

Casimir Cascades in Two-Dimensional Turbulence

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2D Turbulence in Fourier Space

- Navier–Stokes equation for vorticity $\omega = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}$:

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = -\nu \nabla^2 \omega + f.$$

- In Fourier space:

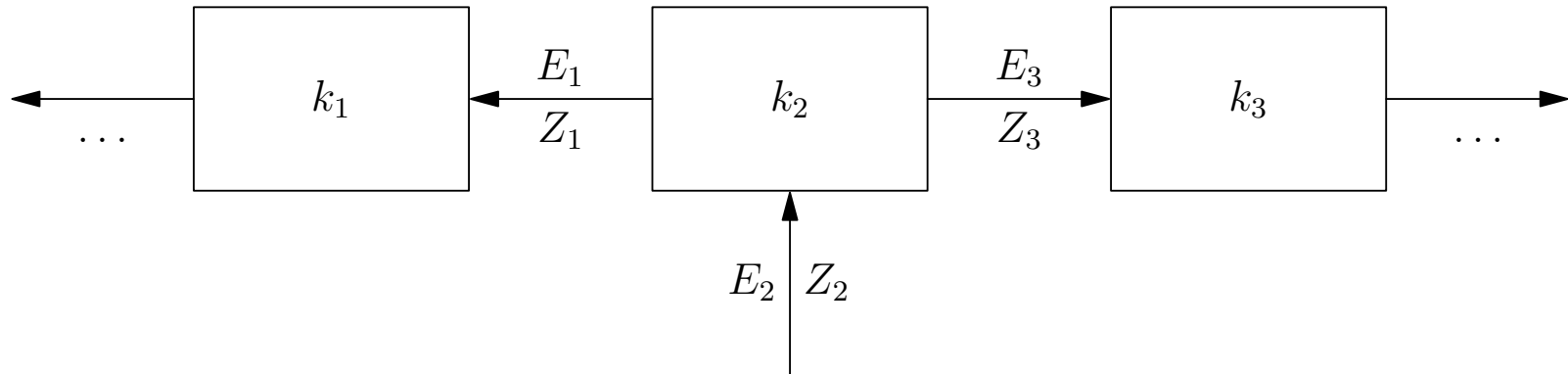
$$\frac{\partial \omega_{\mathbf{k}}}{\partial t} = S_{\mathbf{k}} - \nu k^2 \omega_{\mathbf{k}} + f_{\mathbf{k}},$$

where $S_{\mathbf{k}} = \sum_{\mathbf{p}} \frac{\hat{\mathbf{z}} \times \mathbf{p} \cdot \mathbf{k}}{p^2} \omega_{\mathbf{p}}^* \omega_{-\mathbf{k}-\mathbf{p}}^*.$

- When $\nu = 0$ and $f_{\mathbf{k}} = 0$:

energy $E = \frac{1}{2} \sum_{\mathbf{k}} \frac{|\omega_{\mathbf{k}}|^2}{k^2}$ and enstrophy $Z = \frac{1}{2} \sum_{\mathbf{k}} |\omega_{\mathbf{k}}|^2$ are conserved.

Fjørtoft Dual Cascade Scenario



$$E_2 = E_1 + E_3, \quad Z_2 = Z_1 + Z_3, \quad Z_i \approx k_i^2 E_i.$$

- When $k_1 = k$, $k_2 = 2k$, and $k_3 = 4k$:

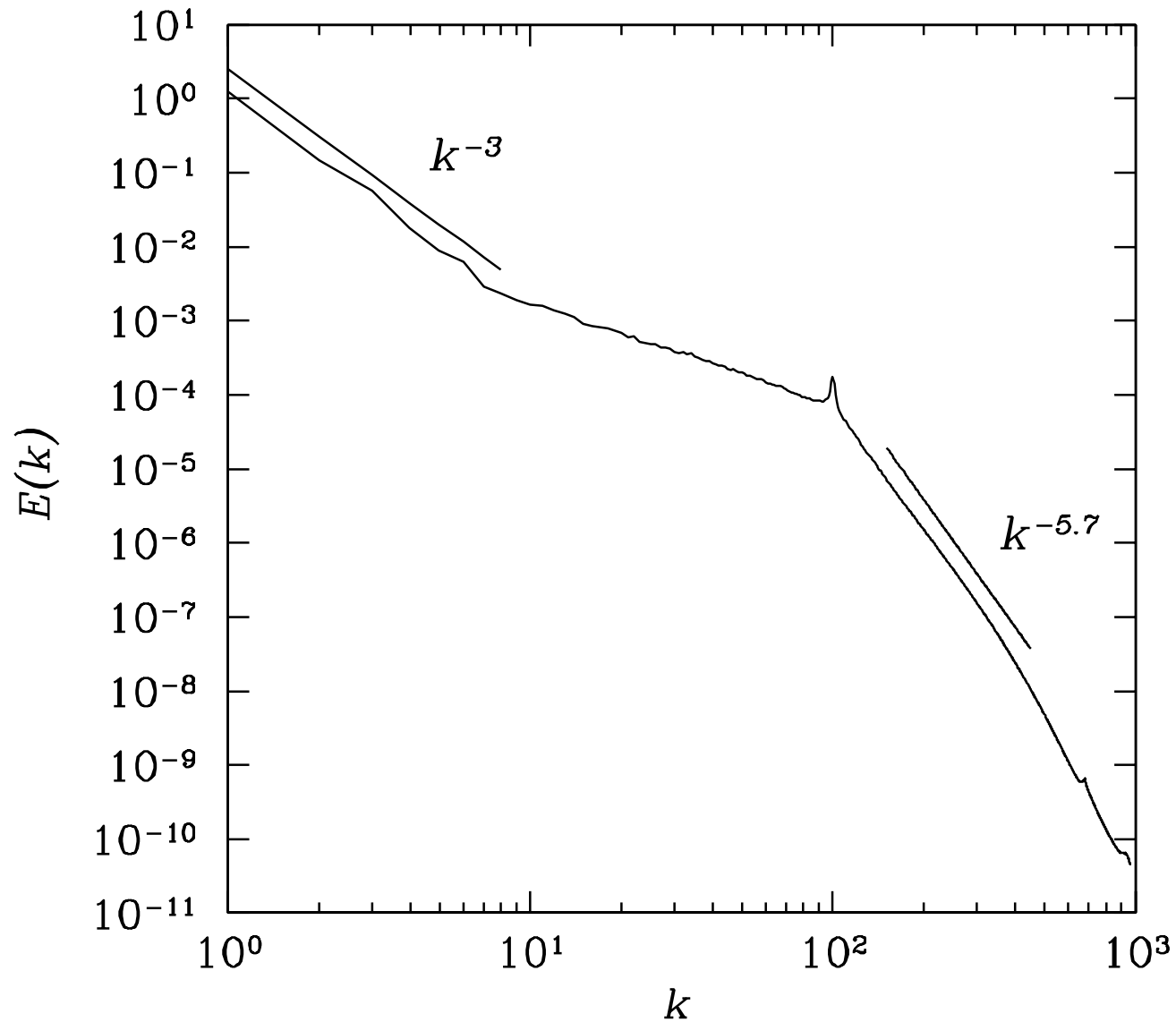
$$E_1 \approx \frac{4}{5}E_2, \quad Z_1 \approx \frac{1}{5}Z_2, \quad E_3 \approx \frac{1}{5}E_2, \quad Z_3 \approx \frac{4}{5}Z_2.$$

- Fjørtoft [1953]: energy cascades to large scales and enstrophy cascades to small scales.

Kraichnan–Leith–Batchelor Theory

- In an infinite domain
[Kraichnan 1967], [Leith 1968], [Batchelor 1969]:
 - large-scale $k^{-5/3}$ energy cascade;
 - small-scale k^{-3} enstrophy cascade.
- In a **bounded** domain, the situation may be quite different...

Long-Time Behaviour in a Bounded Domain



Tran and Bowman, PRE 69, 036303, 1–7 (2004).

Casimir Invariants

- Inviscid unforced two dimensional turbulence has uncountably many other **Casimir invariants**.
- Any continuously differentiable function of the (scalar) vorticity is conserved by the nonlinearity:

$$\begin{aligned}\frac{d}{dt} \int f(\omega) d\mathbf{x} &= \int f'(\omega) \frac{\partial \omega}{\partial t} d\mathbf{x} = - \int f'(\omega) \mathbf{u} \cdot \nabla \omega d\mathbf{x} \\ &= - \int \mathbf{u} \cdot \nabla f(\omega) d\mathbf{x} = \int f(\omega) \nabla \cdot \mathbf{u} d\mathbf{x} = 0.\end{aligned}$$

- Do these invariants also play a fundamental role in the turbulent dynamics, in addition to the quadratic (energy and enstrophy) invariants? Do they exhibit **cascades**?
- Polyakov [1992] has suggested that the higher-order Casimir invariants cascade to large scales, while Eyink [1996] suggests that they might cascade to small scales.

High-Wavenumber Truncation

- Only the quadratic invariants survive high-wavenumber truncation (Montgomery calls them **rugged invariants**).

$$\frac{\partial \omega_{\mathbf{k}}}{\partial t} = \sum_{\mathbf{p}, \mathbf{q}} \frac{\epsilon_{\mathbf{k} \mathbf{p} \mathbf{q}}}{q^2} \omega_{\mathbf{p}}^* \omega_{\mathbf{q}}^*.$$

where $\epsilon_{\mathbf{k} \mathbf{p} \mathbf{q}} = (\hat{\mathbf{z}} \cdot \mathbf{p} \times \mathbf{q}) \delta(\mathbf{k} + \mathbf{p} + \mathbf{q})$.

- Enstrophy evolution:

$$\frac{1}{2} \frac{d}{dt} \sum_{\mathbf{k}} |\omega_{\mathbf{k}}|^2 = \text{Re} \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \frac{\epsilon_{\mathbf{k} \mathbf{p} \mathbf{q}}}{q^2} \omega_{\mathbf{k}}^* \omega_{\mathbf{p}}^* \omega_{\mathbf{q}}^* = 0.$$

- Invariance of $Z_3 = \int \omega^3 dx$ follows from:

$$0 = \sum_{\mathbf{k}, \mathbf{r}, \mathbf{s}} \left[\sum_{\mathbf{p}, \mathbf{q}} \frac{\epsilon_{\mathbf{k} \mathbf{p} \mathbf{q}}}{q^2} \omega_{\mathbf{p}}^* \omega_{\mathbf{q}}^* \omega_{\mathbf{r}}^* \omega_{\mathbf{s}}^* + 2 \text{ other similar terms} \right].$$

- The absence of an explicit $\omega_{\mathbf{k}}$ in the first term means that setting $\omega_{\ell} = 0$ for $\ell > K$ **breaks the symmetry** in the summations!
- This means that high-wavenumber truncation destroys the invariance of Z_3 .
- However, since the missing terms involve $\omega_{\mathbf{p}}$ and $\omega_{\mathbf{q}}$ for \mathbf{p} and \mathbf{q} higher than the truncation wavenumber K , one might expect almost exact invariance of Z_3 for a well-resolved simulation.
- We find that this is indeed the case.

Enstrophy Balance

$$\frac{\partial \omega_{\mathbf{k}}}{\partial t} + \nu k^2 \omega_{\mathbf{k}} = S_{\mathbf{k}} + f_{\mathbf{k}},$$

- Multiply by $\omega_{\mathbf{k}}^*$ and integrate over wavenumber angle \Rightarrow
enstrophy spectrum $Z(k) = \frac{1}{2} \int |\omega_{\mathbf{k}}|^2 k d\theta$ evolves as:

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = T(k) + F(k),$$

where $T(k) = \text{Re} \int S_{\mathbf{k}} \omega_{\mathbf{k}}^* k d\theta$ and $F(k) = \text{Re} \int f_{\mathbf{k}} \omega_{\mathbf{k}}^* k d\theta$.

Nonlinear Enstrophy Transfer Function

$$\frac{\partial}{\partial t} Z(k) + 2\nu k^2 Z(k) = T(k) + F(k).$$

- Let

$$\Pi(k) \doteq \int_k^\infty T(p) dp$$

represent the nonlinear transfer of enstrophy **into** $[k, \infty)$.

- Integrate from k to ∞ :

$$\frac{d}{dt} \int_k^\infty Z(p) dp = \Pi(k) - \epsilon_Z(k),$$

where $\epsilon_Z(k) \doteq \int_k^\infty [2\nu p^2 Z(p) - F(p)] dp$ is the total enstrophy transfer, via dissipation and forcing, **out** of wavenumbers higher than k .

- A positive (negative) value for $\Pi(k)$ represents a flow of enstrophy to wavenumbers higher (lower) than k .
- When $\nu = 0$ and $f_{\mathbf{k}} = 0$:

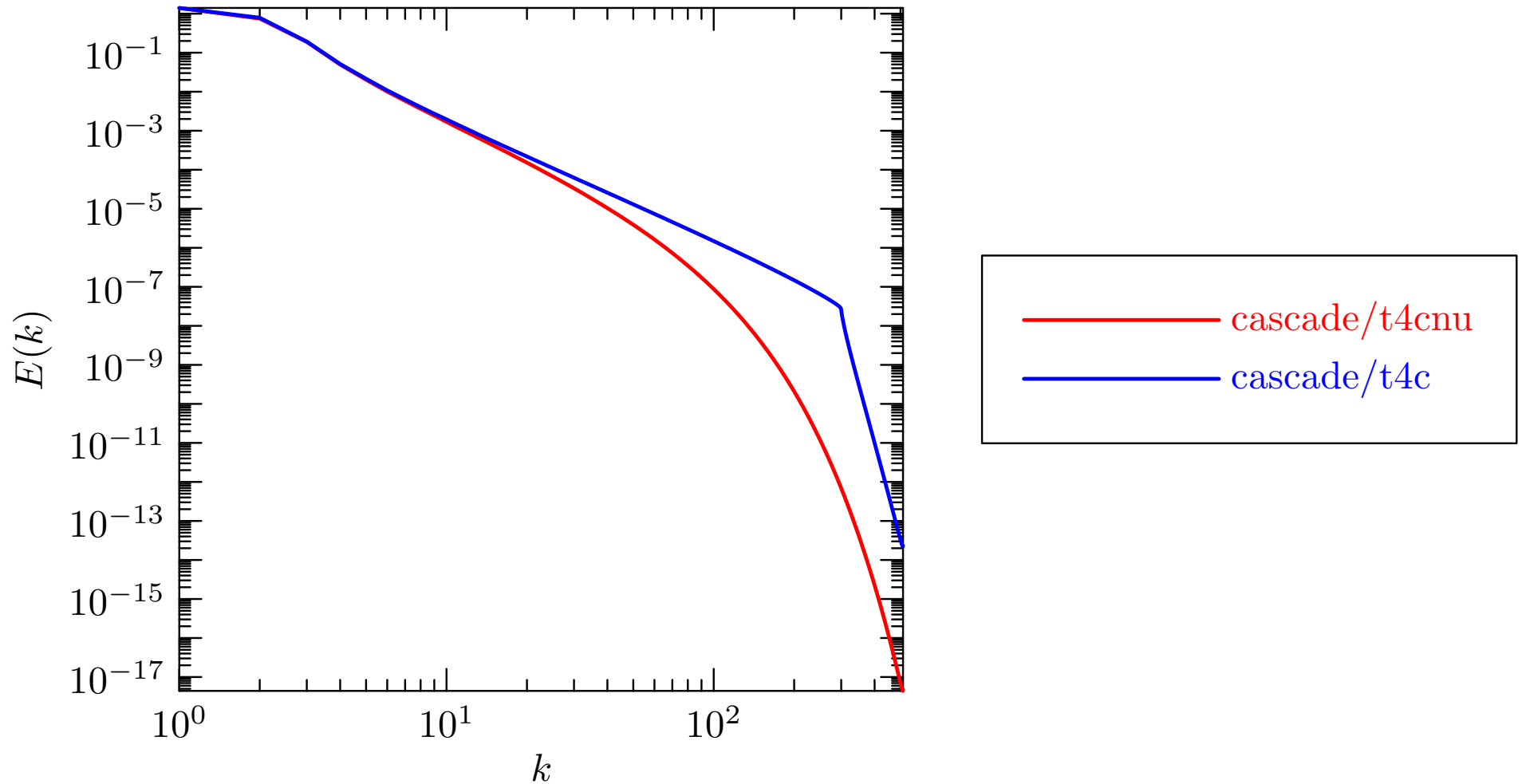
$$0 = \frac{d}{dt} \int_0^\infty Z(p) dp = \int_0^\infty T(p) dp,$$

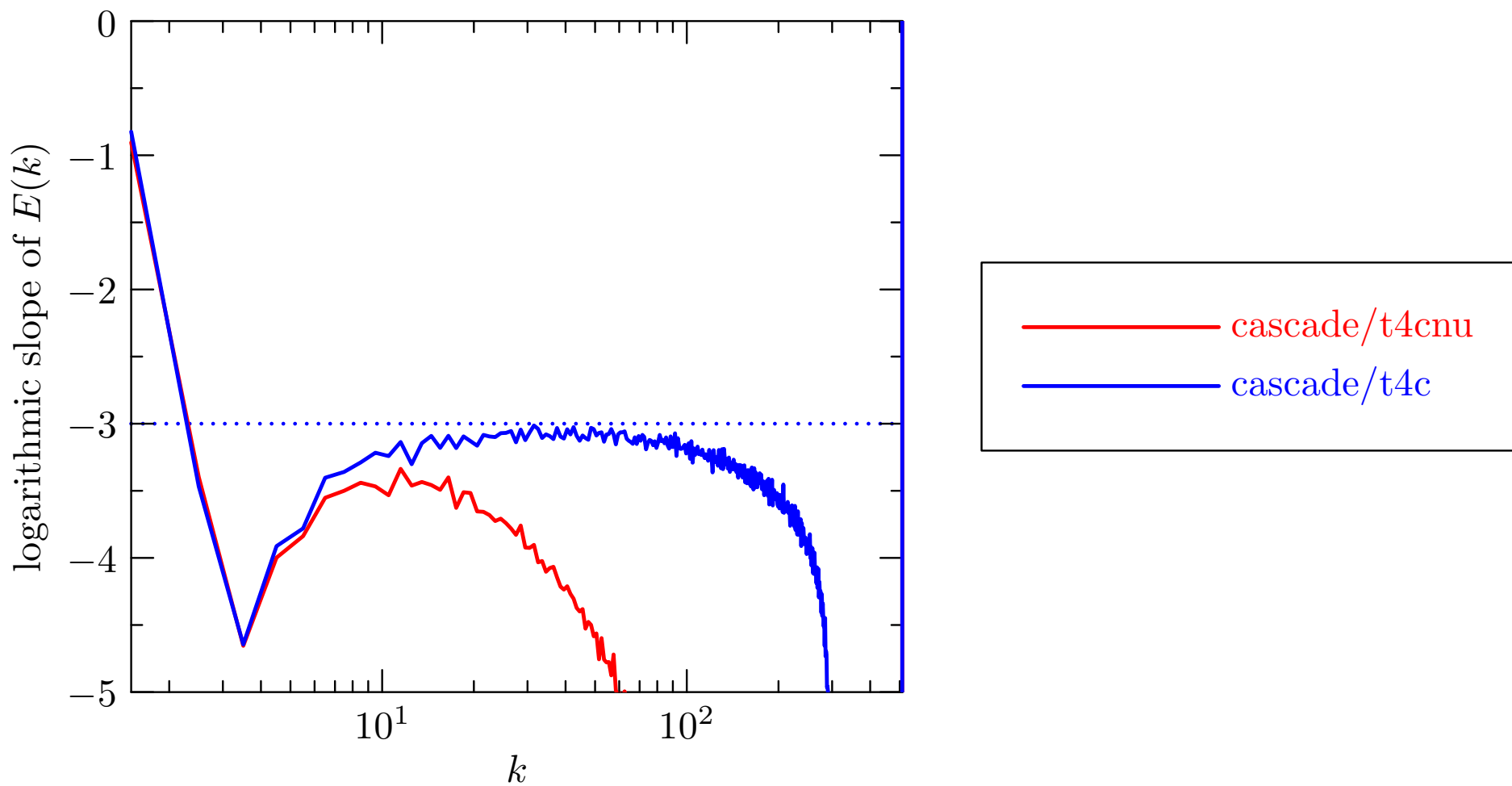
so that

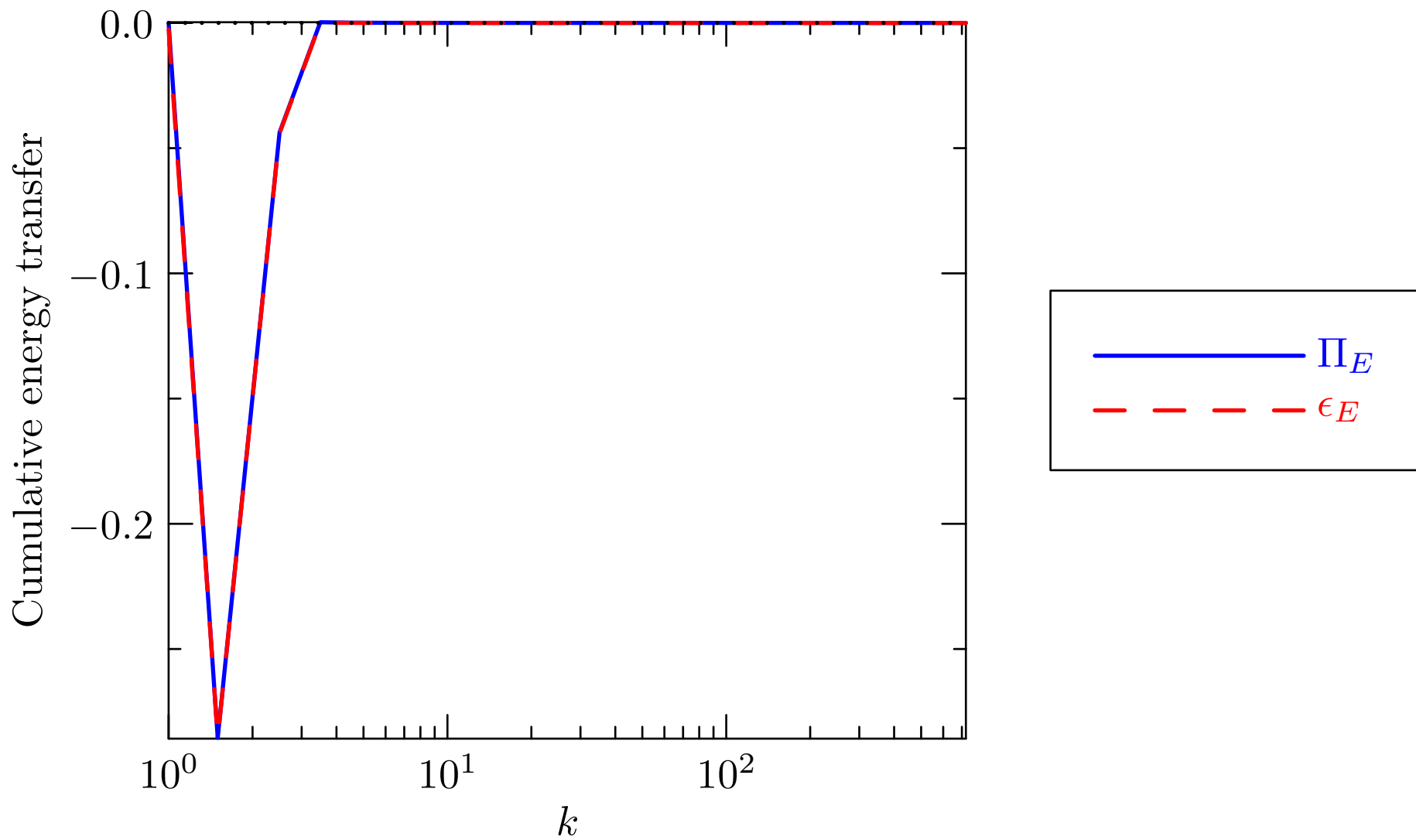
$$\Pi(k) = \int_k^\infty T(p) dp = - \int_0^k T(p) dp.$$

- Note that $\Pi(0) = \Pi(\infty) = 0$.
- In a steady state, $\Pi(k) = \epsilon_Z(k)$.
- This provides an excellent numerical diagnostic for when a steady state has been reached.

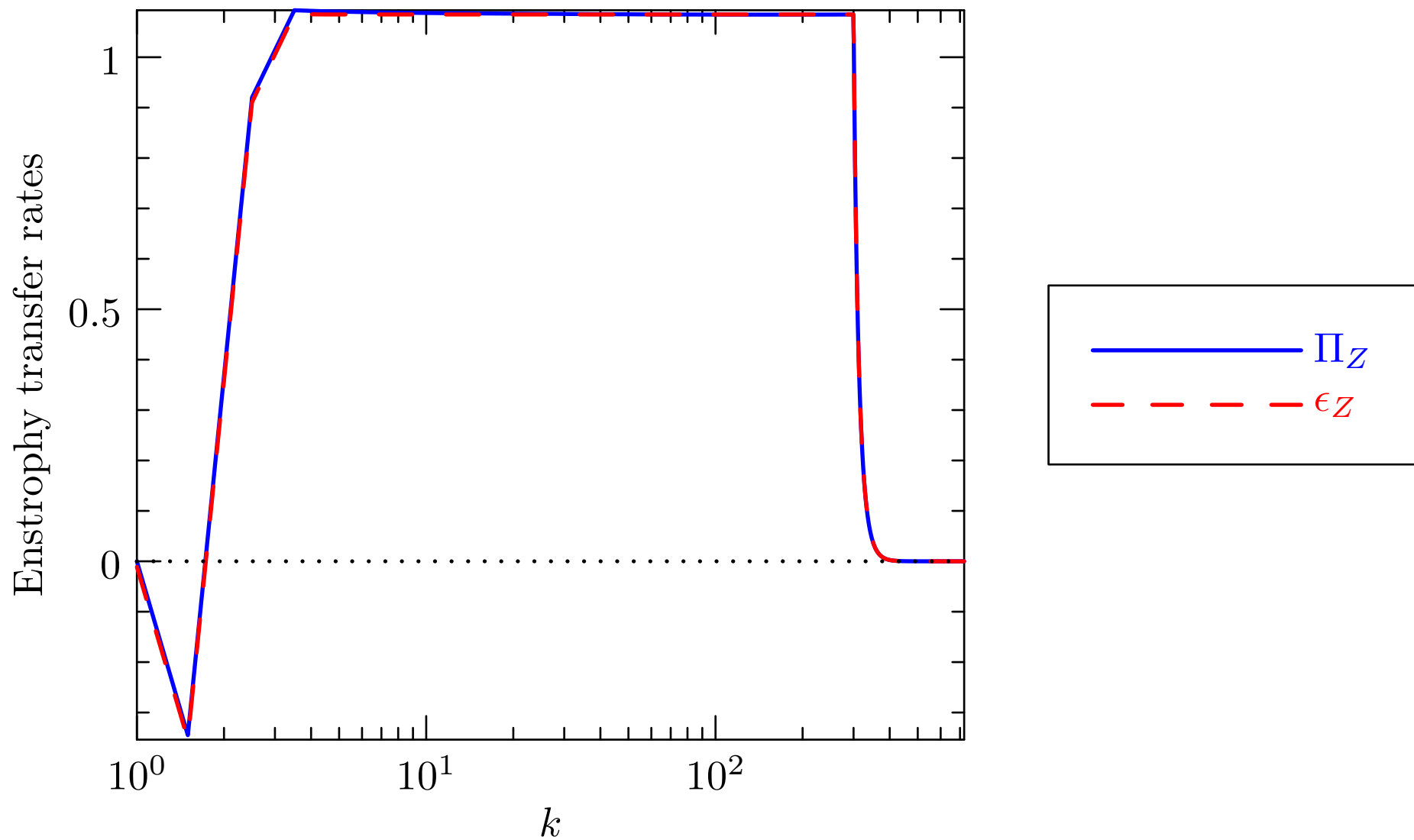
Forcing at $k = 2$, friction for $k < 3$, viscosity for $k \geq k_H = 300$ (1023×1023 dealiased modes)



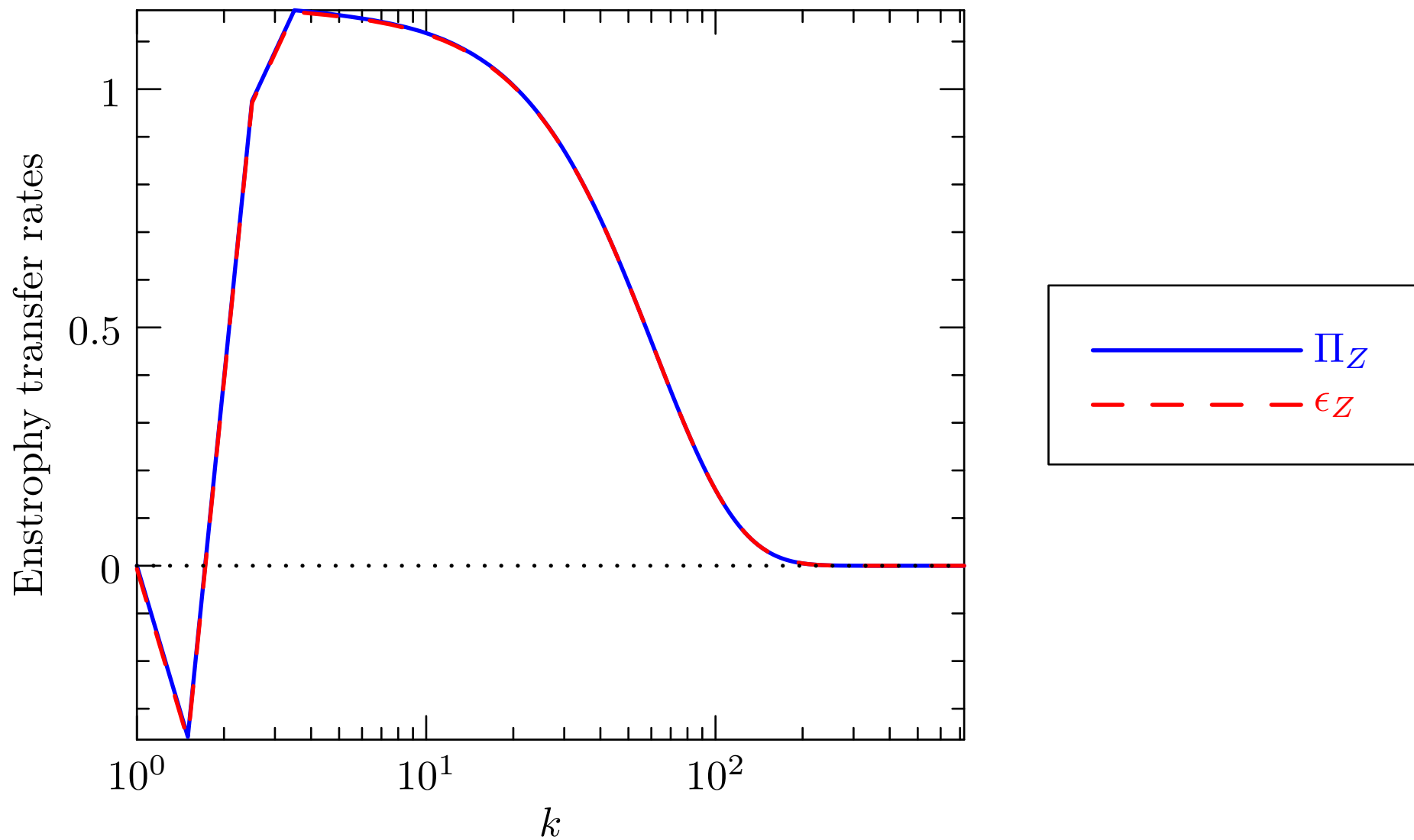




Cutoff viscosity ($k \geq k_H = 300$)

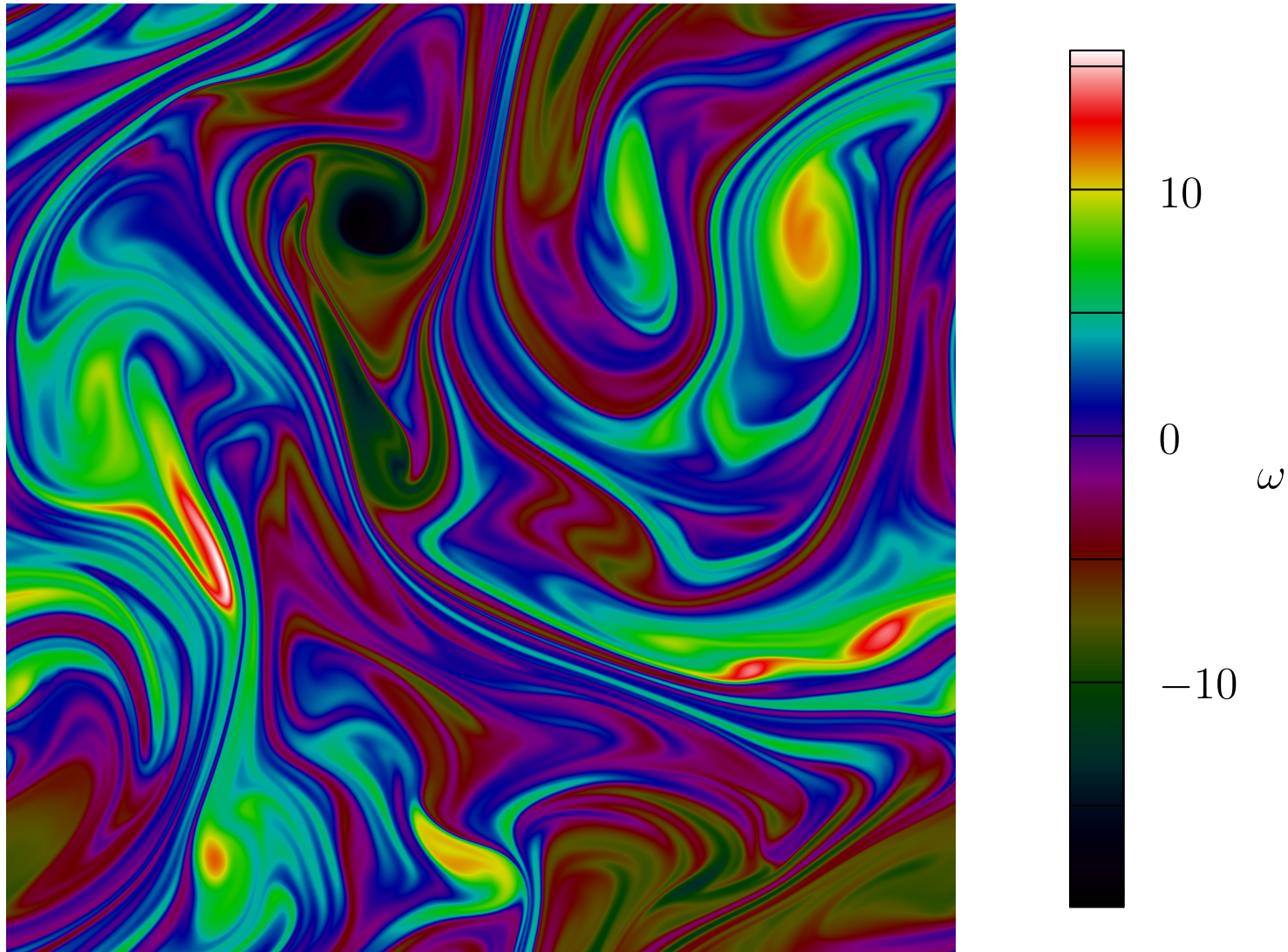


Cutoff viscosity ($k \geq k_H = 300$)

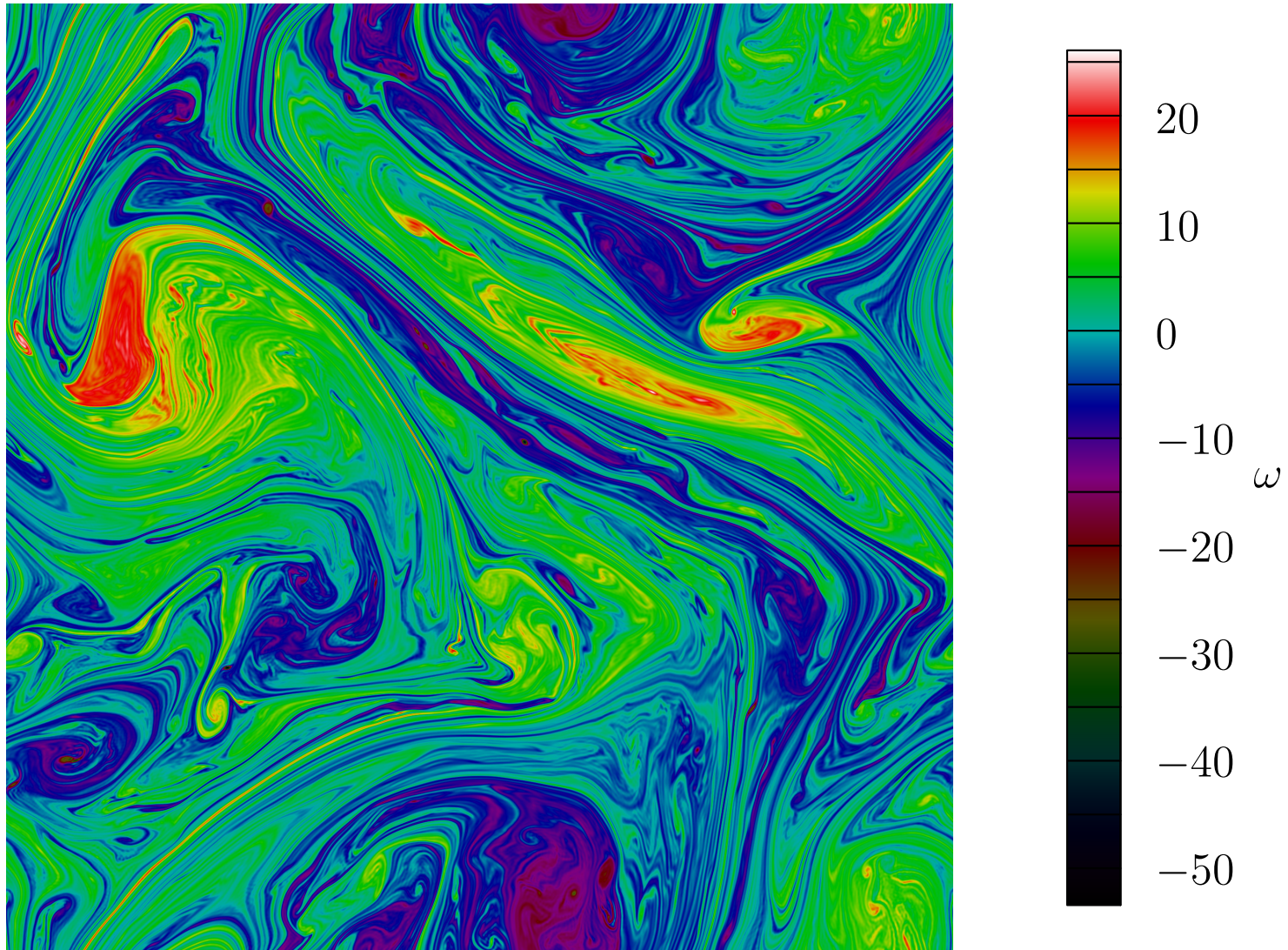


Molecular viscosity ($k \geq k_H = 0$)

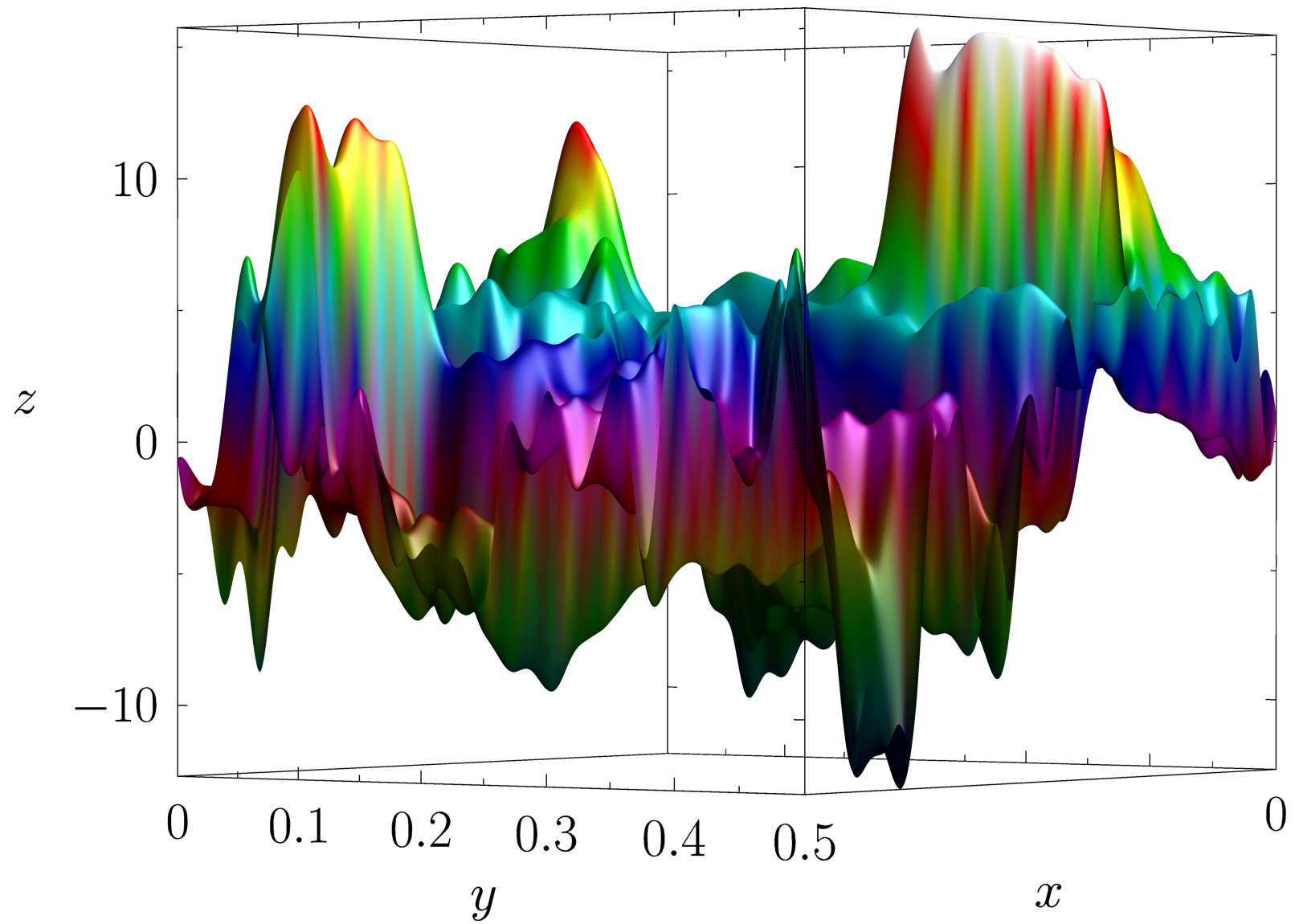
Vorticity Field with Molecular Viscosity



Vorticity Field with Viscosity Cutoff



Vorticity Surface Plot with Molecular Viscosity



Nonlinear Casimir Transfer

- Fourier decompose the fourth-order Casimir invariant

$$Z_4 = N^3 \sum_j \omega^4(x_j) \text{ in terms of } N \text{ spatial collocation points } x_j:$$

$$Z_4 = \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \omega_{\mathbf{k}} \omega_{\mathbf{p}} \omega_{\mathbf{q}} \omega_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}.$$

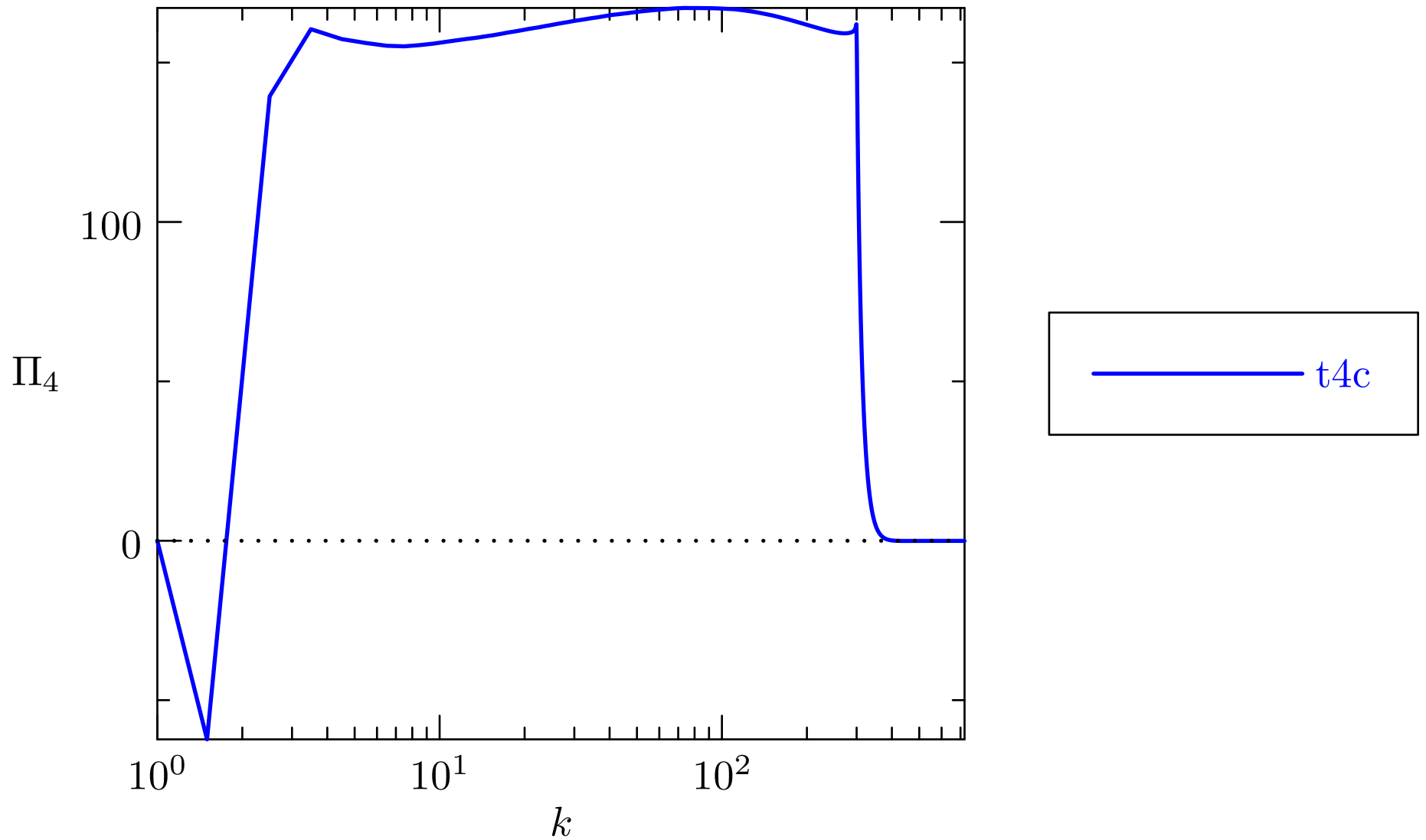
$$\frac{d}{dt} Z_4 = \sum_{\mathbf{k}} \left[S_{\mathbf{k}} \sum_{\mathbf{p}, \mathbf{q}} \omega_{\mathbf{p}} \omega_{\mathbf{q}} \omega_{-\mathbf{k}-\mathbf{p}-\mathbf{q}} + 3\omega_{\mathbf{k}} \sum_{\mathbf{p}, \mathbf{q}} S_{\mathbf{p}} \omega_{\mathbf{q}} \omega_{-\mathbf{k}-\mathbf{p}-\mathbf{q}} \right]$$

$$\frac{d}{dt} Z_4 = N^2 \sum_{\mathbf{k}} \left[S_{\mathbf{k}} \sum_j \omega^3(x_j) e^{2\pi i \mathbf{j} \cdot \mathbf{k} / N} + 3\omega_{\mathbf{k}} \sum_j S(x_j) \omega^2(x_j) e^{2\pi i \mathbf{j} \cdot \mathbf{k} / N} \right]$$

$$\doteq \sum_k T_4(k).$$

Here $S_{\mathbf{k}}$ is the nonlinear source term in $\frac{\partial}{\partial t} \omega_{\mathbf{k}}$.

Downscale Transfer of Z_4



Nonlinear transfer Π_4 of Z_4 averaged over $t \in [250, 740]$.

Dealiasing: Explicit 2/4 Zero Padding

- Computing the transfer of Z_4 requires a ternary convolution: the Fourier transform of the cubic quantity ω^3 .
- Dealiasing a ternary convolution requires a 2/4 zero padding rule (instead of the usual 2/3 rule for a quadratic convolution).

For a truncation wavenumber of 512 in each direction an explicitly dealiased pseudospectral simulation would require a buffer of size 2048×2048 .

- Instead, use *implicit padding* [Bowman & Roberts 2011]: roughly twice as fast, 1/2 of the memory required by conventional explicit padding.
- Memory savings: in d dimensions implicit padding asymptotically uses $(2/3)^{d-1}$ or $(1/2)^{d-1}$ of the memory required by conventional explicit padding.

- Highly optimized implicitly dealiased convolution routines have been implemented as a software layer **FFTW++** on top of the **FFTW** library and released under the Lesser GNU Public License.

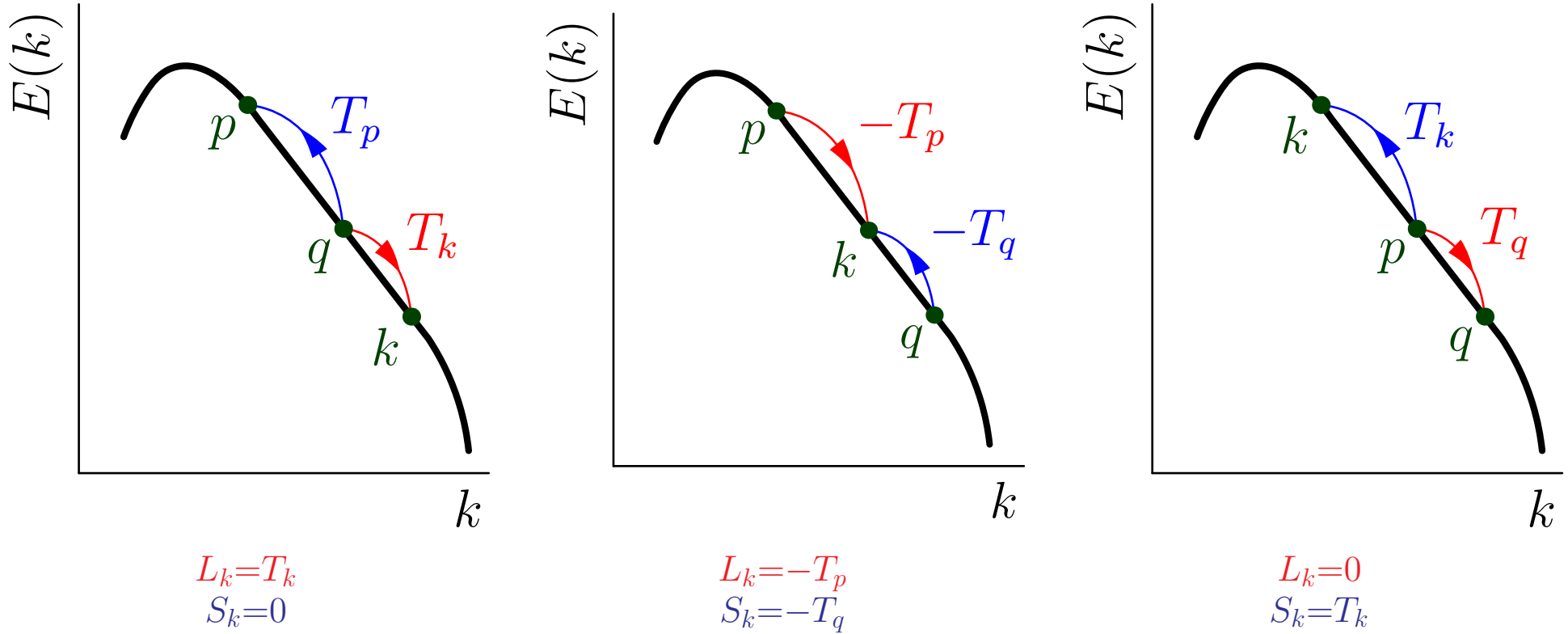
Transfer vs. Flux

- Distinguish between **transfer** and **flux**.
- The mean rate of enstrophy **transfer** to $[k, \infty)$ is given by

$$\Pi(k) = \int_k^\infty T(k) dk = - \int_0^k T(k) dk.$$

- In a steady state, $\Pi(k)$ will trivially be constant within a true inertial range.
- In contrast, the enstrophy **flux** through a wavenumber k is the amount of enstrophy transferred to small scales *via* **triad interactions involving mode k** .

Flux Decomposition for a Single $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ Triad



- Note that energy is conserved: $L_k + S_k = T_k = -T_p - T_q$. Thus

$$L_k = \text{Re} \sum_{\substack{|\mathbf{k}|=k \\ |\mathbf{p}|<k \\ |\mathbf{k}-\mathbf{p}|<k}} M_{\mathbf{k},\mathbf{p}} \omega_{\mathbf{p}} \omega_{\mathbf{k}-\mathbf{p}} \omega_{\mathbf{k}}^* - \text{Re} \sum_{\substack{|\mathbf{k}|=k \\ |\mathbf{p}|<k \\ |\mathbf{k}-\mathbf{p}|>k}} M_{\mathbf{p},\mathbf{k}-\mathbf{p}} \omega_{\mathbf{k}} \omega_{\mathbf{k}-\mathbf{p}} \omega_{\mathbf{p}}^*.$$

Conclusions

- Even though higher-order Casimir invariants do not survive wavenumber truncation, it is possible, with sufficiently well resolved simulations, to check whether they cascade to large or small scales.
- Numerical evidence suggests that in the enstrophy inertial range there is a **direct cascade** of the globally integrated ω^4 inviscid invariant to small scales.
- However, for the globally integrated ω^3 inviscid invariant, we found no systematic cascade: it appears to slosh back and forth between the large and small scales. This is expected since ω^3 does not have a definite sign.
- One should distinguish between **nonlocal transfer** and **flux**. To compute this decomposition efficiently, one needs to develop a **restricted Fast Fourier transform**.

Asymptote: 2D & 3D Vector Graphics Language



Andy Hammerlindl, John C. Bowman, Tom Prince

<http://asymptote.sf.net>

(freely available under the GNU public license)

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