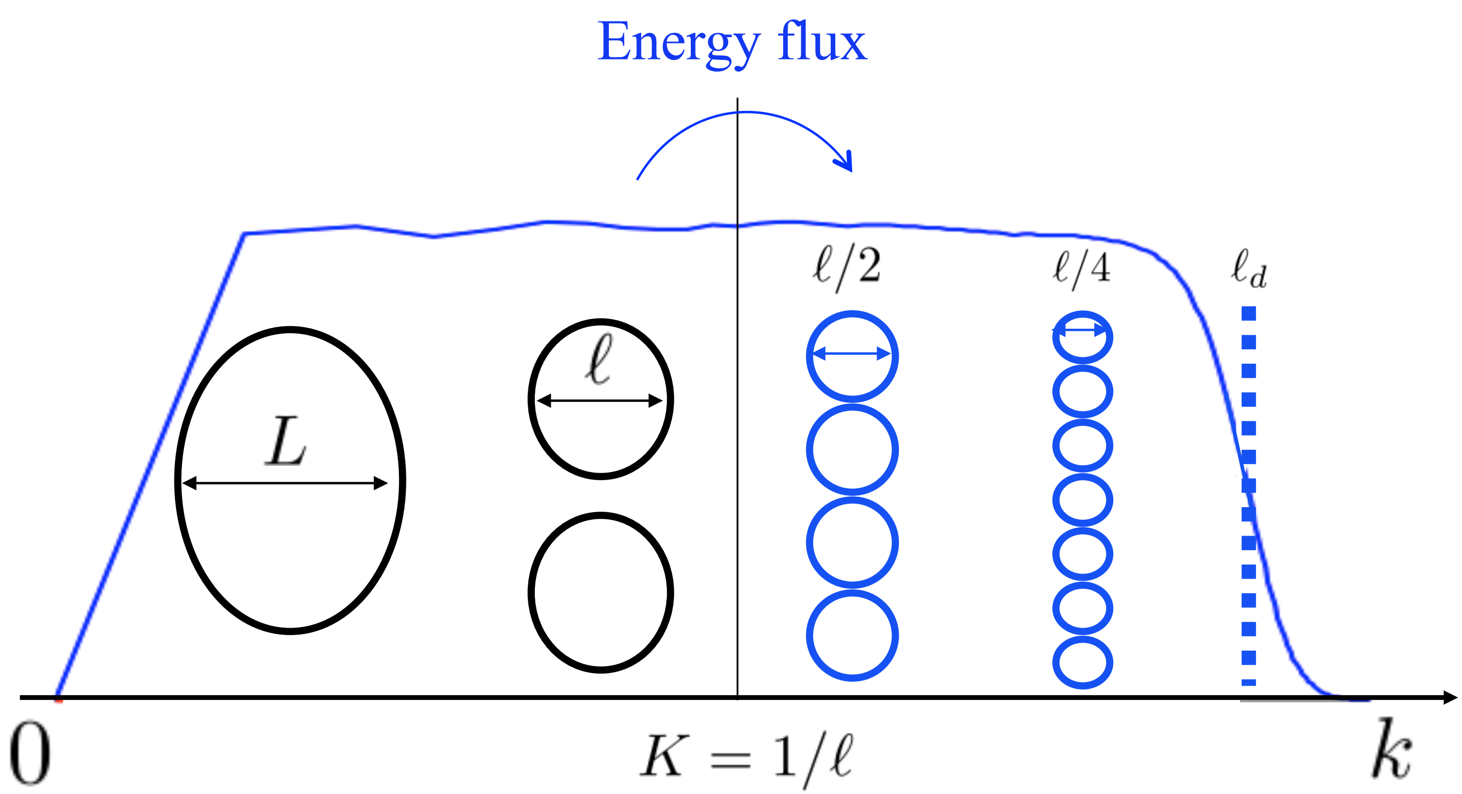


- Fromang & Papaloizou 2007
- Fromang et al. 2007

Alexakis et al. 2005

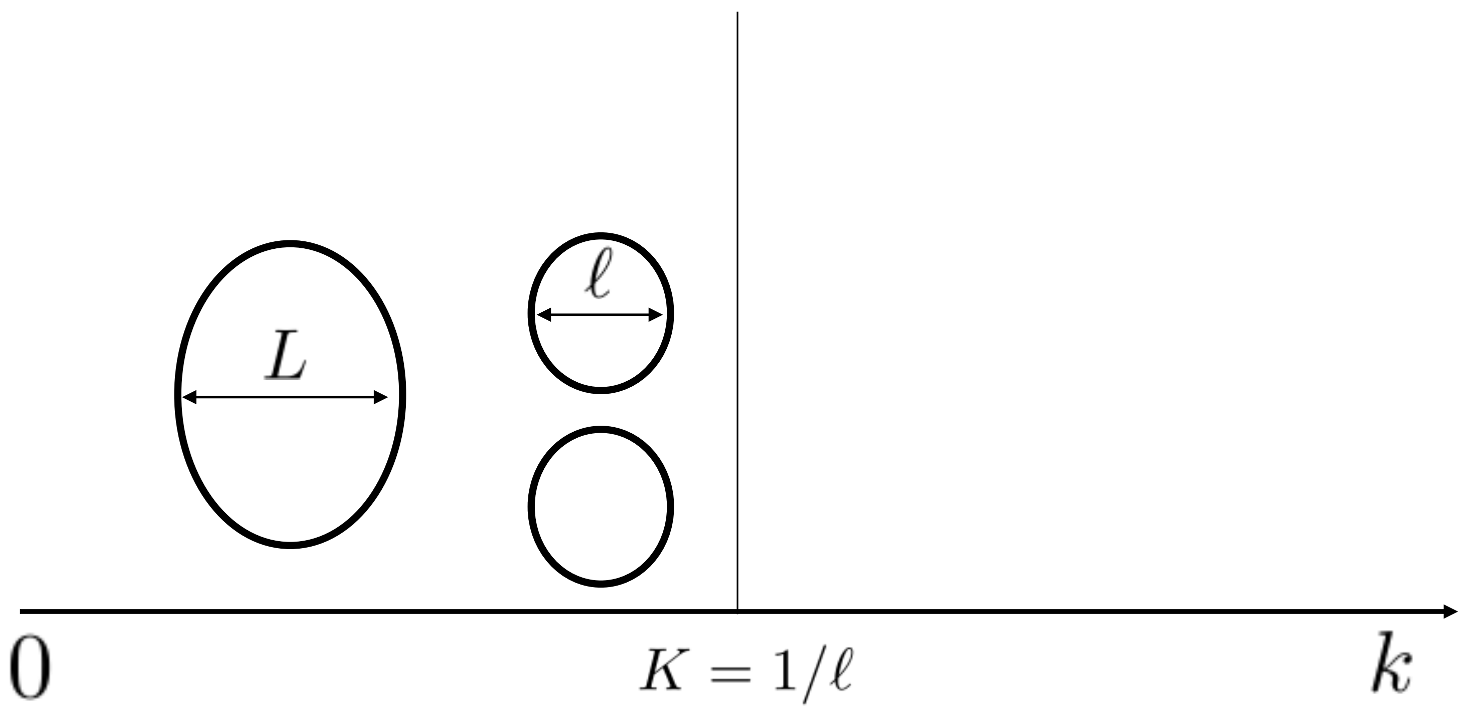
Cascade of Energy

$$\Pi_\ell^E = -\nabla\overline{\mathbf{u}} : \boldsymbol{\tau}_\ell - \overline{\mathbf{J}} \cdot \boldsymbol{\varepsilon}_\ell$$



Cascade of Energy

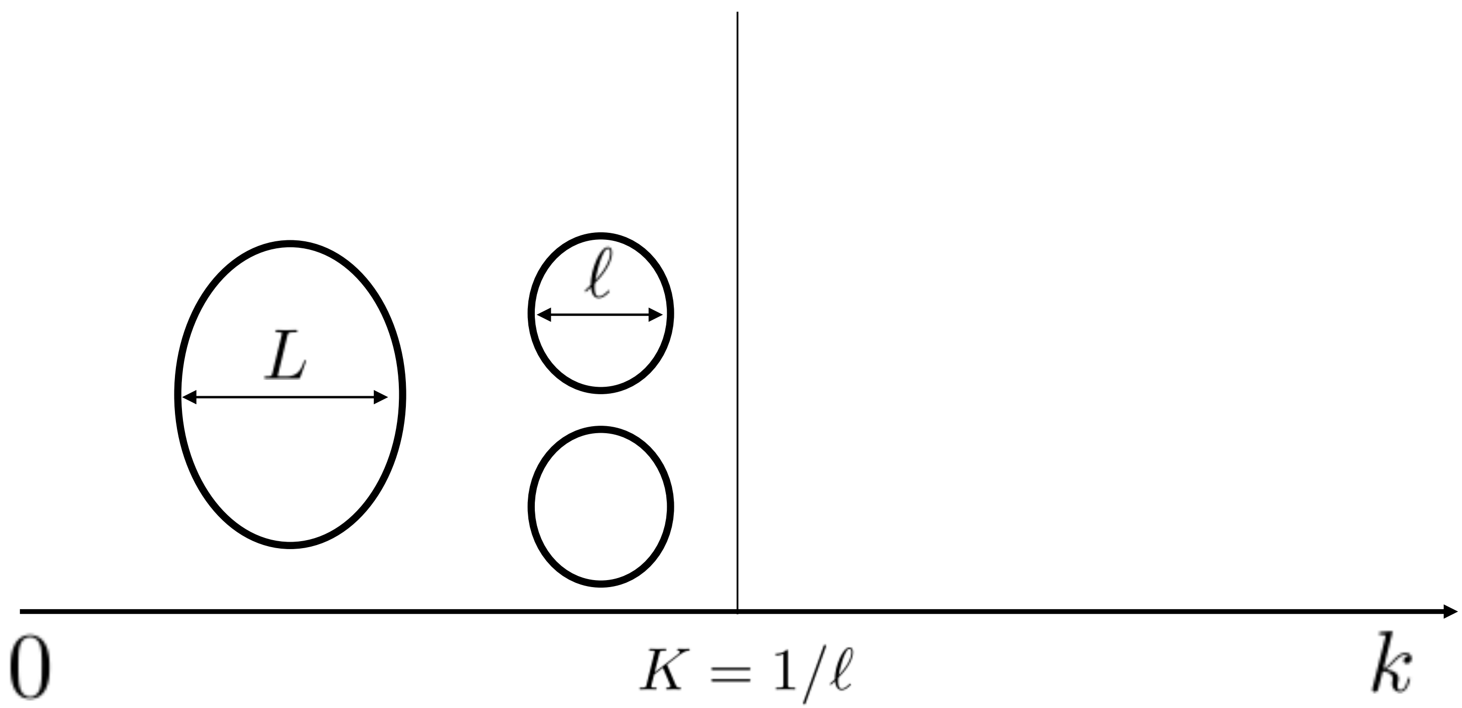
$$\Pi_\ell^E = -\nabla\overline{\mathbf{u}} : \boldsymbol{\tau}_\ell - \overline{\mathbf{J}} \cdot \boldsymbol{\varepsilon}_\ell$$



Cascade of Energy

$$\Pi_\ell^E = -\boldsymbol{\nabla}\overline{\mathbf{u}} : \boldsymbol{\tau}_\ell - \overline{\mathbf{J}} \cdot \boldsymbol{\varepsilon}_\ell$$

$$\boldsymbol{\nabla}\overline{\mathbf{u}} \sim \frac{\delta u(\ell)}{\ell} \sim \ell^{h_u - 1}$$

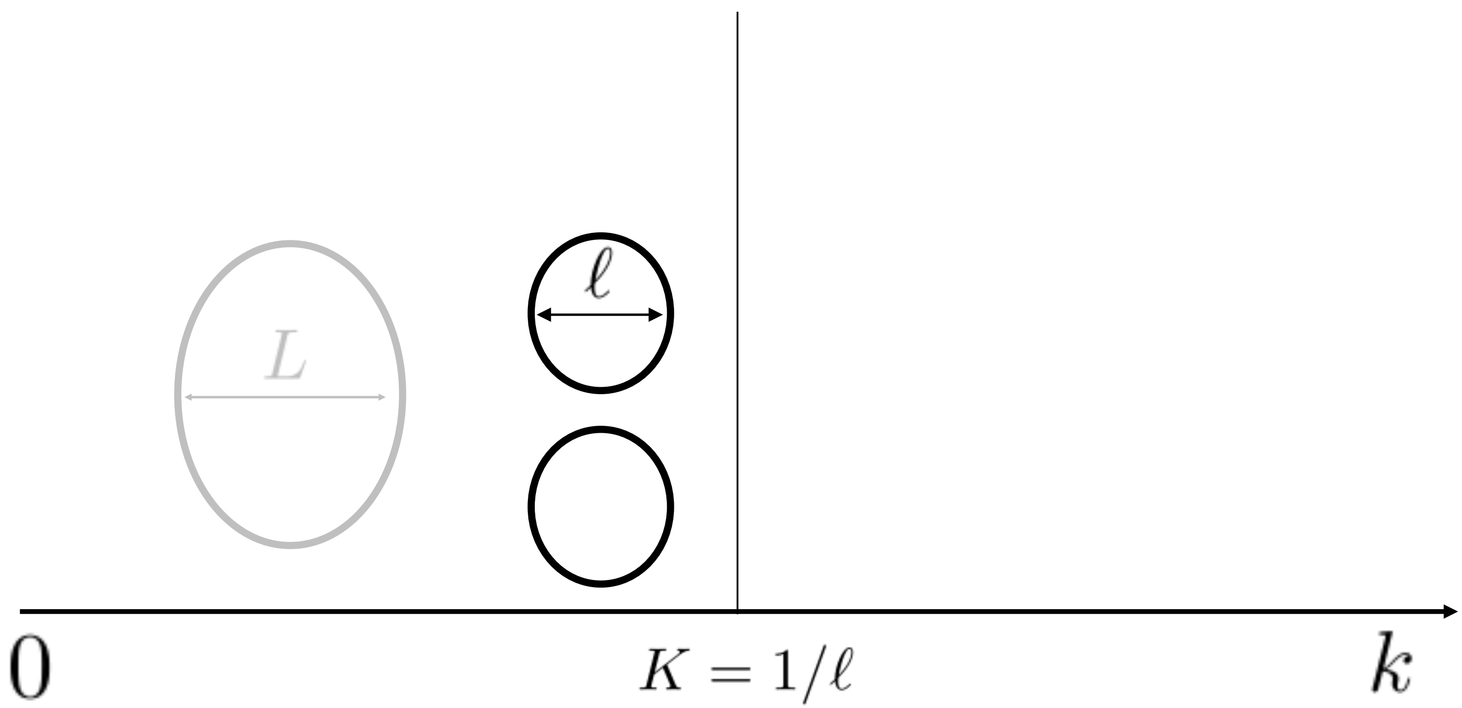


Cascade of Energy

$$\Pi_\ell^E = -\boldsymbol{\nabla}\overline{\mathbf{u}} : \boldsymbol{\tau}_\ell - \overline{\mathbf{J}} \cdot \boldsymbol{\varepsilon}_\ell$$

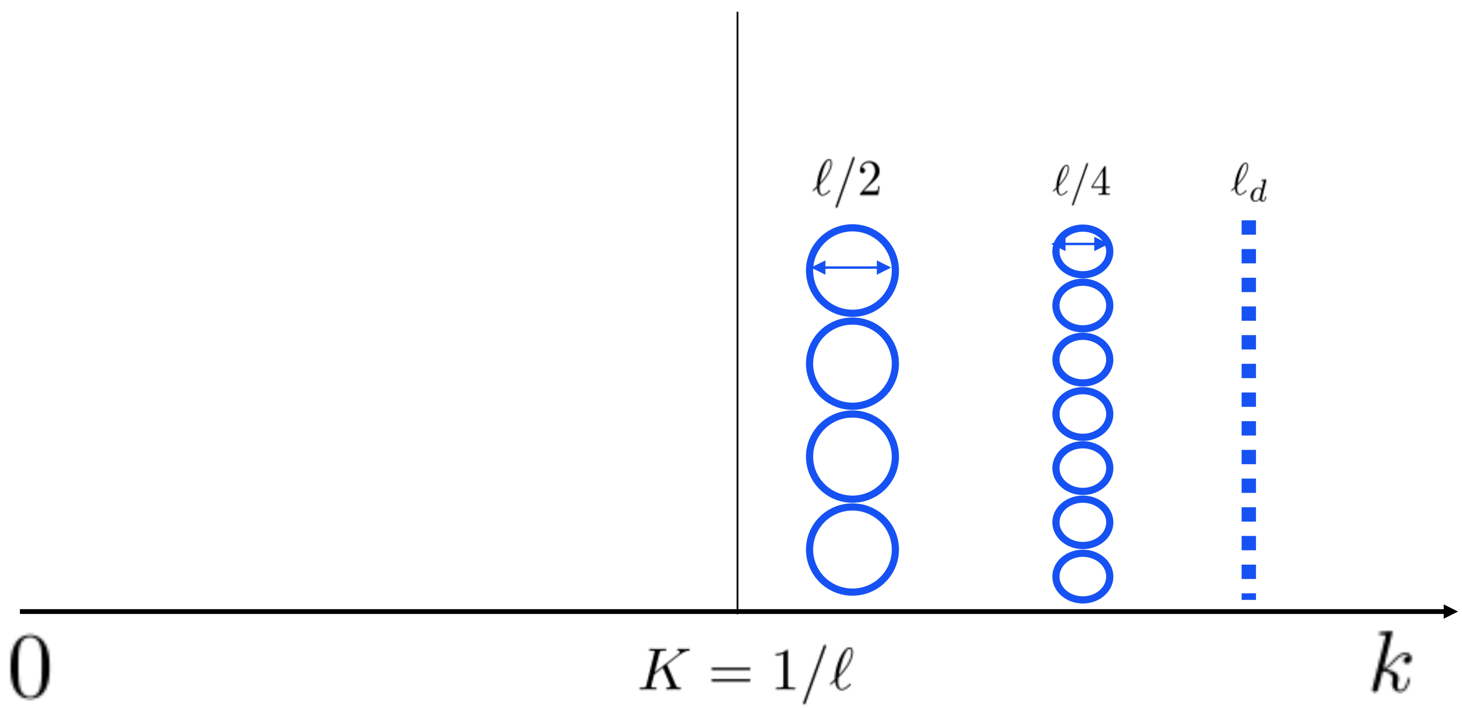
$$\boldsymbol{\nabla}\overline{\mathbf{u}} \sim \frac{\delta u(\ell)}{\ell} \sim \ell^{h_u - 1}$$

$h_u < 1$



Cascade of Energy

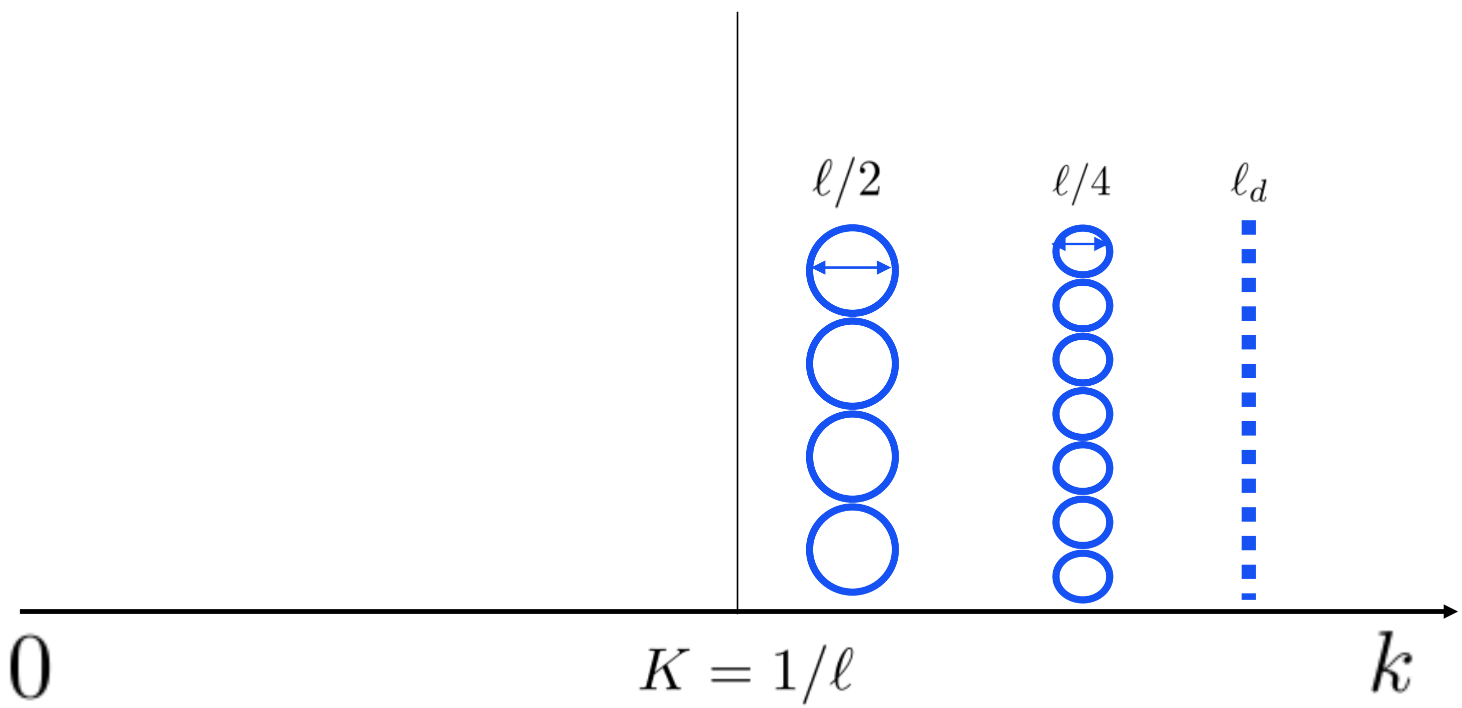
$$\Pi_\ell^E = -\nabla\overline{\mathbf{u}} : \boldsymbol{\tau}_\ell - \overline{\mathbf{J}} \cdot \varepsilon_\ell$$



Cascade of Energy

$$\Pi_\ell^E = -\nabla\overline{\mathbf{u}} : \boldsymbol{\tau}_\ell - \overline{\mathbf{J}} \cdot \boldsymbol{\varepsilon}_\ell$$

$$\boldsymbol{\tau}_\ell \sim \delta u^2(\ell) + \delta B^2(\ell) \sim \ell^{2h_u} + \ell^{2h_b}$$

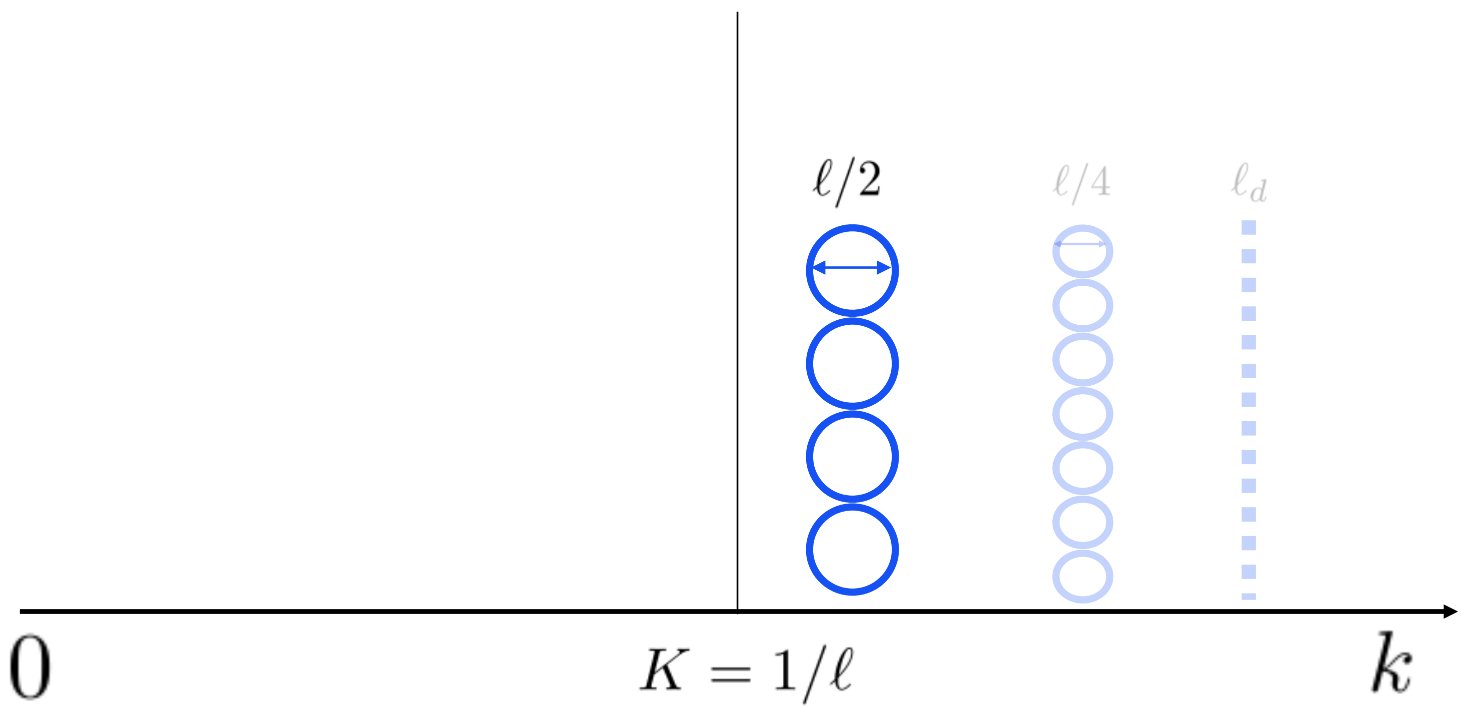


Cascade of Energy

$$\Pi_\ell^E = -\nabla\overline{\mathbf{u}} : \boldsymbol{\tau}_\ell - \overline{\mathbf{J}} \cdot \boldsymbol{\varepsilon}_\ell$$

$$\boldsymbol{\tau}_\ell \sim \delta u^2(\ell) + \delta B^2(\ell) \sim \ell^{2h_u} + \ell^{2h_b}$$

$$\boxed{0 < h_u \;, h_b}$$

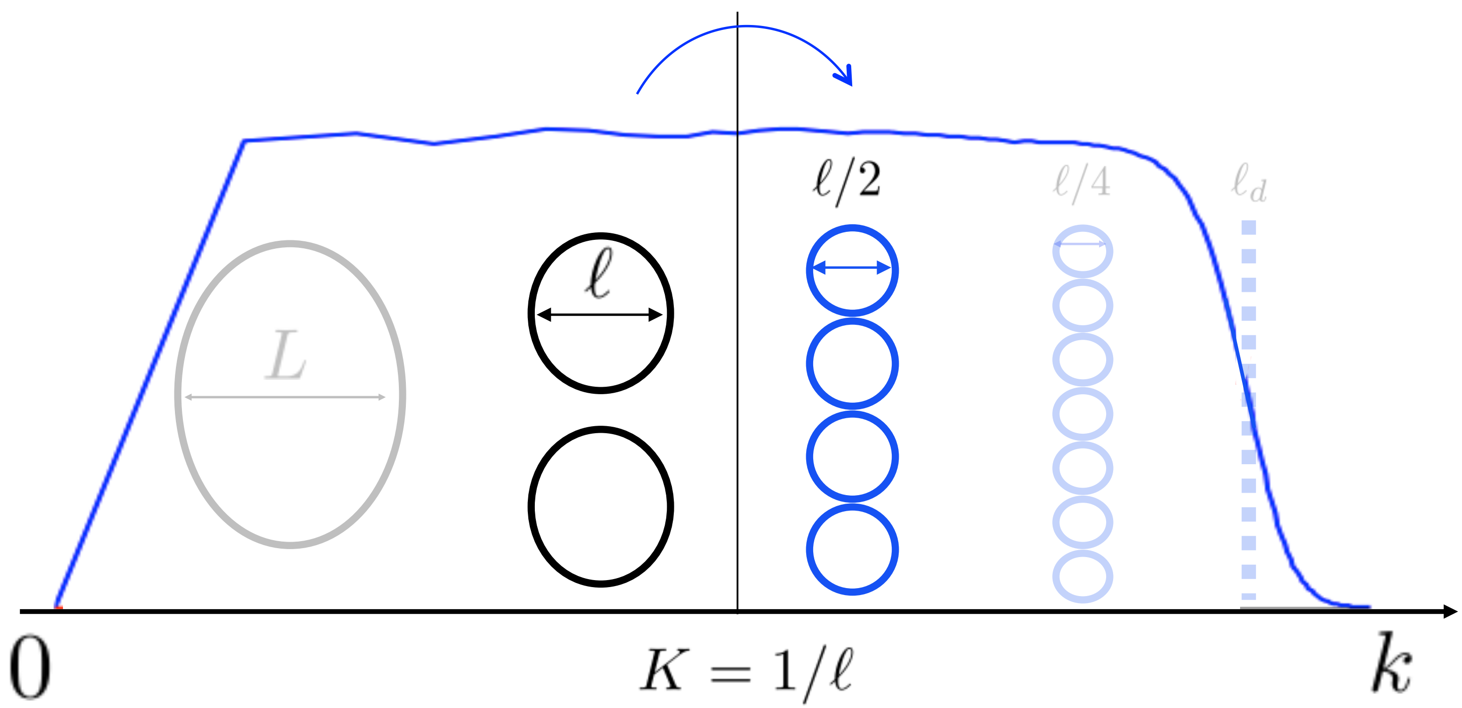


Cascade of Energy

$$\Pi_\ell^E = -\nabla\overline{\mathbf{u}} : \boldsymbol{\tau}_\ell - \overline{\mathbf{J}} \cdot \boldsymbol{\varepsilon}_\ell$$

$$\frac{\delta u(\ell)}{\ell}[\delta u^2(\ell) + \delta B^2(\ell)] \qquad \frac{\delta B(\ell)}{\ell}[\delta u(\ell)\ \delta B(\ell)]$$

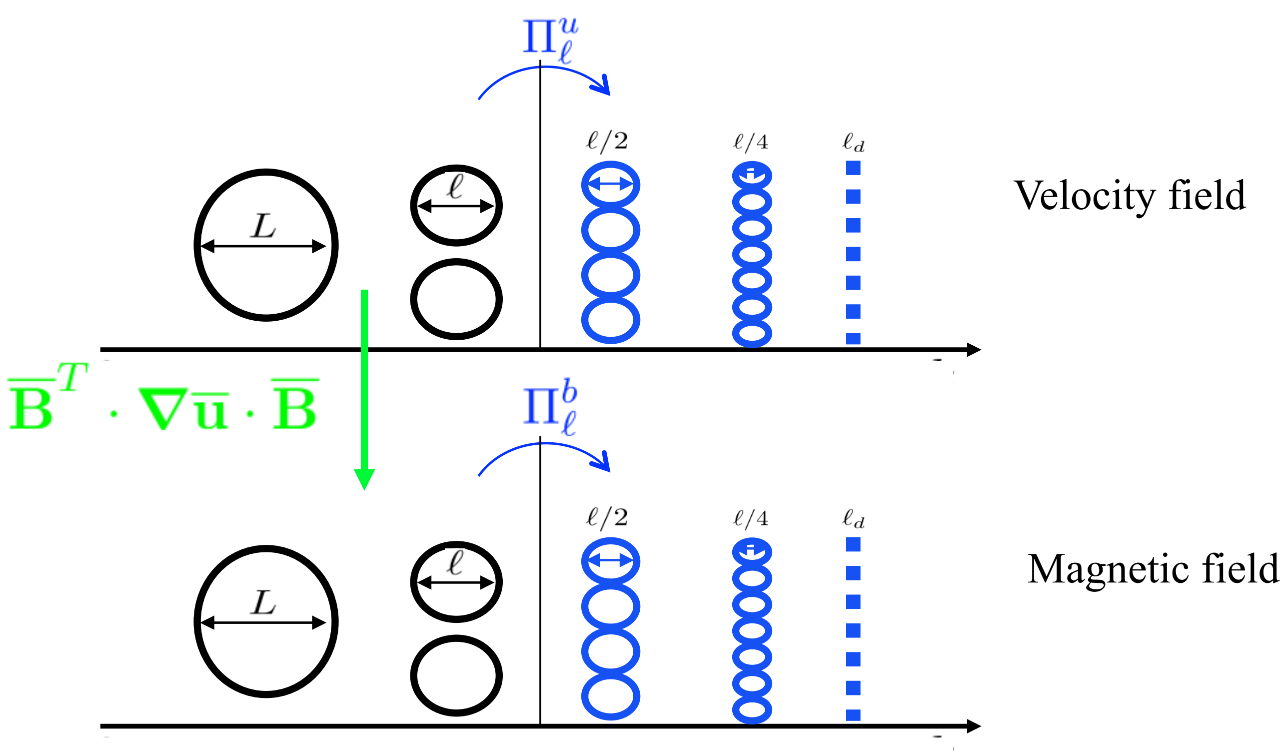
$$0 < h_u,\ h_b < 1$$



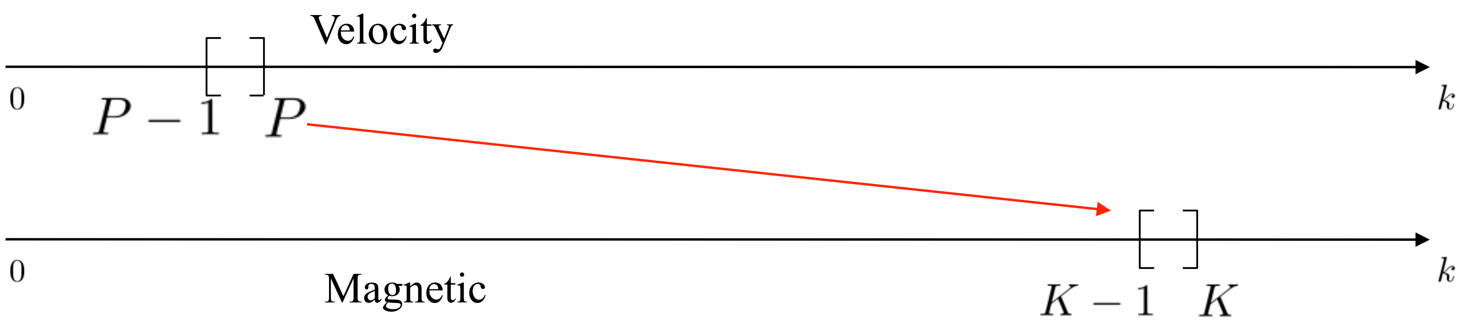
Cascade of Energy

$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla} \cdot [\cdots] = -\Pi_\ell^u - \overline{\mathbf{B}}^T \cdot \boldsymbol{\nabla}\overline{\mathbf{u}} \cdot \overline{\mathbf{B}}$$

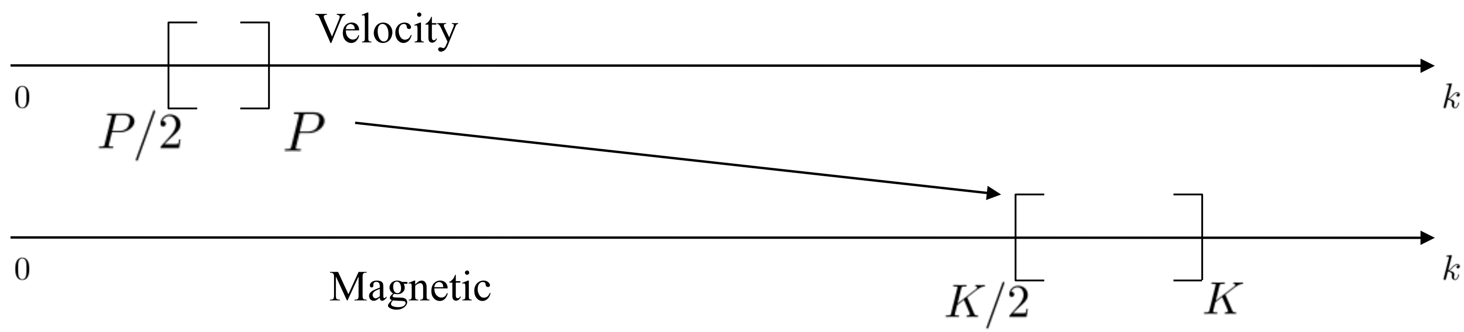
$$\partial_t \frac{|\overline{\mathbf{B}}|^2}{2} + \boldsymbol{\nabla} \cdot [\cdots] = -\Pi_\ell^b + \overline{\mathbf{B}}^T \cdot \boldsymbol{\nabla}\overline{\mathbf{u}} \cdot \overline{\mathbf{B}}$$

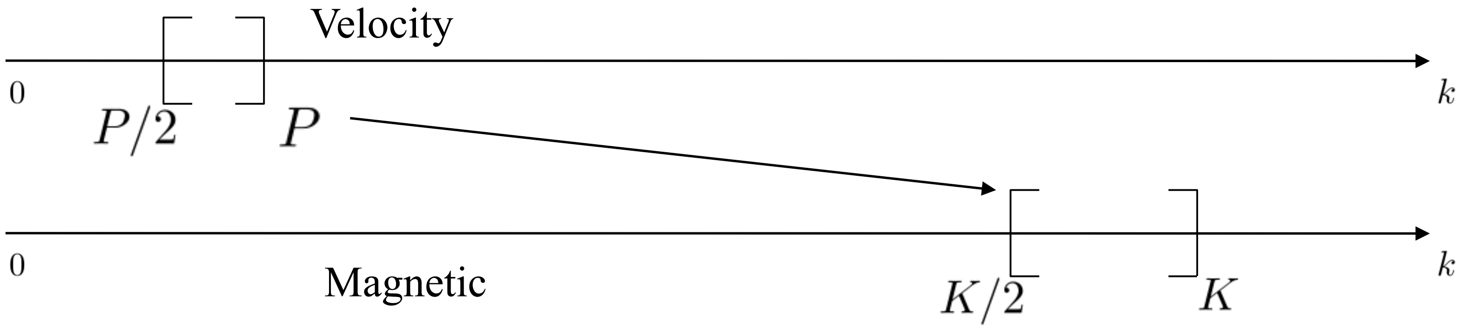


Space-scale localization

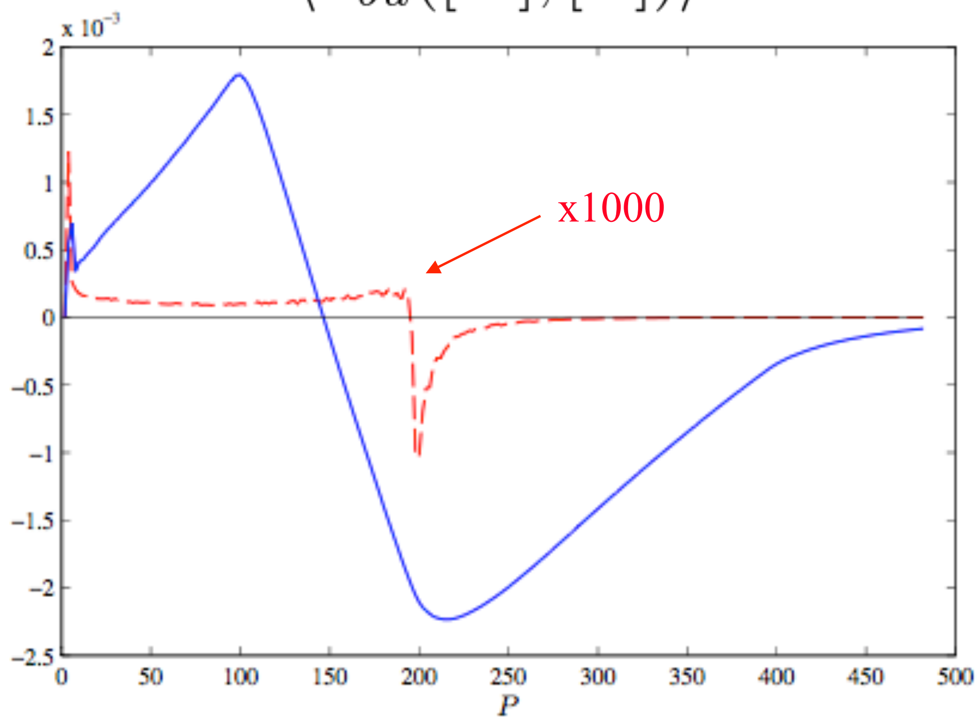


Counter-propagating wave-packets


Velocity
0
P/2
P
k
0
Magnetic
K/2
K
k



$\langle T_{bu}([K],[P])\rangle$ K=200



Proving locality of transfer

<u>Rigorous upperbound</u>

The transfer from [P] into [K]

$$|\langle T_{bu}([K],[P])\rangle| = |\langle \partial_j B_i^{[K]} u_i^{[P]} B_j \rangle|$$

for P<<K is $\leq (\text{const.})\, P^{1-h_u} K^{-2h_b}$

$\sim P^{2/3}$

1024^3 pseudospectral simulation of steady-state MHD turbulence

Fitted slope ~ $P^{0.7}$