Statistical mechanics of two dimensional turbulence and geophysical flows

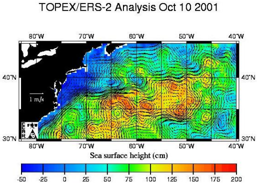
F. BOUCHET (CNRS) – ENS-Lyon

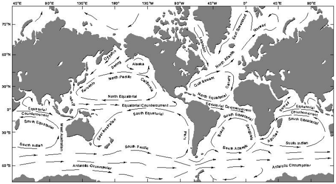
May 2011 - KITP - Santa Barbara

Collaborators

- Equilibrium statistical mechanics of ocean jets and vortices: A. Venaille (PHD student, now in post doc in Princeton with G. Vallis).
- Random change of flow topology (non-equilibrium): E. Simonnet (INLN-Nice) (ANR Statflow)
- Inviscid damping of the 2D-Euler equations: H. Morita (Tokyo university) (ANR Statflow)
- Invariant measures of the 2D Euler and Vlasov equations: M. Corvellec (PHD student, ENS-Lyon)
- Non-equilibrium phase transitions in rotating tank experiments: M. Mathur (MIT-LEGI) and J. Sommeria (LEGI-Grenoble)

The Physical Phenomena





Theoretical ideas:

- Self organization processes. Large number of degrees of freedom (turbulence).
- This has to be explained using statistical physics !!!

Mainly out of equilibrium statistical mechanics. We have to work out new theoretical concepts with such phenomena in mind.

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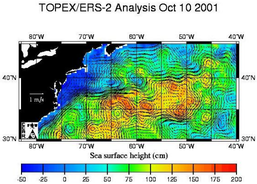

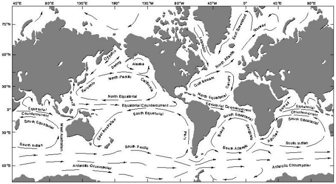

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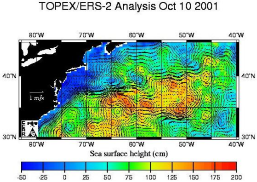

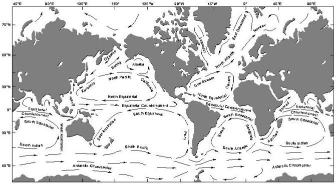

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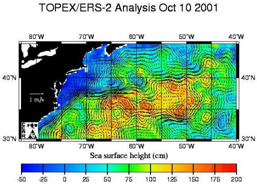

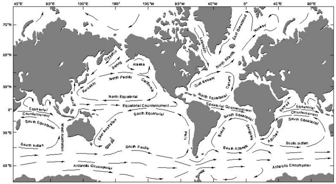

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The 2D Stochastic-Navier-Stokes (SNS) Equations

- The simplest model for two dimensional turbulence
- Navier Stokes equation with random forces

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{\sigma} f_s$$

where $\omega = (\nabla \wedge \mathbf{u}).\mathbf{e}_z$ is the vorticity, f_s is a random force, α is the Rayleigh friction coefficient.

- An academic model with experimental realizations (Sommeria and Tabeling experiments, rotating tanks, magnetic flows, and so on). Analogies with geophysical flows (Quasi Geostrophic and Shallow Water layer models).

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Equilibrium: the 2D Euler Equations

- 2D Euler equations:

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla \omega = 0$$

Vorticity $\omega = (\nabla \wedge \mathbf{u}).\mathbf{e}_z$. Stream function ψ : $\mathbf{u} = \mathbf{e}_z \times \nabla \psi$, $\omega = \Delta \psi$

- Conservative dynamics - Hamiltonian (non canonical) and time reversible
- Stationary solutions of 2D Euler Eq.:

$$\mathbf{u}.\nabla \omega = 0 \text{ or equivalently } \omega = f(\psi)$$

- Stationary solutions of 2D Euler equations play a crucial role. Degeneracy: what does select f?
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$$\text{Energy:}\quad E[\omega] = \frac{1}{2}\int_{\mathscr{D}} d^2x\, \mathbf{v}^2 = E_0$$

$$\text{Casimir's functionals:}\quad \mathscr{C}_s[\omega] = \int_{\mathscr{D}} d^2x\, s(\omega)$$

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The 1-1/2 Layer Quasi-Geostrophic Model

It describes a layer of fluid in the limit of small Rossby numbers (strong rotation compared to nonlinear terms)

$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0 \; ; \; \mathbf{v} = \mathbf{e}_z \times \nabla \psi \; ; \; q = \Delta \psi - \frac{\psi}{R^2}$$

The Rossby deformation radius:

$$R = \frac{\sqrt{gH}}{2\Omega}$$

Outline

1. Statistical mechanics of ocean jets and vortices
   - Equilibrium statistical mechanics
   - Ocean rings (mesoscale eddies)
   - Strong mid-basin ocean jets
2. Non-equilibrium phase transitions
   - Experiments
   - Random changes of flow topology in the 2D S-Navier-Stokes Eq. (F. B., E. Simonnet and H. Morita)
   - Random transition in experiments (M.M., J.S., and F.B.)
3. Irreversible relaxation of the 2D Euler equations
   - Irreversibility in fluid mechanics
   - Nonlinear damping for the 2D Euler Eq. (F.B, and H.M.)
   - The Kolmogorov Flow

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Statistical Mechanics for 2D and Geophysical Flows

- Statistical hydrodynamics ? Very complex problems.
- Example: Intermittency in 3D turbulence ; phenomenological approach, simplified models (Kraichnan).
- It may be much simpler for 2D or geophysical flows: conservative systems.
- Statistical equilibrium: A very old idea, some famous contributions Onsager (1949), Joyce and Montgomerry (1970), Caglioti Marchioro Pulvirenti Lions (1990), Robert Sommeria (1991), Miller (1991), Eyink and Spohn (1994), Kiessling and Lebowitz (1994).

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Robert-Sommeria-Miller (RSM) Theory

Equilibrium statistical mechanics : the most probable vorticity field

- A probabilistic description of the vorticity field q: $\rho\left(\mathbf{x},\sigma\right)$ is the local probability to have $q\left(\mathbf{x}\right)=\sigma$ at point $\mathbf{x}$
- A measure of the number of microscopic field q corresponding to a probability ρ (Liouville and Sanov theorems):

$$\text{Boltzmann-Gibbs Entropy:} \quad \mathscr{S}\left[\rho\right]\equiv-\int_{\mathscr{D}}d\mathbf{x}\int_{-\infty}^{+\infty}d\sigma\,\rho\log\rho$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0,d)=\sup_{\{\rho|N[\rho]=1\}}\left\{\mathscr{S}[\rho]\mid E\left[\overline{q}\right]=E_0\ ,D\left[\rho\right]=d\ \right\}\ (\text{MVP}).$$

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$$q=f\left(\psi\right)$$

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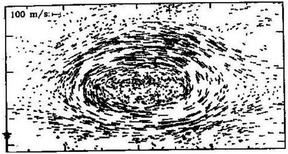
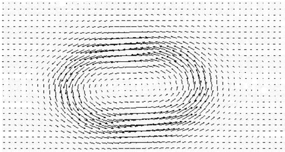
Great Red Spot of Jupiter

Real Flow and Statistical Mechanics Predictions (1-1/2 layer QG model)



Observation data (Voyager)



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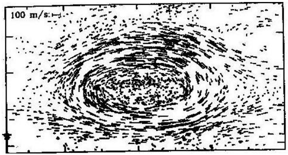

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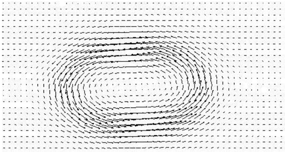

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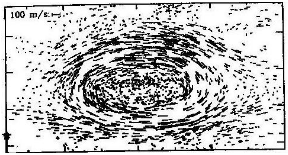

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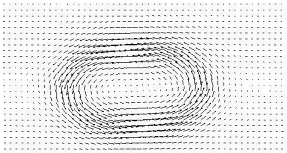

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The Simplest Ocean Model
The 1-1/2 Layer Quasi-Geostrophic Model

We describe the upper layer of an ocean by the Quasi Geostrophic model (one and half layer):

$$\frac{\partial q}{\partial t}+\mathbf{v}\cdot\nabla q=0\;;\;\mathbf{v}=\mathbf{e}_z\times\nabla\psi\;;\;q=\Delta\psi-\frac{\psi}{R^2}+\tilde{\beta}y$$

- An extremely rough model of an ocean. The simplest one

Two simple questions.

1. What can equilibrium statistical mechanics teach us about mesoscale eddies?
2. Does it exist statistical equilibria with strong eastward mid basin jets (Gulf-Stream Kuroshio)?

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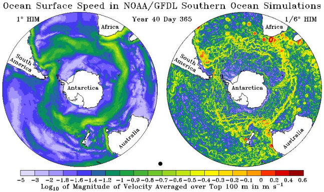
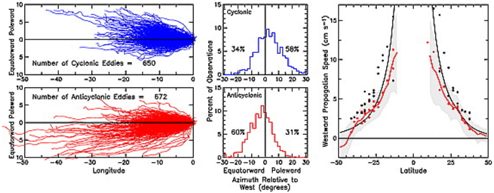
Gulf Stream rings - Agulhas rings - Meddies - etc ...





Hallberg-Gnanadesikan - JPO 2006

Chelton and co. - GRL 2007

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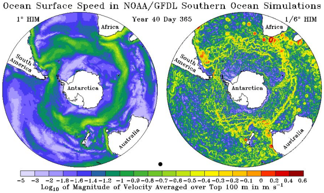

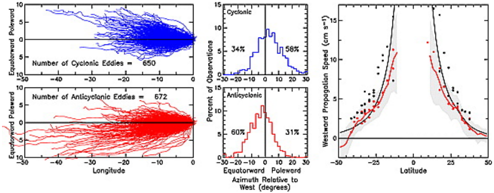

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The 1-1/2 Layer Quasi-Geostrophic Model

First case: no beta effect $\tilde{\beta} = 0$

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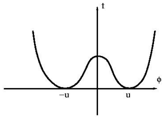
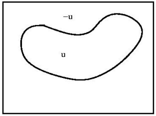
Variational Problem for Statistical Equilibria
(The case of the 1-1/2 layer Quasi Geostrophic model)

Variational problem: limit $R \to 0$. $(\phi = \psi / R^2)$.

$$\begin{cases} \min\left\{ F_R[\phi] \mid \text{with } A[\phi] \text{ given} \right\} \\ \text{with } F_R[\phi] = \int_D d\mathbf{r} \left[\frac{R^2 (\nabla\phi)^2}{2} + f(\phi) \right] \text{ and } A[\phi] = \int_D d\mathbf{r}\, \phi . \end{cases}$$





The function f: two minima

Phase coexistence

- An analogy with first order phase transitions.
- Modica (90'), function with bounded variations.

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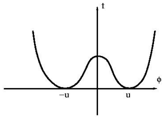

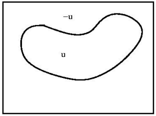

The function f: two minima

Phase coexistence

- An analogy with first order phase transitions.
- Modica (90'), function with bounded variations.

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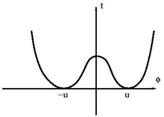

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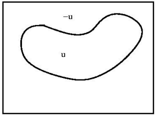

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Reduction to a One Dimensional Variational Problem

An isoperimetrical problem

- A variational problem for the jet shape (interface)

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- Laplace equation:

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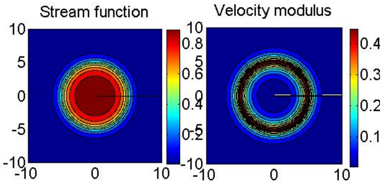
Ocean Rings are Statistical Equilibria

Analogy with first order phase transition: strong jets + minimization of the length for a fixed area

$$F_R\left[\phi_R\right] = 2Re_c L \text{ and } \frac{e_c}{r} = u\alpha_1$$

- Without topography: vortices are rings (like bubbles in usual thermodynamics).



Statistical Mechanics Explains the Westward Drift

QG 1-1/2 layer model with beta effect $h(y) = \tilde{\beta} y$ in a channel geometry

- In a domain with translational symmetry (channel), using Noether's theorem we obtain a new invariant:

$$L = \int \mathrm{d}^2\mathbf{r}\, yq$$

- Statistical equilibria with this new invariant: $q = (s')^{-1}(-\beta\psi + \gamma y)$
- Equilibria are steady solution of the QG eq. in a reference frame drifting with constant velocity $V_\gamma \mathbf{e}_x$.

$$V_\gamma = -\tilde{\beta} R^2 \mathbf{e}_x \text{ (westward drift)}$$

- In the literature, $-\tilde{\beta} R^2 \mathbf{e}_x$ is referred as the speed of non dispersive baroclinic waves. (Chelton and co. GRL 2007 - McWilliams and Flierl JPO 1979).

Conclusion: Ocean rings are statistical equilibria. Stat mech explains their observed drift properties.

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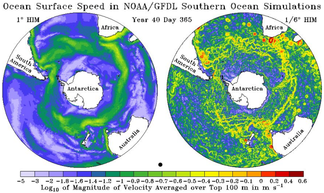
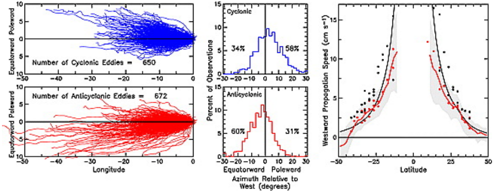
Ocean Rings (Mesoscale Ocean Vortices)

Gulf Stream rings - Agulhas rings - Meddies - etc ...





Hallberg-Gnanadesikan - JPO 2006

Chelton and co. - GRL 2007

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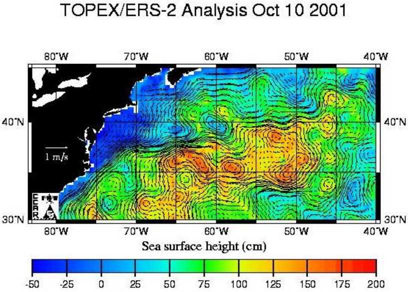
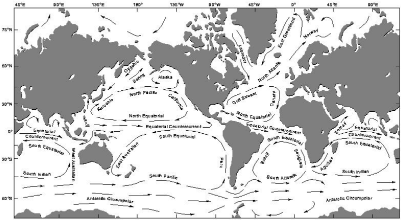
Outline

Is the map of ocean currents a statistical equilibrium ?



North Atlantic sea surface height



A sketch of ocean currents

Variational Problem for Statistical Equilibria
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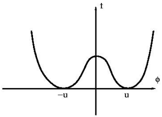

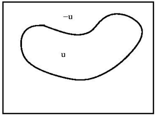

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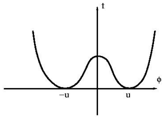

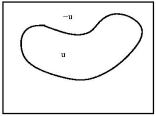

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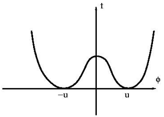

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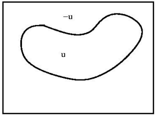

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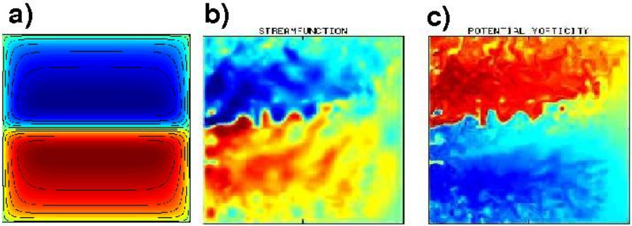
Strong Eastward Jets are Statistical Equilibria

Statistical equilibria of the QG 1-1/2 layer in a closed basin $h(y) = 0$

$$F_R[\phi_R] = 2Re_c L \text{ and } \frac{e_c}{r} = u\alpha_1$$



- The states with negative PV to the north (eastward jet), and positive PV to the south (westward jet) are equivalent.
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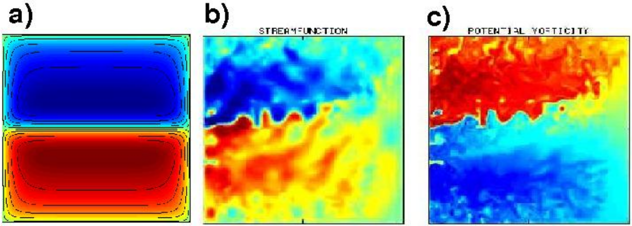

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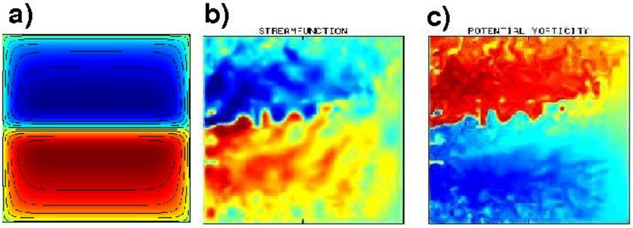

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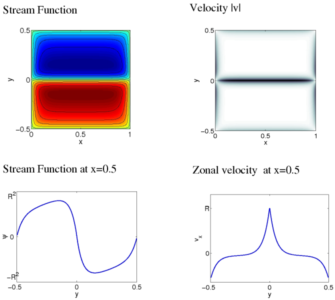
Strong Eastward Jets are not GLOBAL Statistical Equilibria

$$F_R\left[\phi_R\right] = 2Re_c L - 2Ru \int_{A_+} dl\, \beta y$$

$$F_{Eastward} > F_{Westward}$$



Conclusion: With a beta effect, global statistical equilibria are the ones with westward jets. The bad ones !.

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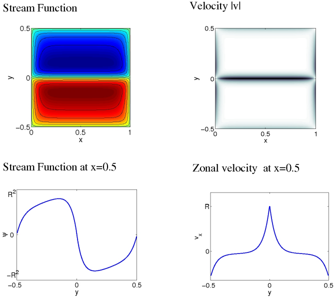

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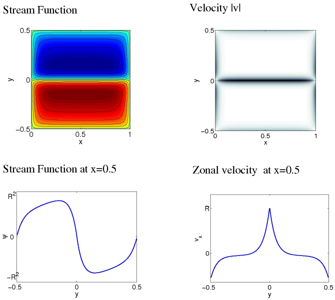

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Strong Eastward Jets may be metastable Statistical Equilibria

We study the local stability of strong eastward jets with topography $h(y) = \tilde{\beta} y = \beta y - \psi_2 / R^2$ with $\psi_2 = ay$.

$$F_R[\phi_R] = 2Re_c L - 2Ru \int_{A_+} dl\, \tilde{\beta} y$$

$$\delta^2 \mathscr{F} \geq \left[-2u\tilde{\beta} + Re\left(\frac{k\pi}{L_x} \right)^2 \right] \int dx\, (\delta l)^2 .$$

We have local free energy minima (for all δl, $\delta^2 \mathscr{F} \geq 0$) if

$$\tilde{\beta} < \tilde{\beta}_c = \frac{1}{2} \frac{e}{u} \frac{\pi^2}{L_x^2} R.$$

Conclusion 1: With a low effective beta effect, eastward jets are metastable statistical equilibria.

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Equilibrium Statistical Mechanics and Oceans

Some first results in the QG 1-1/2 layer model - What else ?

- Ocean mesoscale vortices (rings) are statistical equilibria.
- Global statistical equilibria are flows with strong midbasin westward jets (incompatible with the actual forcing by the wind).
- Strong eastward jets are probably marginally unstable, from a statistical mechanics point of view.

- Perspectives: Kuroshio bistability - More properties of the rings - Circumpolar antarctic current - Less naive models.

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Other Recent Results for the Equilibrium RSM Theory

- Simplified variational problems for the statistical equilibria of 2D flows. F. Bouchet, Physica D, 2008
- Invariant measures of the 2D Euler and Vlasov equations. F. Bouchet and M. Corvellec, J. Stat. Mech., 2010
- Equilibrium phase transitions in 2D and geophysical flows A. Venaille and F. Bouchet, J. Stat. Phys., 2011

- Phase transitions, ensemble inequivalence and Fofonoff flows. A. Venaille and F. Bouchet, Phys. Rev. Lett.
- Are strong mid-basin eastward jets (Gulf Stream, Kuroshio) statistical equilibria? A. Venaille and F. Bouchet, accepted for publication in JPO

Antoine Venaille

The 2D Stochastic-Navier-Stokes (SNS) Equations

- The simplest model for two dimensional turbulence.
- Navier Stokes equation with a random force

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{\sigma} f_s \qquad (1)$$

where $\omega = (\nabla \wedge \mathbf{u}).\mathbf{e}_z$ is the vorticity, f_s is a random force, α is the Rayleigh friction coefficient.

- An academic model with experimental realizations (Sommeria, Tabeling, Ecke experiments, rotating tanks, magnetic flows, soap films, and so on). Analogies with geophysical flows.

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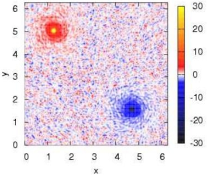
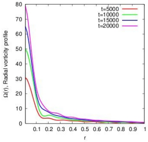
Numerical Simulation of the 2D Stochastic-NS Eq.

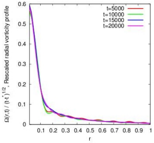


Self similar growth of a dipole structure, for the 2D S-NS Eq.

Left : vorticity field.

Bottom : vorticity profiles.





M Chertkov, C Connaughton, I Kolokolov, V Lebedev (PRL 2007)

Also M. G. Shats, H. Xia, H. Punzmann and G. Falkovich (PRL 2007)

The 2D Stochastic Navier-Stokes Equations

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega + \sqrt{\nu} f_s$$

- Some recent mathematical results: Kuksin, Sinai, Shirikyan, Bricmont, Kupianen, Hairer, etc;
  - Existence of a stationary measure μ_ν. Existence of $lim_{\nu\to 0}\ \mu_\nu$,
  - In this limit, almost all trajectories are solutions of the 2D Euler equations.
- We would like to obtain more physical results:
  - What is the link of this limit $\nu \to 0$ with the RSM theory?
  - Will we stay close to some steady solutions of the 2D Euler equations?
  - Can we describe these statistically stationary states and their properties?

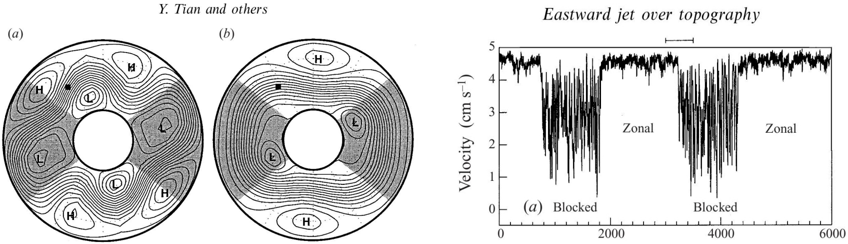
Outline

1. Statistical mechanics of ocean jets and vortices
   - Equilibrium statistical mechanics
   - Ocean rings (mesoscale eddies)
   - Strong mid-basin ocean jets
2. Non-equilibrium phase transitions
   - Experiments
   - Random changes of flow topology in the 2D S-Navier-Stokes Eq. (F. B., E. Simonnet and H. Morita)
   - Random transition in experiments (M.M., J.S., and F.B.)
3. Irreversible relaxation of the 2D Euler equations
   - Irreversibility in fluid mechanics
   - Nonlinear damping for the 2D Euler Eq. (F.B, and H.M.)
   - The Kolmogorov Flow

Non-Equilibrium Phase Transitions in Real Flows

Rotating tank experiments (Quasi Geostrophic dynamics)

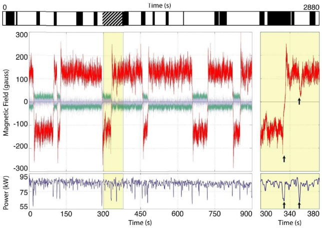
Transitions between blocked and zonal states:



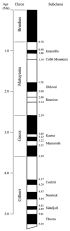
Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

Random Transitions in Other Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)



VKS experiment



Earth

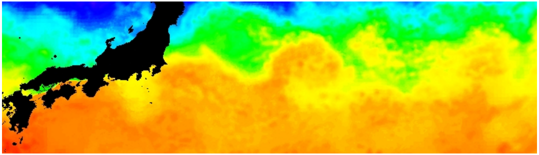
(VKS experiment)

Other examples :

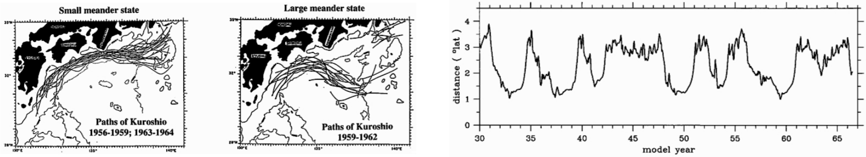
- Turbulent convection, Van Karman and Couette turbulence.
- Multistability in the atmosphere, weather regimes, and so on.

Non-Equilibrium Phase Transitions in Real Flows

The Kuroshio current bistability (two layer Quasi-Geostrophic or primitive equations dynamics)



See surface temperature of the pacific ocean, east of Japan



Kuroshio paths and bistability timeserie

Outline

The 2D Stochastic Navier-Stokes Equations

- The 2D Stochastic Navier Stokes equations:

$$\frac{\partial \omega}{\partial t}+\mathbf{u}.\nabla\omega=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s, \quad (2)$$

  where f_s is a random force (white in time, smooth in space).
- We use very small Rayleigh friction, to observe large scale energy condensation (this is not the inverse cascade regime).
- We study the limit: $\lim_{\alpha\to 0}\lim_{\nu\to 0}$ $(\nu \ll \alpha)$ $(Re \gg R_\alpha \gg 1)$ (Weak forces and dissipation).
- We have time scale separations:

turnover time $= 1 \ll 1/\alpha =$ forcing or dissipation time.

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Large Scale Structures and Euler Eq. Steady States

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s \qquad (3)$$

- Time scale separation: magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards steady solutions of the Euler Eq.:

$$\mathbf{u}.\nabla\omega = 0 \text{ or equivalently } \omega = f(\psi)$$

  where the Stream Function ψ is given by: $\mathbf{u} = \mathbf{e}_z \times \nabla\psi$.
- Steady states of the Euler equation will play a crucial role. Degeneracy: what does select f ?

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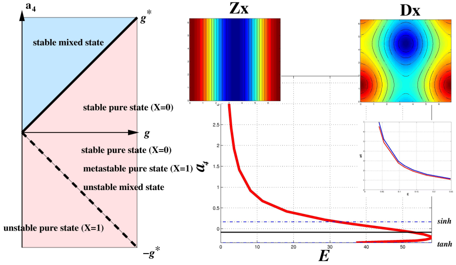
Steady States of Euler Eq. as Maxima of Variational Problems

Energy-Casimir Variational Problems

$$S(E) = \max_{\omega}\left\{ \int_{\mathscr{D}} d\mathbf{r}\, s(\omega) \,\middle|\, \frac{1}{2}\int_{\mathscr{D}} d\mathbf{r}\, \frac{\mathbf{v}^2}{2} = E \right\}.$$

- Numerical results: Z. Yin, D. C. Montgomery, and H. J. H. Clercx, Phys. Fluids (2003).
- Maxima: $\omega = \Delta\psi = \left(s^{'}\right)^{-1}(\beta\psi)$ (stable steady states of the Euler Eq.).
- In the following, normal form analysis with $s(\omega) = -\frac{\omega^2}{2} + a_4\frac{\omega^4}{4} + \dots$
- Geometry parameter $g = E(\lambda_1 - \lambda_2) \propto (L_x - L_y)$.

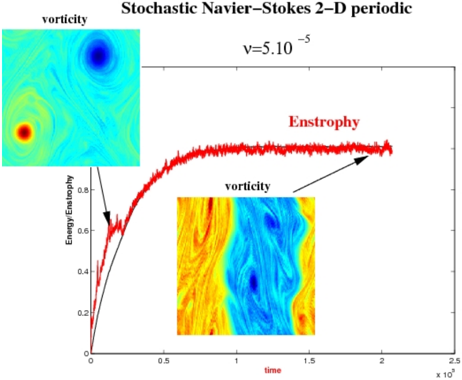
Steady States for the 2D-Euler Eq. (doubly periodic)



Bifurcation analysis: degeneracy removal, either by the domain geometry (g) or by the nonlinearity of the vorticity-stream function relation (f, parameter a_4).

Derivation: normal form for an Energy-Casimir variational problem. A general degeneracy removal mechanism.

Numerical Simulation of the 2D Stochastic NS Eq.



Very long relaxation times. 10^5 turnover times.

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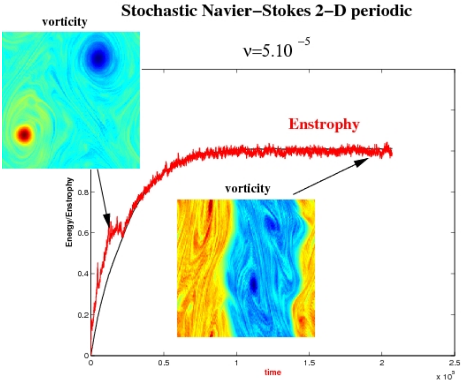

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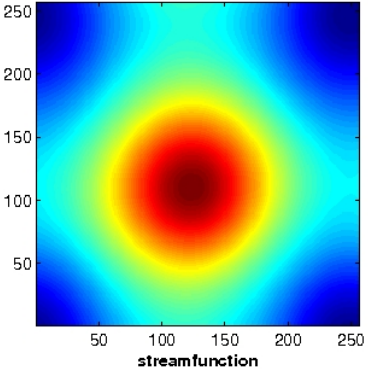
Non-Equilibrium Stationary States: Dipoles



Are we close to some steady states of the Euler Eq.?

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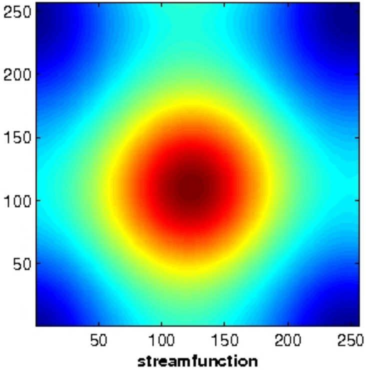
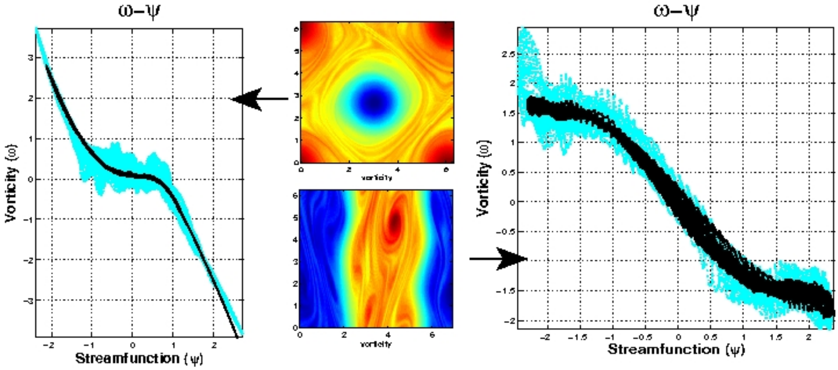


Are we close to some steady states of the Euler Eq.?

Vorticity-Streamfunction Relation



Conclusion: we are close to steady states of the Euler Eq.

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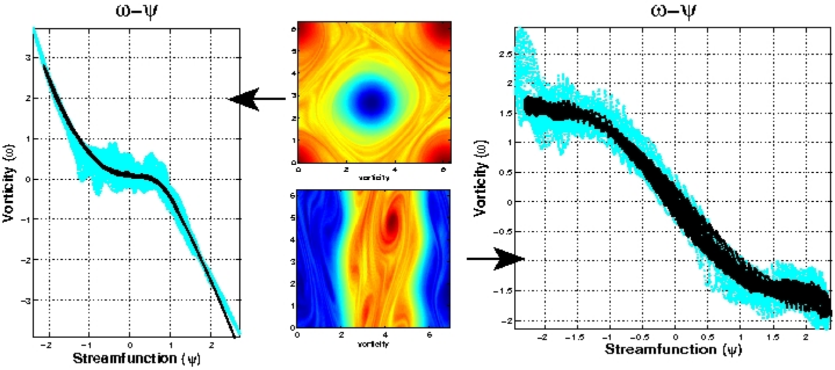

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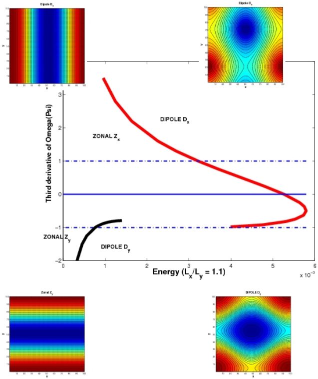
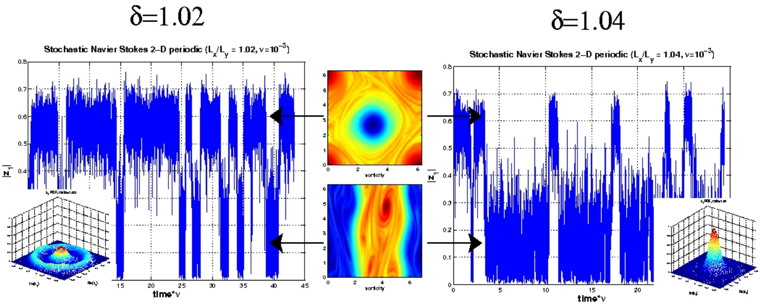


A second order phase transition.

Non-Equilibrium Phase Transition

The time series and PDF of the Order Parameter



Order parameter : $z_1 = \int dxdy\, \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$.

Outline

Experimental Applications

- Using the equilibrium theory, we can predict the existence of non-equilibrium phase transitions.
- Or phase transitions governed by the domain geometry, the topography, or the energy.
- Prediction of flow topology change in Quasi-Geostrophic and Shallow Water dynamics (rotating tank experiments).

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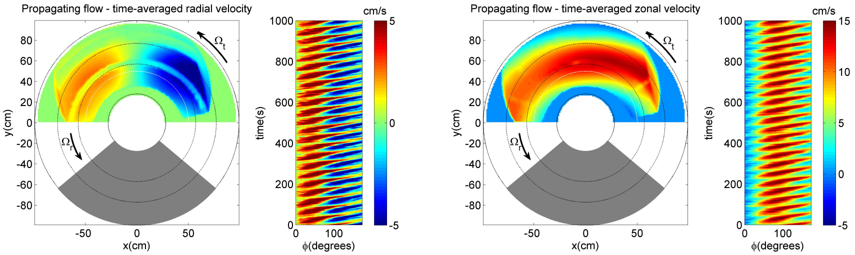
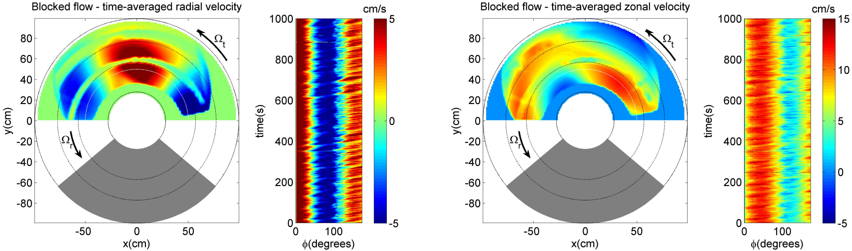
Bistability in rotating tank



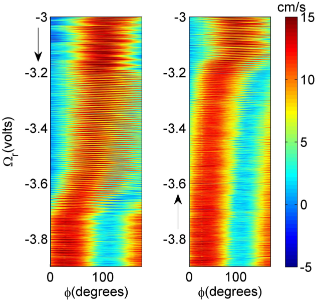
The zonal (propagating) state

Bistability in rotating tank



The blocked state

Bistability in rotating tank experiments
M. Mathur, J. Sommeria (LEGI)



Bistability (hysteresis) in rotating tank experiments

Outline

1. Statistical mechanics of ocean jets and vortices
   - Equilibrium statistical mechanics
   - Ocean rings (mesoscale eddies)
   - Strong mid-basin ocean jets
2. Non-equilibrium phase transitions
   - Experiments
   - Random changes of flow topology in the 2D S-Navier-Stokes Eq. (F. B., E. Simonnet and H. Morita)
   - Random transition in experiments (M.M., J.S., and F.B.)
3. Irreversible relaxation of the 2D Euler equations
   - Irreversibility in fluid mechanics
   - Nonlinear damping for the 2D Euler Eq. (F.B, and H.M.)
   - The Kolmogorov Flow

Irreversibility in Fluid Mechanics and in Turbulence

Do we need viscosity to explain irreversible behavior of turbulent flows ?

- In many fluid mechanics or turbulence textbooks, it is stated, for example, that "Viscosity, whatever small, is necessary to explain the irreversible behavior of turbulent flows".
- Based on "D'Alembert's Paradox" (Euler and Lagrange theorems) (about potential flows) and Prandtl boundary layer analysis.
- The reversibility paradox of very small Reynolds number flows.

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Outline

The 2D Euler equations

Dynamics of small perturbations of a steady state

- 2D Euler equations

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = 0,$$

- Hamiltonian and time reversible.
- Base state: a steady state $\mathbf{v}_0 = U(y)\mathbf{e}_x$, with vorticity ω_0: $\mathbf{v}_0.\nabla \omega_0 = 0$.
- The 2D Euler dynamics close to $\mathbf{v}_0$, $\omega = \omega_0(y) + \omega'$ and $\mathbf{v} = U(y)\mathbf{e}_x + \mathbf{v}'$
- Irreversible behavior. Asymptotic behavior for large times.

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Nonlinear Landau damping

Clément Mouhot, and Cédric Villani, 2010

- Vlasov equation (dynamics of electrons in a plasma). μ-space density $f(x,p,t)$:

$$\frac{\partial f}{\partial t}+p\frac{\partial f}{\partial x}-\frac{dV}{dx}\frac{\partial f}{\partial p}=0.$$

- Hamiltonian and time reversible. A transport equation by a non-divergent flow, like the 2D Euler equations.
- Base state: a steady state $f=f_0(p)$. Understanding of the linearized equation by Landau (1946)
- Proof of the irreversible convergence, for large times, of f (weak topology) and ρ (strong topology) towards homogeneous densities (Mouhot, and Villani, 2010)

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The Linearized Euler Eq. close to Shear flows

- Base flow : $\mathbf{v}_0(\mathbf{r}) = U(y)\mathbf{e}_x$. The linearized Euler equation :

$$\frac{\partial \omega'}{\partial t} + ikU(y)\omega' - ik\psi U''(y) = 0, \qquad (4)$$

with $\omega'(x,y,t) = \omega'(y,t)\exp(ikx)$ and $\omega' = \frac{d^2\psi}{dy^2} - k^2\psi$.

- Laplace transform : $\phi(y,c,\varepsilon) = \int_0^\infty dt\, \Psi(y,t)\exp(ik(c+i\varepsilon)t)$

$$\left(\frac{d^2}{dy^2} - k^2\right)\phi - \frac{U''(y)}{U(y)-c-i\varepsilon}\phi = \frac{\omega'(y,0)}{ik(U(y)-c-i\varepsilon)} \qquad (5)$$

- This is the celebrated Rayleigh equation. A one century old classical problem in fluid mechanics, applied mathematics and mathematics. Rayleigh (1842-1919)

- Large time asymptotic is related to the limit $\varepsilon \to 0$
- Singularity of the equation : critical layer $U(y_c) = c$

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Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary streamlines: $U'(y_0) = 0$

- Mathematical methods: Laplace transform and detailed analysis of singularities due to the critical layers and stationary streamlines.
- By contrast with what was previously believed, we can deal with the difficulty related to the stationary streamlines.

Theory: a) Asymptotic oscillatory vorticity field

$$\omega'(y,t) \underset{t\to\infty}{\sim} \omega_\infty(y)\exp(ikU(y)t) + \mathcal{O}\left(\frac{1}{t^\alpha}\right)$$

b) DEPLETION OF VORTICITY FLUCTUATIONS:
For any stationary streamline of the flow (y_0 such that $U'(y_0) = 0$)

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- + Prediction of the asymptotic vorticity $\omega_\infty(y)$.

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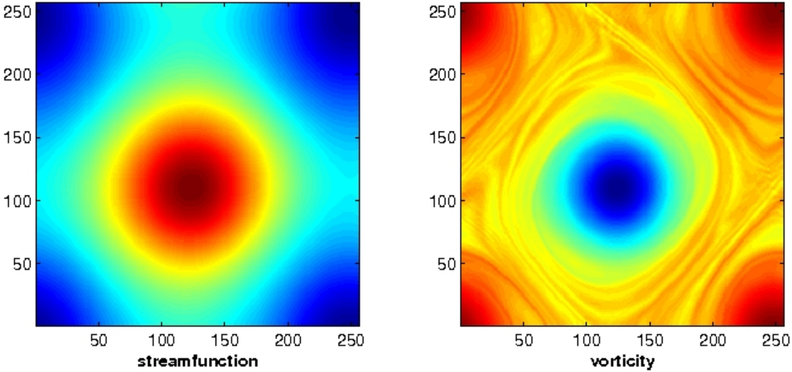
Non-Equilibrium Stationary States: Dipoles



Depletion of vorticity fluctuations in vortices

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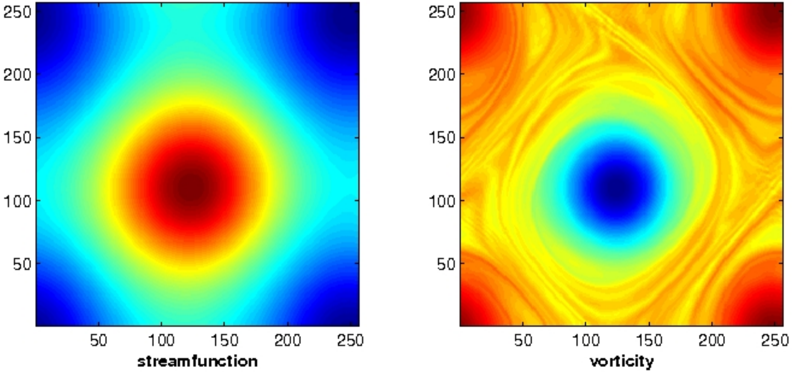

Depletion of vorticity fluctuations in vortices

Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary streamlines: the velocity field

Theorem: asymptotically algebraically decaying velocity field

$$v_x(y,t) \underset{t\to\infty}{\sim} \frac{v_{x,\infty}(y)}{t}\exp\left(-ikU(y)t\right)$$

$$v_y(y,t) \underset{t\to\infty}{\sim} \frac{v_{y,\infty}(y)}{t^2}\exp\left(-ikU(y)t\right)$$

- What about stationary streamlines? They should give contributions of order $1/t^{1/2}$!
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Irreversible Relaxation of 2D Euler Eq.

Relaxation towards Young measures. Young measure entropy

- $\Omega(x,y,0) = \Omega_0(y) + \varepsilon\omega(x,y,0)$. Then we can prove that, for the 2D linearized Euler eq.

$$\Omega(x,y,0) \underset{t\to\infty}{\sim} \Omega_0(y) + \varepsilon\omega_\infty(x-U(y)t,y,0).$$

- We define the measure

$$\rho(\sigma,y) = \frac{1}{T(y)}\int_0^{T(y)} dt\,\delta\left(\sigma - [\Omega_0(y) + \varepsilon\omega_\infty(x-U(y)t,y,0)]\right)$$

Then (theorem for the linearized equation)

$$\Omega(x,y,0) \underset{t\to\infty}{\to} Y[\rho(\sigma,y)],$$

where $Y[\rho]$ is the Young measure defined by ρ.

- Conjecture: this also true for the 2D Euler eq. (nonlinear)
- This could define the notion of an entropy for the dynamical solutions to the 2D Euler equations and its irreversible evolution.

Outline

An Example: the Kolmogorov Flow

- $U(y) = \cos(y)$ in the doubly periodic domain $(0, 2\pi/\delta)\, x\, (0, 2\pi)$; δ is the aspect ratio
- Two stationary streamlines $U'(y_0) = 0$, for $y_0 = 0$ or $y_0 = \pi$
- Usual criteria for stability (Rayleigh, Arnold) do not apply
- The Kolmogorov flow is stable for $\delta > 1$ (Lyapounov stability), spectrally and linearly stable (easily proved).
- This flow has no neutral modes.

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Asymptotic behavior of the 2D Euler Eq.

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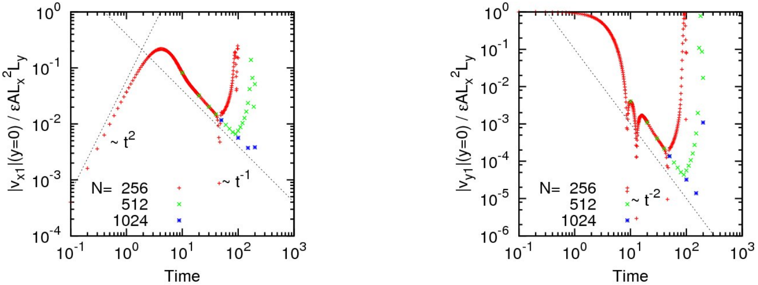

Evolution of the perturbation velocity, components $v_x(t)$ and $v_y(t)$, advected by a constant shear flow $U(y)$ with stationary streamlines

The velocity perturbation converges to zero (asymptotic stability) even without dissipation

F. Bouchet, and H. Morita, Physica D 2010,

Asymptotic Behavior of the Euler Eq.: Conclusions

- Asymptotically oscillating vorticity fields
- Algebraic decay of the velocity field with $1/t$ or $1/t^2$ laws, whatever the cases (except at stationary streamlines).
- All cases of base flow with any type of shear have been treated.
- Depletion of the vorticity perturbation at the stationary streamlines
- Axisymmetric vortices should behave the same way.

Publications

1. F. Bouchet, Physica D, 2008 Simplified variational problems for the statistical equilibria of 2D flows.
2. F. Bouchet, and E. Simonnet, PRL (March 2009), Random changes of flow topology in 2D and geophysical turbulence.
3. A. Venaille, and F. Bouchet, PRL (March 2009), Phase transitions, ensemble inequivalence and Fofonoff flows.
4. F. Bouchet, and M. Corvellec, JStatMech, 2010, Invariant measures of the 2D Euler and Vlasov equations
5. F. Bouchet, and H. Morita, Physica D 2010, Asymptotic stability of the 2D Euler and of the 2D linearized Euler equations
6. A. Venaille, and F. Bouchet, accepted in J. Phys. Oceanography. Oceanic rings and jets as statistical equilibrium states
7. A. Venaille, and F. Bouchet, accepted in J. Stat. Phys., Solvable phase diagrams and ensemble inequivalence for two-dimensional and geophysical turbulent flows
8. F. Bouchet, and A. Venaille, accepted for publication in Physics Reports, Statistical mechanics of two-dimensional and geophysical flows

Summary

Messages :

- Ocean rings are statistical equilibria - Ocean strong mid basin eastward jets are marginally unstable statistical equilibria
- We predicted and observed non-equilibrium phase transitions for the 2D-Stochastic Navier Stokes equations
- We observed such non-equilibrium phase transition in experiments
- We predicted the long time asymptotics of the inviscid damping of the 2D Euler equations