Spontaneous Stochasticity,
Flux-Freezing, and Fast Turbulent Reconnection

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March 9, 2011

Richardson Two-Particle Diffusion

Volcanic ash plume over Kīlauea volcano

Meteorologist, physicist and applied mathematician Lewis Fry Richardson proposed in 1926 that particle-pairs advected by turbulence (e.g. a pair of soot particles in a volcanic plume) would have mean-square separation increasing with time as the cube power

$$\langle |\mathbf{x}_1(t) - \mathbf{x}_2(t)|^2 \rangle \sim t^3.$$

This is Richardson's t^3-law.

Scale-Dependent Eddy-Diffusivity

Reference.	K cm.^{2} sec^{-1}	l cm.
K from molecular diffusion of oxygen into nitrogen (Kaye and Laby's 'Physical and Chemical Constants'). For l see preceding discussion.	$1{\cdot}7 \times 10^{-1}$	5×10^{-2}
K at 9 metres above ground from anemometers at heights of 2, 16 and 32 metres (W. Schmidt, 'Wien. Akad. Sitzb.,' IIa, vol. 126, p. 773 (1917)).	$3{\cdot}2 \times 10^{3}$	$1{\cdot}5 \times 10^{3}$
K from anemometers at heights of 21 to 305 metres (Åkerblom, F., 'Nova Acta Reg. Soc. Upsaliensis' (1908)).	$1{\cdot}2 \times 10^{5}$	$1{\cdot}4 \times 10^{4}$
K from pilot balloons at heights between 100 and 800 metres (Taylor, 'Phil. Trans.,' A, vol. 215, p. 21 (1914), also Hesselberg and Sverdrup, 'Leipzig Geophys. Inst.,' Ser. 2, Heft 10 (1915)).	6×10^{4}	5×10^{4}
K from tracks of balloons either manned (L. F. Richardson, 'Weather Prediction by Numerical Process,' p. 221) or not manned (Richardson & Proctor, 'Royal Meteorological Society Memoirs,' No. 1).	10^{8}	2×10^{6}
Volcano ash, same reference as last	5×10^{8}	5×10^{6}
Diffusion due to cyclones regarded as deviations from the mean circulation of the latitude (Defant, 'Geog. Ann.,' H. 3, also (1921), 'Wien. Akad. Wiss. Sitzb.,' IIa, vol. 130, p. 401 (1921)).	10^{11}	10^{8}

Richardson's table of raw data

Richardson's approach was semi-empirical. By estimating "effective diffusivity" $K = \langle|\Delta\mathbf{x}|^2\rangle/t$ as a function of $\ell = \sqrt{\langle|\Delta\mathbf{x}|^2\rangle}$, he found from data that

$$K(\ell) \sim K_0 \ell^{4/3}.$$

He proposed that the probability density function of the separation vector $\boldsymbol{\ell} = \mathbf{x}_1 - \mathbf{x}_2$ would satisfy a diffusion equation

$$\partial_t P(\boldsymbol{\ell}, t) = \frac{\partial}{\partial \ell_i}\left(K(\ell)\frac{\partial P}{\partial \ell_i}(\boldsymbol{\ell}, t)\right)$$

with scale-dependent 2-particle eddy-diffusivity. This equation predicts at long times that

$$\langle|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^2\rangle \sim t^3,$$

averaging over velocity realizations.

Similarity Solution

Richardson (1926) observed that there is an exact similarity solution of his equation, given by the stretched-exponential PDF

$$P_*(\boldsymbol{\ell}, t) = \frac{A}{(K_0 t)^{9/2}} \exp\left(-\frac{9\ell^{2/3}}{4K_0 t}\right)$$

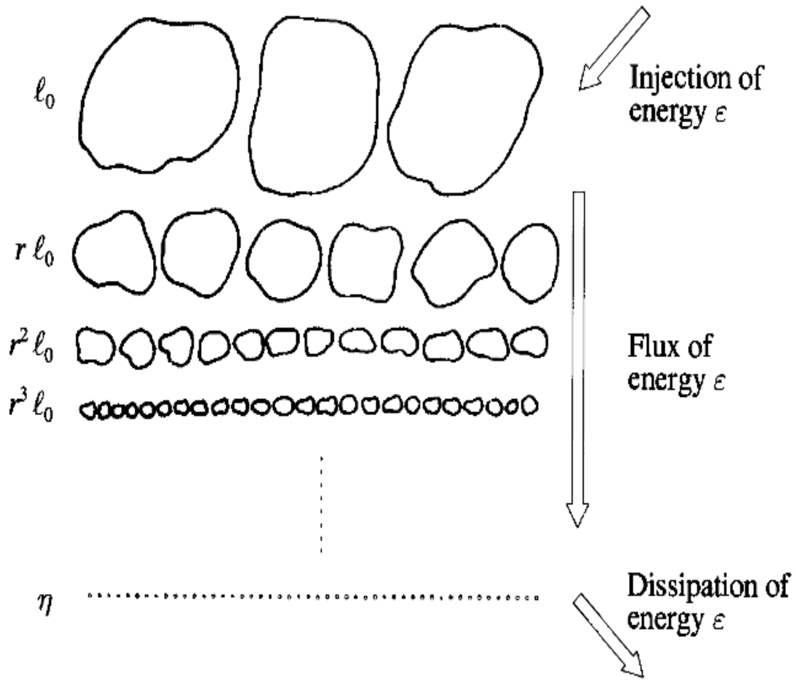
in three space dimensions. All solutions approach this self-similar form asymptotically at long times.

Averaging ℓ^2 with respect to this density yields

$$\langle \ell^2(t) \rangle = \gamma_0 t^3$$

with $\gamma_0 = 1144 K_0^3/81$.

Kolmogorov Cascade Picture



A cartoon of the Kolmogorov cascade

In the Kolmogorov (1941) picture, velocity differences across eddies of size ℓ have magnitude

$$\delta u(\ell) \sim (\varepsilon\ell)^{1/3}.$$

This increases with ℓ, so that larger turbulent eddies have larger velocities.

A pair of particles as they separate thus experience greater relative velocities as they move further apart. The outcome is an explosive separation

$$\langle \ell^2(t)\rangle \sim g_0 \varepsilon t^3,$$

even much faster than ballistic ($\propto t^2$).

The (presumed universal) constant g_0 is now usually called "Richardson's constant".

Advection by Kolmogorov Velocity

A toy calculation: Assume that $\ell(t)$ satisfies

$$\frac{d}{dt}\ell(t) = \delta u(\ell) = \frac{3}{2}(g_0\varepsilon\ell)^{1/3}.$$

Separation of variables gives the exact solution

$$\ell(t) = \left[\ell_0^{2/3} + (g_0\varepsilon)^{1/3}(t - t_0)\right]^{3/2}.$$

For $t - t_0 \gg \ell_0^{2/3}/(g_0\varepsilon)^{1/3} \equiv T_0$

$$\ell^2(t) \sim g_0\varepsilon t^3.$$

Fate of Particles Initially at the Same Point?

An odd feature of the previous result is that, if $\ell_0 = 0$, then

$$\ell^2(t) = g_0 \varepsilon (t - t_0)^3 > 0.$$

Two particles started at the *same* point at time t_0 separate to a finite distance at any time $t > t_0$!

The same oddity may be seen in Richardson's similarity solution, which satisfies at initial time $t_0 = 0$

$$P_*(\boldsymbol{\ell}, 0) = \delta^3(\boldsymbol{\ell}).$$

All particles start with separation $\ell(0) = 0$. However, $P_*(\boldsymbol{\ell}, t)$ is a smooth density for $t > 0$, so that $\ell(t) > 0$ with probability one.

Breakdown of Laplacian Determinism

According to Richardson's results, Lagrangian fluid particles that are advected by the fluid velocity $\mathbf{u}(\mathbf{x},t)$ starting at $\mathbf{x}_0$

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t),t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

have the property that there is more than one solution. Doesn't this violate the theorem on uniqueness of solutions of initial-value problems for ODE's? No!

Loophole: The theorem requires that $\mathbf{u}(\mathbf{x},t)$ be $\mathbf{x}-$differentiable. A turbulent velocity field in a Kolmogorov inertial range is only Hölder continuous

$$|\mathbf{u}(\mathbf{x}_1,t) - \mathbf{u}(\mathbf{x}_2,t)| \leq C|\mathbf{x}_1 - \mathbf{x}_2|^h$$

with exponent $h \doteq 1/3$.

Kraichnan White-Noise Advection Model

All the previous facts were noted in a seminal paper of Bernard, Gawędzki, and Kupiainen (1998). They studied a soluble model of advection by a Gaussian random velocity field with zero mean and covariance

$$\langle u_i^\nu(\mathbf{x},t)u_j^\nu(\mathbf{x}',t')\rangle = [D_0^\nu\delta_{ij} - S_{ij}^\nu(\mathbf{x}-\mathbf{x}')]\delta(t-t')$$

temporal white-noise and spatially rough for $r_\nu \ll r \ll L$,

$$S_{ij}^\nu(\mathbf{r}) = D_1[(1+h)\delta_{ij} - h\widehat{r}_i\widehat{r}_j]r^{2h}$$

with $0 < h < 1$, but smooth for $r < r_\nu$. The velocity realizations $\mathbf{u}^\nu(\mathbf{x},t)$ are incompressible and only Hölder continuous with exponent h for $\nu \equiv D_1 r_\nu^{2h} \to 0$.

Richardson's 2-particle diffusion equation holds exactly within this model

$$\partial_t P(\mathbf{r},t) = \frac{\partial}{\partial r_i}\left(S_{ij}^\nu(\mathbf{r})\frac{\partial P}{\partial r_j}(\mathbf{r},t)\right)$$

with $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. Note Richardson's original equation corresponds to $h = 2/3$ (a peculiarity of the white-noise approximation).

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Random Advection Problem

Bernard et al. (1998) study the problem of stochastic particle advection,

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{u}^{\nu}(\mathbf{x}(t), t) + \sqrt{2\kappa}\,\boldsymbol{\eta}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

perturbed by a Gaussian white-noise proportional to $\sqrt{2\kappa}$.

The transition probability for a single particle in a fixed (non-random) velocity realization $\mathbf{u}$ can be written as a path-integral:

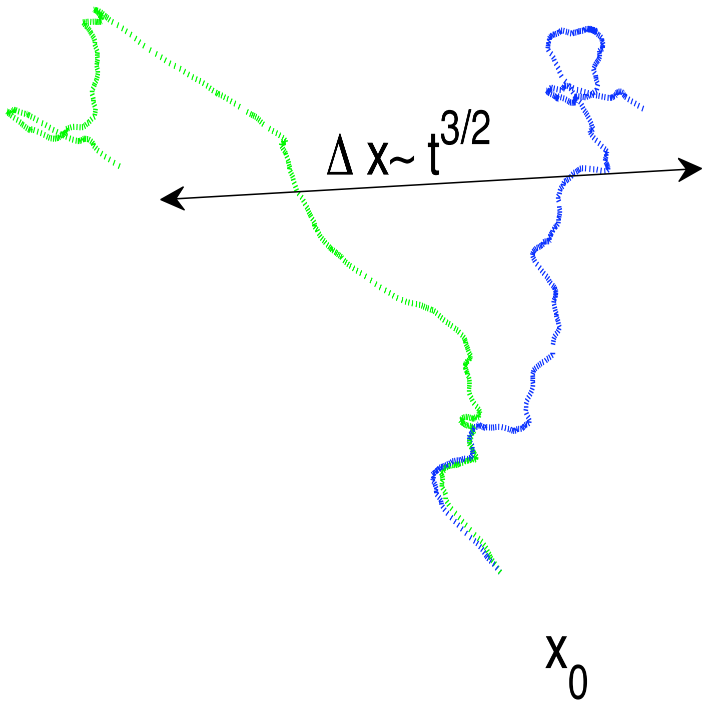
$$P_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x}, t|\mathbf{x}_0, t_0) = \int_{\mathbf{x}(t_0)=\mathbf{x}_0}^{\mathbf{x}(t)=\mathbf{x}} \mathcal{D}\mathbf{x}\ \exp\left(-\frac{1}{4\kappa}\int_{t_0}^{t} d\tau\, |\dot{\mathbf{x}}(\tau) - \mathbf{u}^{\nu}(\mathbf{x}(\tau), \tau)|^2\right)$$

In the limit $\nu, \kappa \to 0$ (or infinite Reynolds-number $Re = u_{rms}L/\nu$) with $Pr = \nu/\kappa$ fixed one would naively expect collapse to a delta-function,

$$P_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x}, t|\mathbf{x}_0, t_0) \to \delta^3(\mathbf{x} - \mathbf{x}(t)),$$

with $\mathbf{x}(t)$ the unique solution of $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t),\ \mathbf{x}(t_0) = \mathbf{x}_0$.

Spontaneous Stochasticity



The distribution does not collapse! At least in the Kraichnan model there is a nontrivial limiting distribution $P_{\mathbf{u}}(\mathbf{x}, t|\mathbf{x}_0, t_0)$ over an infinite family of solutions to the (deterministic) initial-value problem $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t),\ \ \mathbf{x}(t_0) = \mathbf{x}_0$.

There is an obvious analogy with *spontaneous symmetry-breaking*, e.g. a non-vanishing mean-magnetization in a ferromagnet even in the limit of zero external magnetic field.

More Chaotic Than Chaos!

Compare with what happens for a smooth velocity field ($h = 1$):

$$\frac{d}{dt}\ell(t) = \delta u(\ell) \doteq A\ell.$$

Separation of variables now gives the exact solution

$$\ell(t) = \ell_0 e^{A(t-t_0)}.$$

The trajectories never "forget" their initial separations: $\ell(t) \to 0$ as $\ell_0 \to 0$ for any $t > t_0$.

For smooth dynamical systems there is at most exponential deviation of trajectories. The exponential growth rate

$$\lambda = \lim_{t\to\infty} \lim_{\ell_0\to 0} \frac{1}{t-t_0} \ln\left(\frac{\ell(t)}{\ell_0}\right)$$

is the Lyapunov exponent and $\lambda > 0$ is the signature of chaos. Any imprecision in the initial data is exponentially magnified, leading to loss of predictability at long enough times.

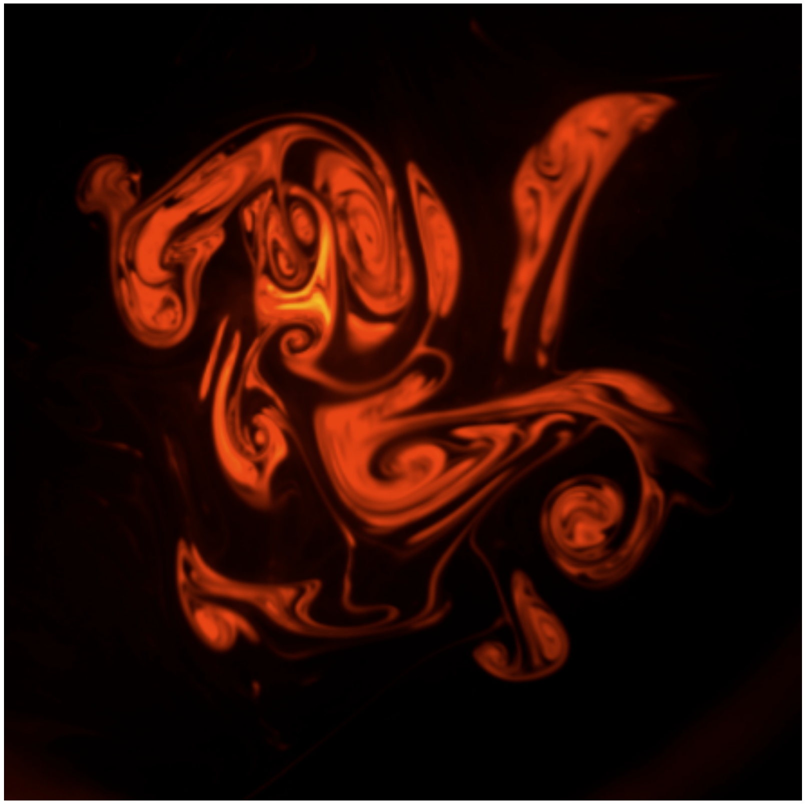
Spontaneous stochasticity corresponds instead to $\lambda = +\infty$. The solution is unpredictable for *all* future times, even with infinitely precise knowledge of the initial conditions!

Passive Scalar Advection

The scalar advection-diffusion equation

$$\partial_t\theta + (\mathbf{u}^\nu\cdot\boldsymbol{\nabla})\theta = \kappa\triangle\theta,$$

describes the evolution of dye and other passive tracers in a turbulent flow:



The exact solution is given by the Feynman-Kac formula

$$\theta(\mathbf{x},t) = \int d^3x_0\,\theta(\mathbf{x}_0,t_0)P^{\nu,\kappa}_{\mathbf{u}}(\mathbf{x}_0,t_0|\mathbf{x},t)$$

$$= \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a}\ \theta(\mathbf{a}(t_0),t_0)$$

$$\exp\left(-\frac{1}{4\kappa}\int_{t_0}^{t} d\tau\,|\dot{\mathbf{a}}(\tau)-\mathbf{u}^\nu(\mathbf{a}(\tau),\tau)|^2\right)$$

for $t_0 < t$. Now the stochastic equation

$$\dot{\mathbf{a}}(\tau) = \mathbf{u}^\nu(\mathbf{a}(\tau),\tau) + \sqrt{2\kappa}\,\boldsymbol{\eta}(\tau)$$

is solved backward in time from t to t_0, with final condition $\mathbf{a}(t) = \mathbf{x}$.

Dissipative Anomaly

Note that

$$\frac{d}{dt}\int d^3x\ \theta^2(\mathbf{x},t) = -2\kappa\int d^3x\ |\boldsymbol{\nabla}\theta(\mathbf{x},t)|^2,$$

so that, naively, the integral is conserved for $\kappa = 0$.

However, in the infinite Reynolds-number, fixed Prandtl-number limit ($\nu, \kappa \to 0$)

$$\theta(\mathbf{x},t) = \int d^3x_0\, \theta(\mathbf{x}_0,t_0) P_{\mathbf{u}}(\mathbf{x}_0,t_0|\mathbf{x},t).$$

Heuristically, molecular diffusion is replaced by turbulent diffusion. In particular, the scalar intensity is still dissipated

$$\int d^3x\ \theta^2(\mathbf{x},t) < \int d^3x\ \theta^2(\mathbf{x},t_0), \quad t > t_0$$

even as $\nu, \kappa \to 0$!

This is the scalar analogue of a dissipative anomaly of Onsager (1949) for fluid turbulence.

The Lagrangian mechanism is "spontaneous stochasticity."

Laboratory Experiment

M. Bourgoin et al. Science **311** 835–38 (2006)

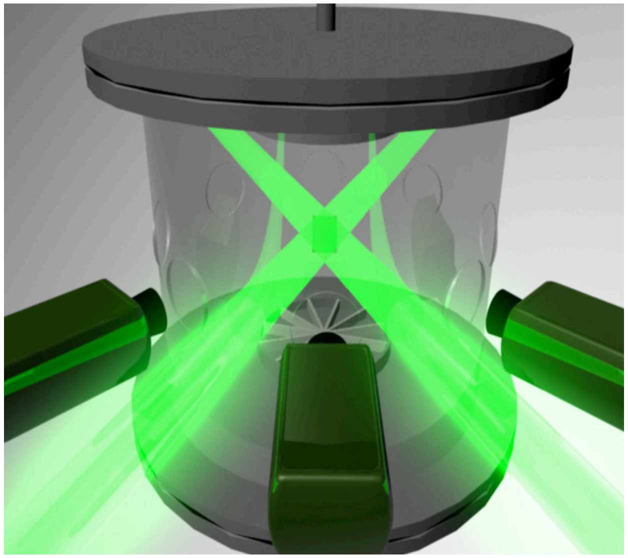
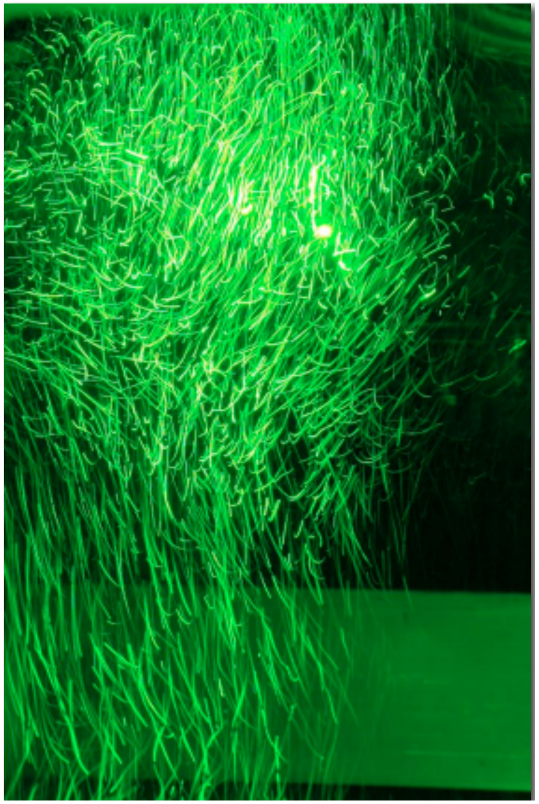


Figure 1. Sketch of the experimental setup. The three cameras have an angular separation of approximately 45° and are arranged in the forward scattering direction from both lasers.



Experimental Results

Reynolds numbers up to $Re_\lambda = 815$

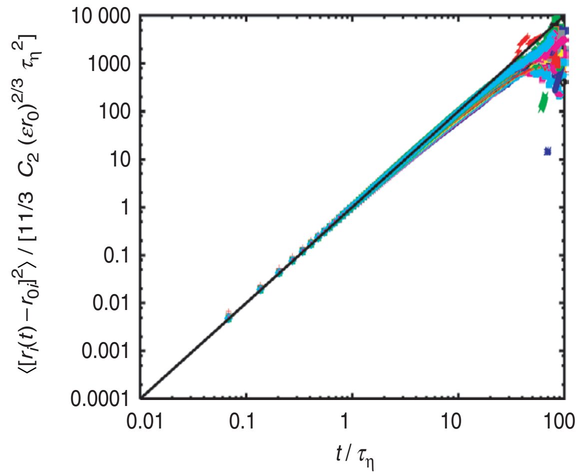


Figure 3. Scale collapse of the mean-square particle separation at $R_\lambda = 815$. The relative dispersion $\langle|\mathbf{r}(t) - \mathbf{r}_0|^2\rangle$ scaled by $(11/3)C_2(\epsilon r_0)^{2/3}$ is plotted for 50 different bins of initial separations, ranging from 0–1 mm (≈0–43η) to 49–50 mm (≈2107–2150η). The solid line is a pure t^2 power law, and is not a fit. The data collapse on to the t^2 law almost perfectly [21].

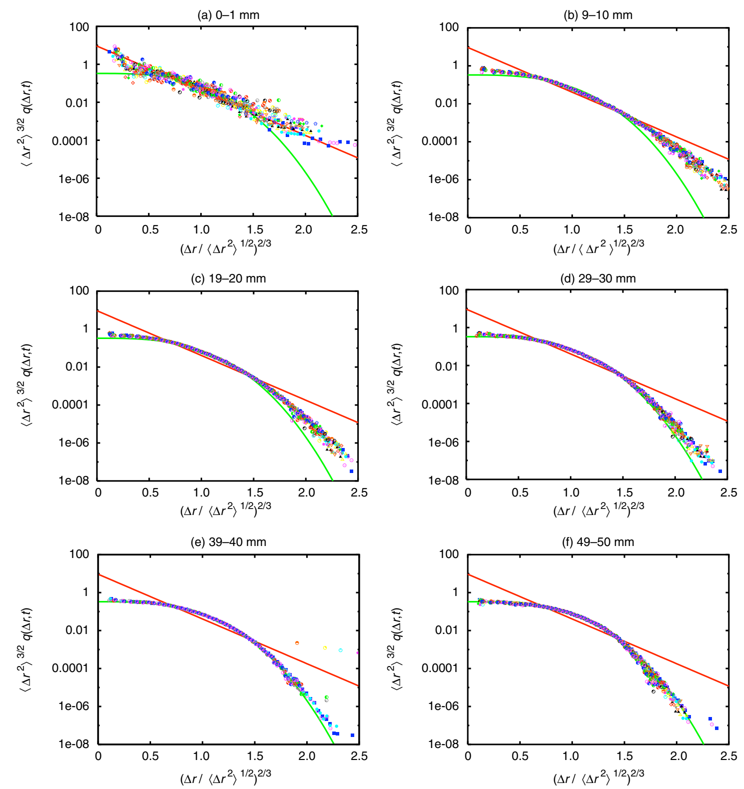


Figure 10. The distance neighbour function for different initial separations at $R_\lambda = 815$. The red straight line is Richardson's predicted PDF, while the green curved line is Batchelor's. The symbols show the experimental measurements. Each plot shows a different initial separation; for each initial separation, PDFs from 20 times ranging from τ_η to $20\tau_\eta$ are shown.

Modified Richardson Scaling

More success in looking for the modified Richardson scaling law:

$$\langle r^{2/3}(t)\rangle - r_0^{2/3} \sim C_R \varepsilon^{1/3} t.$$

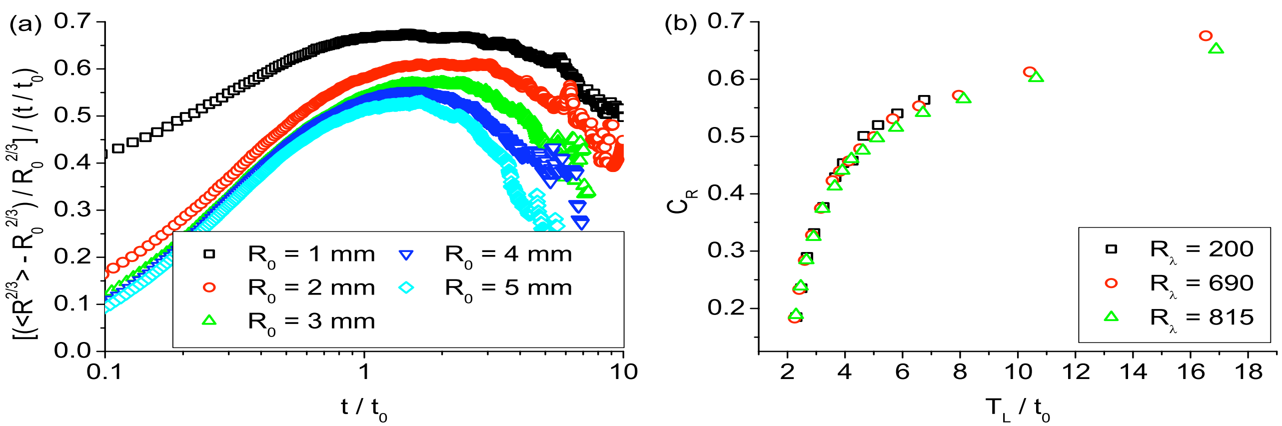


Figure 1. (a) The separation of two particles in time, compared with a modified Richardson–Obukhov law, for different initial separations, ranging from 1 to 5 mm. The Reynolds number is fixed at $R_\lambda = 690$, with a Kolmogorov scale of $\eta = 30\,\mu$m. We observe similar behavior at other Reynolds numbers. (b) The change of C_{R} with T_{L}/t_0, shown for three Reynolds numbers.

Note that $t_0 \equiv (r_0^2/\varepsilon)^{1/3}$.

Göttingen Turbulence Tunnel

Apparatus	P (bar)	ν (m^2/s)	u' (m/s)	ϵ (m^2/s^3)	ℓ (m)	λ (μm)	η (μm)	τ_η (ms)	R_λ
SF_6 tunnel	15	1.5×10^{-7}	1.0	1.2	0.45	1400	7.3	0.36	9600
air tunnel	1	1.5×10^{-5}	1.2	3.9	0.4	9100	172	2.0	730
SF_6 tank	15	1.5×10^{-7}	1.0	5.5	0.094	648	5.0	0.17	4360
water tank	1	8×10^{-7}	2.2	59	0.094	1000	9.7	0.12	2800

Numerical Simulations

L. Biferale et al. Phys. Fluids **17** 115101 (2005) employed numerical simulations of the Navier-Stokes fluid equations on a 1024^3 grid at Reynolds numbers up to $Re_\lambda = 284$

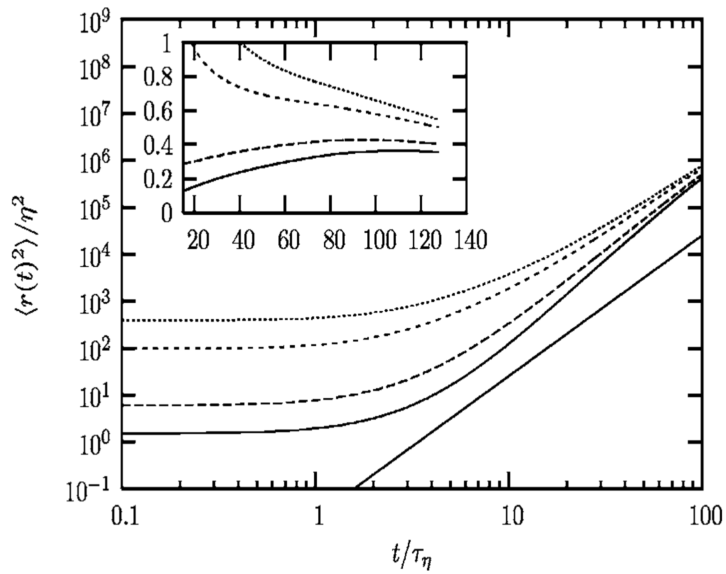


FIG. 1. The evolution of $\langle r(t)^2\rangle/\eta^2$ vs t/τ_η for the initial separations $r_0 = 1.2\eta$, $r_0=2.5\eta$, $r_0=9.8\eta$, and $r_0=19.6\eta$. The straight line is proportional to t^3. Inset: $\langle r(t)^2\rangle/\varepsilon t^3$ for the same four initial separations starting from $t/\tau_\eta \sim 15$.

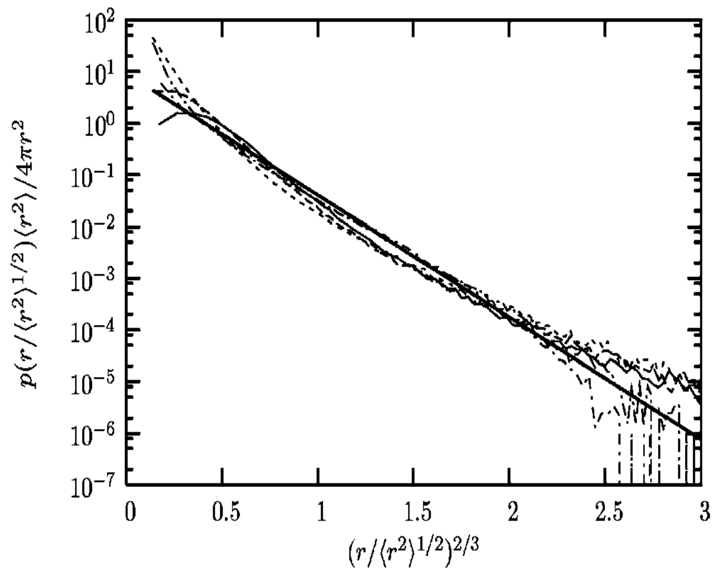


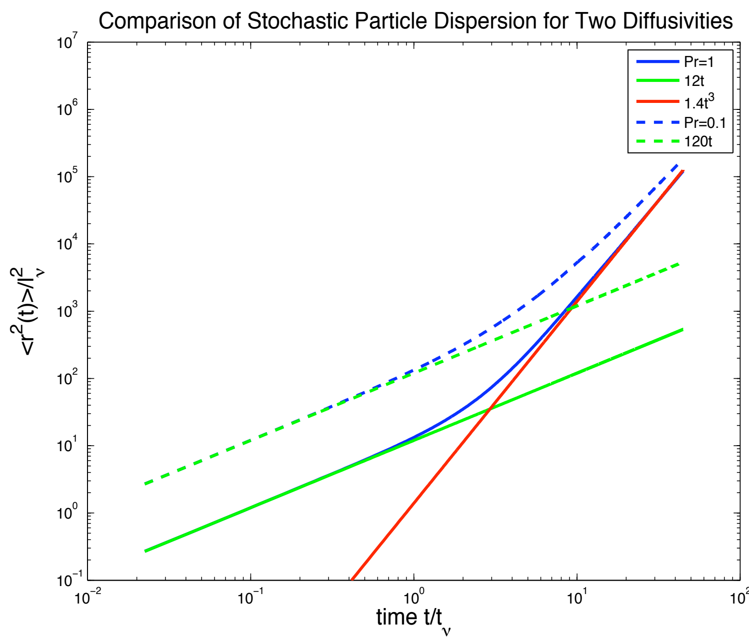
FIG. 2. Comparison of the Richardson PDF with the DNS data. The curves refer to data for $r_0=1.2\eta$ at $t=5.2\tau_\eta$ (solid line), $t=7\tau_\eta$ (long dashed line), $t=14\tau_\eta$ (short dashed line), $t=42\tau_\eta$ (dotted line), and $t=70\tau_\eta$ (dot-dashed line). The thick solid line is the Richardson PDF (2).

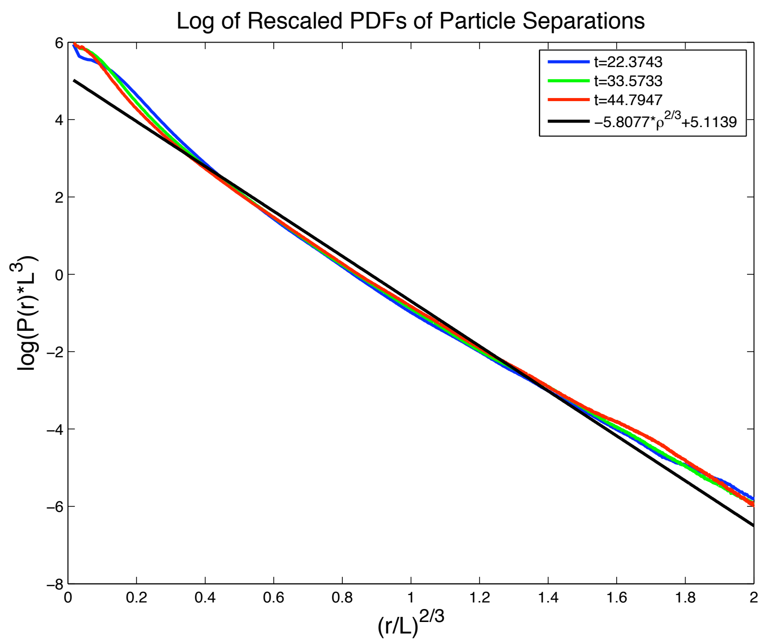
My Numerical Results

I have performed a study using turbulence data from numerical simulations on a 1024^3 grid at Reynolds number $Re_\lambda = 433$, available on-line in the public database at Johns Hopkins:

`http://turbulence.pha.jhu.edu`

I track 1024 particles (5×10^5 pairs) that all satisfy $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t) + \sqrt{2\kappa}\,\boldsymbol{\eta}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$. Below $\kappa = \nu$ and 10ν. The results are averaged over 256 initial points $\mathbf{x}_0$:





Magnetohydrodynamics

Hannes Alfvén, physics Nobel laureate (1970) — "for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics"

Sufficiently collisional plasmas and liquid metals obey the magnetohydrodynamic equations:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{4\pi\rho}(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} + \nu \triangle \mathbf{u}$$

$$\partial_t \mathbf{B} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{B} = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{u} + \lambda \triangle \mathbf{B}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0,$$

for an incompressible plasma. $\nu = \mu/\rho$ is the kinematic viscosity and $\lambda = \eta c^2/4\pi$ is the magnetic diffusivity for the (Spitzer) resistivity η.

The first MHD equation is Newton's 2nd law and the second MHD equation—called the induction equation—is a combination of Faraday's law and Ohm's law.

Magnetic Flux-Freezing

Ideal MHD ($\lambda = 0$) predicts that magnetic flux is "frozen-in" to the plasma (Alfvén, 1942): magnetic field lines are advected by the plasma fluid velocity.

Proof: The Lagrangian time-derivative (moving with the fluid) of $\mathbf{B}$ is, by the ideal induction equation,

$$\frac{d}{dt}\mathbf{B} = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{u}.$$

Compare with the evolution equation of an infinitesimal line-element $\delta\boldsymbol{\ell} = \mathbf{x}_1 - \mathbf{x}_2$ advected by the fluid:

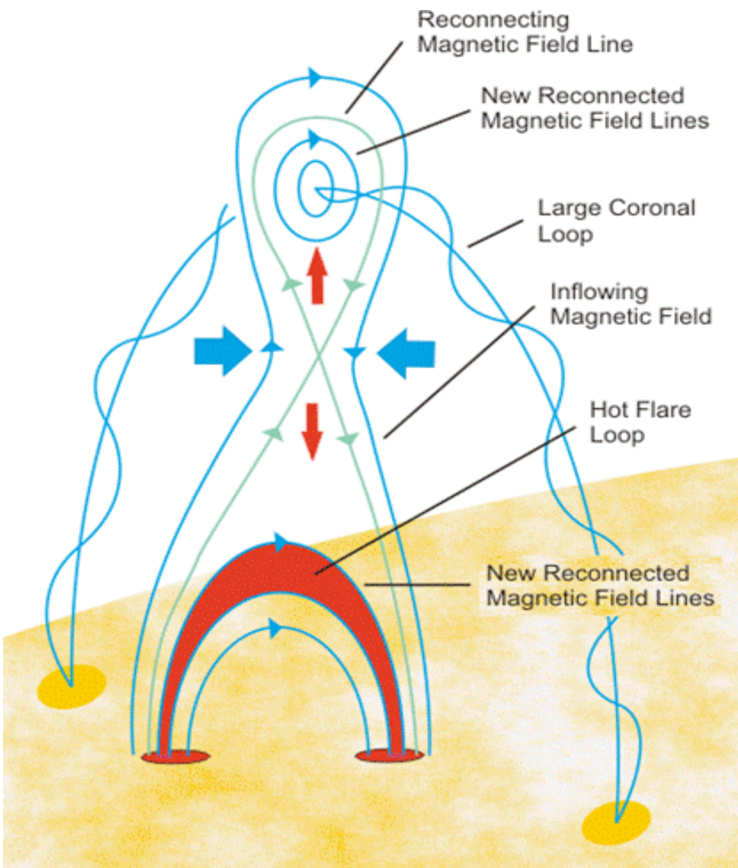
$$\frac{d}{dt}\delta\boldsymbol{\ell} = \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2 = \mathbf{u}(\mathbf{x}_1, t) - \mathbf{u}(\mathbf{x}_2, t) \doteq (\delta\boldsymbol{\ell}\cdot\boldsymbol{\nabla})\mathbf{u}.$$

The equations are identical!

"The most important property of an ideal plasma is flux freezing." —R. M. Kulsrud, *Plasma Physics for Astrophysics* (2005)

Flux-freezing is used to explain,e.g., the magnetization of white dwarfs and neutron stars, the low angular momentum of stars, the spiral structure of lines of force in the solar wind, etc.,etc.!

The Magnetic Reconnection Problem



"Flux-frozenness" often fails! Topology changes of magnetic field occur at very fast rates in solar flares and coronal mass ejections, e.g. releases times in flares range from 15 minutes to several hours.

Other examples:

* If flux-freezing held during star-formation, the magnetic pressure of in-falling field-lines would be so great as to prevent gravitational collapse altogether.

*The tangled line-structure in small-scale dynamos would quench the exponential growth of magnetic field, etc.

Is an MHD Solution Possible?

It is usually argued that magnetic reconnection cannot be explained by resistive MHD, because the violations of flux-freezing are too slow at high conductivity:

"Flux freezing is a very strong constraint on the behavior of magnetic fields in astrophysics. As we show in chapter 3, this implies that lines do not break and their topology is preserved. The condition for flux freezing can be formulated as follows: In a time t, a line of force can slip through the plasma a distance

$$\ell = \sqrt{\frac{\eta c^2 t}{4\pi}} \tag{1}$$

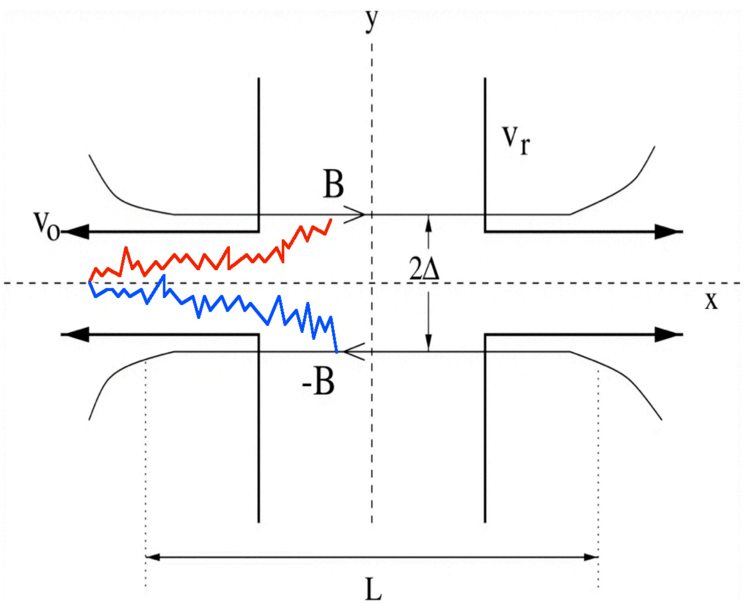
If this distance ℓ is small compared to δ, the scale of interest, then flux freezing holds to a good degree of approximation."—R. M. Kulsrud (2005), Ch.13, Magnetic Reconnection

With $\ell = 10^4$ km and $\eta c^2/4\pi = 10^4$ cm/sec, Kulsrud's equation (1) predicts a time-scale for solar flares of about 3 million years!

Most attempts to explain *fast magnetic reconnection* appeal to microscopic mechanisms besides Spitzer resistivity which can decouple plasma particles and field lines over larger distances, e.g. anomalous resistivity, Hall MHD effect, etc.

Sweet-Parker Reconnection Solution

By assuming a magnetic geometry with very thin current sheets, Sweet and Parker obtained an accelerated rate of reconnection.



In an Alfvén crossing time $t_A = L/v_A$, **resistive diffusion** of field-lines gives

$$\Delta \simeq \sqrt{\lambda t_A} \simeq L/\sqrt{S}$$

with $S = v_A L/\lambda$; e.g. see Kulsrud, (2005).

With $v_0 \simeq v_A$, mass conservation $v_0 \Delta = v_R L$ implies that

$$v_R = v_A/\sqrt{S}.$$

Sweet-Parker (1958) theory is recovered.

At the high Lundquist numbers S typical in astrophysics, Sweet-Parker reconnection is still far too slow. E.g. in solar flares with $S = 3 \times 10^{12}$ and $v_A = 3 \times 10^7$ cm/sec, the release time is predicted to be about 2 years!

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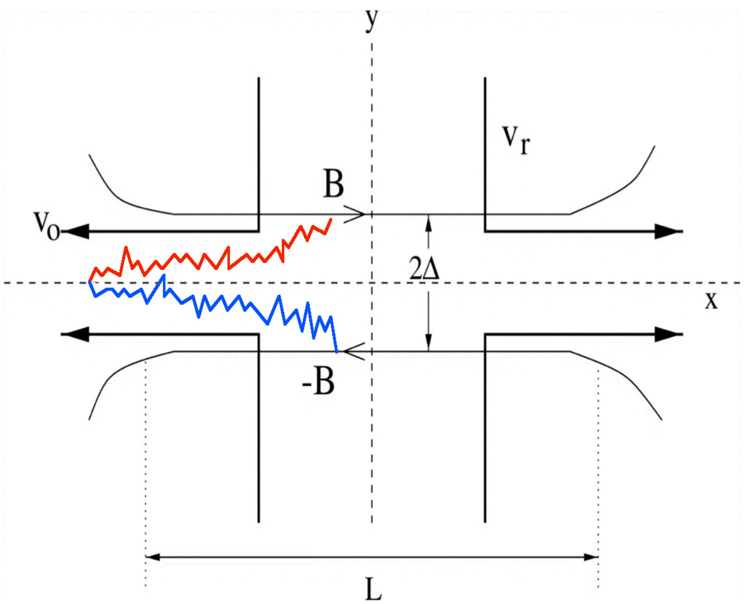


In an Alfvén crossing time $t_A = L/v_A$, **resistive diffusion** of field-lines gives

$$\Delta \simeq \sqrt{\lambda t_A} \simeq L/\sqrt{S}$$

with $S = v_A L/\lambda$; e.g. see Kulsrud, (2005).

With $v_0 \simeq v_A$, mass conservation $v_0 \Delta = v_R L$ implies that

$$v_R = v_A/\sqrt{S}.$$

Sweet-Parker (1958) theory is recovered.

At the high Lundquist numbers S typical in astrophysics, Sweet-Parker reconnection is still far too slow. E.g. in solar flares with $S = 3 \times 10^{12}$ and $v_A = 3 \times 10^7$ cm/sec, the release time is predicted to be about 2 years!

But the Sweet-Parker model is laminar: what about turbulence?

Stochastic Flux Freezing for Resistive MHD

The exact solution of the induction equation

$$\partial_t \mathbf{B} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{B} = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{u} + \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \lambda\triangle\mathbf{B}, \quad \mathbf{B}(t_0) = \mathbf{B}_0$$

is given by a "sum-over-histories", or *stochastic Lundquist formula*

$$\mathbf{B}(\mathbf{x},t) = \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a}\ \mathbf{B}_0(\mathbf{a}(t_0))\cdot\mathbf{J}(\mathbf{a},t,t_0) \exp\left(-\frac{1}{4\lambda}\int_{t_0}^{t} d\tau\, |\dot{\mathbf{a}}(\tau) - \mathbf{u}^{\nu}(\mathbf{a}(\tau),\tau)|^2\right)$$

where the matrix $\mathbf{J}$ satisfies the ODE along the stochastic trajectory $\mathbf{a}(\tau)$

$$\frac{d}{d\tau}\mathbf{J}(\mathbf{a},\tau,t_0) = \mathbf{J}(\mathbf{a},\tau,t_0)\boldsymbol{\nabla}_x\mathbf{u}(\mathbf{a}(\tau),\tau) - \mathbf{J}(\mathbf{a},\tau,t_0)(\boldsymbol{\nabla}_x\cdot\mathbf{u})(\mathbf{a}(\tau),\tau),, \quad \mathbf{J}(\mathbf{a},t_0,t_0) = \mathbf{I}.$$

See G. L. Eyink, "Stochastic line motion and stochastic flux conservation for nonideal hydromagnetic models," J. Math. Phys. **50** 083102 (2009)

Note that the final condition $\mathbf{a}(t) = \mathbf{x}$ on the path-integral trajectories implies that they correspond to solutions of the stochastic equation

$$\dot{\mathbf{a}}(\tau) = \mathbf{u}(\mathbf{a},\tau) + \sqrt{2\lambda}\,\boldsymbol{\eta}(\tau)$$

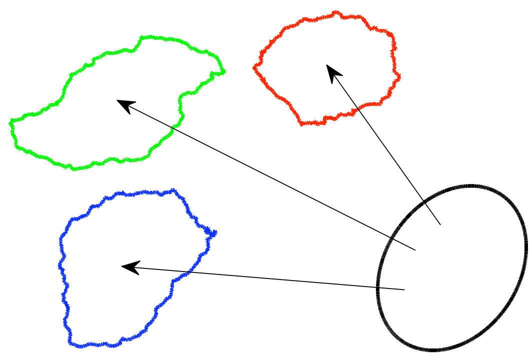
integrated backward in time from t to t_0.

Stochastic Alfvén Theorem

Flux-freezing holds *on average*:

$$\int_S \mathbf{B}(\mathbf{r},t')\cdot d\mathbf{S}(\mathbf{r}) = \left\langle \left[\int_{\mathbf{a}_{t',t}(S)} \mathbf{B}(\mathbf{a},t)\cdot d\mathbf{S}(\mathbf{a}) \right] \right\rangle.$$

Here $\mathbb{E}$ is average over the ensemble of random loops at earlier time:



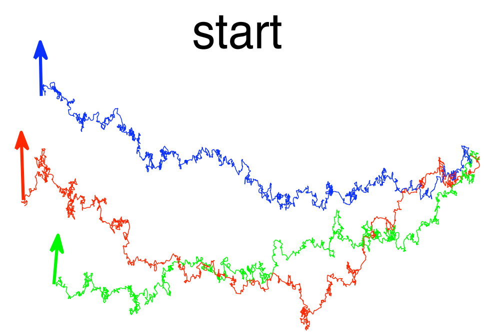
Note $\mathbf{a}_{t',t} = \mathbf{x}_{t',t}^{-1}$ are "back-to-label maps" for the stochastic forward flows.

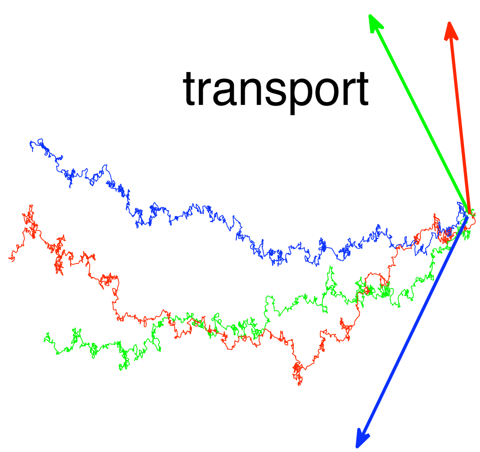
The above result holds for all smooth surfaces S and for any pair of times $t_0 \leq t < t' \leq t_f$ if and only if the magnetic field $\mathbf{B}(\mathbf{r},t)$ (for *any* velocity $\mathbf{u}$) satisfies the induction equation $\partial_t \mathbf{B} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \lambda \triangle \mathbf{B}$ (Eyink, 2009a).

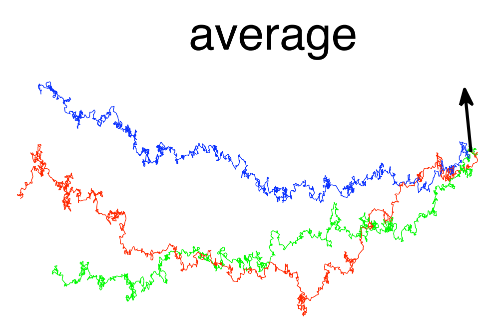
Stochastic Lundquist Formula

$$\mathbf{B}(\mathbf{x},t) = \mathbb{E}\left[\left. \frac{\mathbf{B}(\mathbf{a},t_0)\boldsymbol{\cdot}\boldsymbol{\nabla}_a \mathbf{x}_{t,t_0}(\mathbf{a})}{\det(\boldsymbol{\nabla}_a \mathbf{x}_{t,t_0}(\mathbf{a}))} \right|_{\mathbf{x}_{t,t_0}(\mathbf{a})=\mathbf{x}} \right]$$

Eyink (2009, 2010)







2-Particle Dispersion in Nonlinear MHD Turbulence

Unlike in hydrodynamic turbulence, there are expected to be signficant effects of the Lorentz force in nonlinear MHD turbulence. Particle separations will be different parallel and perpendicular to the magnetic field.

The laws of 2-particle dispersion will depend upon theory of MHD turbulence. Assuming the Goldreich-Sridhar (1995) scaling, $\delta u(r_\perp) \sim (\varepsilon r_\perp)^{1/3}$, one obtains the Richardson-type law

$$\ell_\perp^2 \sim \varepsilon t^3$$

for the transverse slippage of magnetic field lines in MHD turbulence. Note that separation should be slower along the field, $\ell_\parallel^2 \sim (\varepsilon/v_A)^2 t^4$, because of the greater smoothness for parallel displacements, $\delta u(r_\parallel) \sim (\varepsilon r_\parallel/v_A)^{1/2}$.

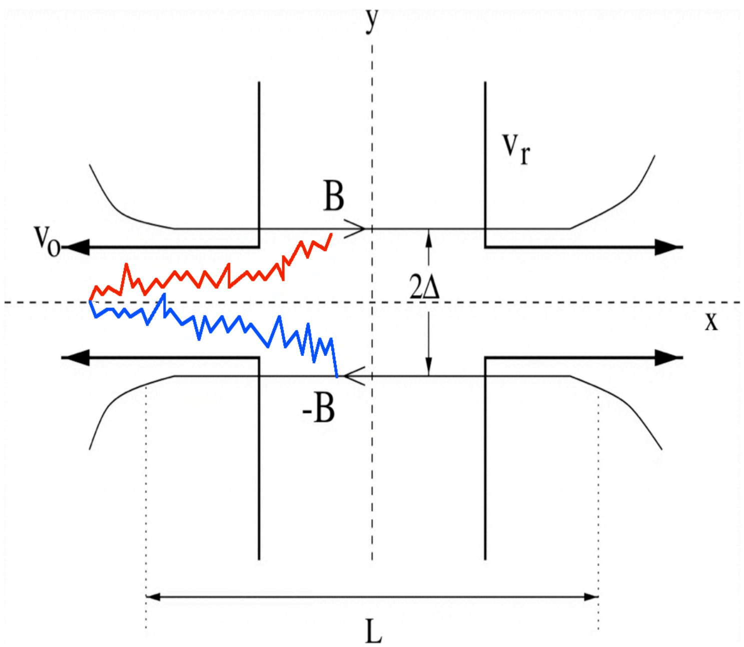
These predictions are in qualitative agreement with recent numerical studies at moderate Reynolds numbers:

A. Busse and W.-C. Müller, Diffusion and dispersion in magnetohydrodynamic turbulence: The influence of mean magnetic fields, Astron. Nachr. **329** 714 (2008)

Further investigation at higher Reynolds numbers is required.

Magnetic Reconnection — Turbulent

Assume that the reconnection occurs in a background MHD plasma turbulence with rms velocity $u_L < v_A$ and integral length or injection scale L_f.



Richardson diffusion of field-lines instead gives

$$\Delta \simeq \sqrt{\varepsilon t_A^3} \simeq L(L/L_f)^{1/2} M_A^2$$

using $\varepsilon = \frac{u_L^4}{v_A L_f}$ (Kraichnan, 1965) and $M_A \equiv \frac{u_L}{v_A}$.

With $v_0 \simeq v_A$, mass conservation $v_0 \Delta = v_R L$ implies that

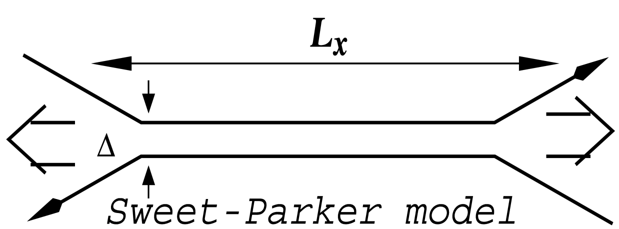
$$v_R = v_A (L/L_f)^{1/2} M_A^2.$$

Now **Lazarian-Vishniac (1999)** theory is obtained, for the case $L < L_f$. *The reconnection rate is independent of resistivity!*

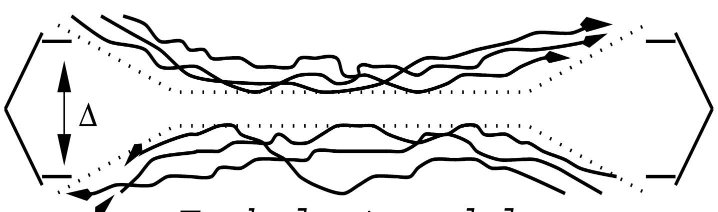
Estimating for solar flares that $L_f \simeq L$ and $u_L \simeq 0.1 v_A$ (Bemporad, 2008) one obtains a release time of about one hour.

For more details, see Eyink, Lazarian & Vishniac (2011).

Line-Wandering and Lazarian-Vishniac (1999) Theory



Sweet-Parker model



Turbulent model

Moving a distance s along field-lines, one finds that a pair of lines initially a distance $\ell_\perp^{(0)}$ apart at $s = 0$ separate at the rate

$$\frac{d}{ds}\ell_\perp \simeq \frac{\delta b_\ell}{B_0} \simeq \frac{\delta u_\ell}{v_A}.$$

Using the GS95 scaling $\delta u_\ell \approx u_L \left(\frac{\ell_\perp}{L_f}\right)^{1/3} M_A^{1/3}$, one can solve to obtain

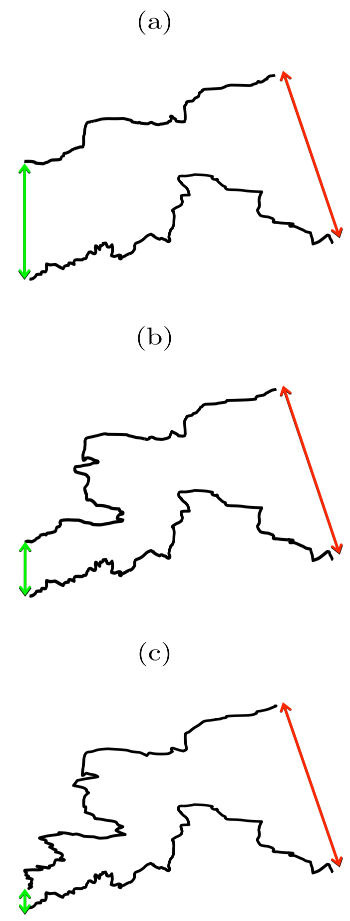
$$\ell_\perp^2 \simeq (s^3/L_f) M_A^4$$

when $L_f > s \gg \ell_\perp^{(0)}$. *This is an analogue of Richardson diffusion for magnetic field-lines!*

Spontaneous Stochasticity of Magnetic Field-Lines

Consider in (a) a pair of magnetic field-lines with initial separations $\ell_\perp^{(0)}$ indicated by green arrows and final separations $\ell_\perp(s)$ indicated by red arrows, after traversing distance s along the lines. Suppose that the initial separation $\ell_\perp^{(0)}$ is gradually decreased toward zero inside the inertial range, indicated in (b) and (c) by the shortened green arrows.

When the magnetic field is turbulent and rough, then the final separation indicated by red arrows need not decrease to zero! This is the property of spontaneous stochasticity of the magnetic field-lines themselves.



Other Contributions to Line-Diffusion

In principle, all of the other terms in the *Generalized Ohm's Law*

$$\mathbf{E} = -\frac{1}{c}\mathbf{u}\times\mathbf{B} + \frac{\mathbf{J}\times\mathbf{B}}{nec} - \frac{\nabla\cdot\mathbf{P}_e}{nec} + \frac{m_e}{ne^2}\left(\frac{\partial\mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{uJ} + \mathbf{Ju})\right) + \eta\mathbf{J}$$

contribute to the slippage of magnetic field-lines.

For example, in *resistive Hall MHD* where

$$\mathbf{E} = -\frac{1}{c}\mathbf{u}\times\mathbf{B} + \frac{\mathbf{J}\times\mathbf{B}}{nec} + \eta\mathbf{J}$$

the Hall electric field leads to field-lines being stochastically frozen-in to the electron fluid

$$d\mathbf{x}_{t,t_0}(\mathbf{a}) = \mathbf{u}_e(\mathbf{x}_{t,t_0}(\mathbf{a}),t)dt + \sqrt{2\lambda}\,d\mathbf{W}(t), \qquad \mathbf{x}_{t_0,t_0}(\mathbf{a}) = \mathbf{a},$$

with

$$\mathbf{u}_e = \mathbf{u} - \mathbf{J}/en.$$

See Eyink (2009).

Hall Term Negligible Compared with Turbulent Advection!

The Hall velocity $\mathbf{u}^H = \mathbf{J}/ne = c\boldsymbol{\nabla}\times\mathbf{B}/4\pi ne$ (a dissipation-range variable) in a turbulent plasma may have a very large magnitude, but also a very *short-range correlation* in space.

For example, assume small-scale equipartition of velocity and magnetic fields, so that $\delta\mathbf{B}(\mathbf{r}) \sim \sqrt{4\pi\rho}\,\delta\mathbf{u}(\mathbf{r})$ for $r \ll L_f$. If the velocity-increments scale as $\langle\delta u_i(\mathbf{r})\delta u_j(\mathbf{r})\rangle \sim Ar^{2h}$ in an inertial range with roughness exponent h, then

$$\langle u_i^H(\mathbf{r})u_j^H(0)\rangle \sim \left(\frac{c}{4\pi ne}\right)^2 \triangle\langle\delta B_i(\mathbf{r})\delta B_j(\mathbf{r})\rangle$$

$$\sim \left(\frac{c}{4\pi ne}\right)^2 4\pi\rho\cdot Ar^{-2(1-h)} \ll Ar^{2h} = \langle\delta u_i(\mathbf{r})\delta u_j(\mathbf{r})\rangle$$

for $r^2 \gg c^2 m_i/4\pi ne^2 = \delta_i^2$, with δ_i *the ion inertial-length/ion skin depth*.

Thus, the Hall velocity contributes negligibly to stochastic relative motion for $r \gg \delta_i$ and resistive diffusion contributes negligibly for $r \gg r_\eta \equiv (\lambda^3/\varepsilon)^{1/4}$

$$\frac{d\mathbf{r}}{dt} = \delta\mathbf{u}(\mathbf{r}) - \delta\mathbf{J}(\mathbf{r})/en + \sqrt{2\lambda}\delta\boldsymbol{\eta}(t) \simeq \delta\mathbf{u}(\mathbf{r}).$$

When $r \gg \delta_i$ and r_η, the dominant effect is turbulent relative advection!

Conclusions

(i) Because of the "forgetting" of initial separations in turbulent Richardson diffusion, Lagrangian particle dynamics becomes *intrinsically stochastic*. Fluid particle trajectories are random even in a fixed velocity realization.

(ii) Flux-freezing in turbulent MHD solutions is thus fundamentally altered and becomes intrinsically stochastic. Magnetic flux-conservation holds neither in the standard (deterministic) sense nor is entirely broken.

(iii) Stochastic flux-freezing underlies the mechanism of the Lazarian-Vishniac (1999) theory of fast turbulent reconnection. The LV99 predictions were originally obtained from the spontaneous stochasticity property of the field-lines themselves for a rough, turbulent magnetic field.

(iv) Spontaneous stochasticity has many other important implications both in MHD (e.g. turbulent magnetic dynamo) and in hydrodynamic turbulence.

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Appendix: The Batchelor Problem—Kinematic Magnetic Dynamo

ENHANCEMENT OF A MAGNETIC FIELD BY A CONDUCTING FLUID

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Submitted June 29, 1967

Zh. Eksp. Teor. Fiz. 53, 1806–1813 (November, 1967)

A simple model of turbulent motion of a conducting liquid fluid is considered, in which the flow velocity has a Gaussian distribution function and the time for establishment of diffusion of the liquid particles is zero. In this case an exact solution of the problem of amplification of a spontaneous magnetic field can be derived. The instability criterion and magnetic field increment are obtained.

INTRODUCTION

WE discuss in the present article the question of the enhancement of a spontaneous magnetic field by a turbulent conducting liquid. The general picture of the interaction between the magnetic field and the conducting liquid was proposed by Batchelor[1], who started from the analogy with a velocity vortex in an incompressible liquid. The turbulent motion of the liquid influences the magnetic field in two ways: On the one hand, the matter stretches the force lines and increases the magnetic energy, and on the other hand the matter increases the diffusion coefficient[2,3] and by the same token it increases the rate of damping of the magnetic field. The latter effect is determined by the diffusion velocity of the liquid particles, and the former by the diffusion velocity of the liquid particles relative to one another (the runaway velocity). We consider below a certain artificial model, which, however, enables us to find the exact solution and in which it is possible to trace in detail the roles of the noted effects.

The magnetic field is described by the diffusion equation

$$\partial \mathbf{H} / \partial t = \mathrm{rot}[\mathbf{u}\mathbf{H}] + \lambda\Delta\mathbf{H}, \quad \mathrm{div}\,\mathbf{H} = 0, \tag{1*}$$

where $\mathbf{u}$ is the liquid flow velocity and λ is the magnetic diffusion coefficient. The turbulent motion of the liquid is assumed stationary, isotropic, and possessing sufficiently simple correlation properties. For the mean values of the Fourier components of the velocity

$$\mathbf{u}(\mathbf{r}, t) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{r}} \mathbf{u}(\mathbf{k}, t), \tag{2}$$

$$\mathbf{u}(\mathbf{k}, t) = (2\pi)^{-3} \int d\mathbf{r} e^{-i\mathbf{k}\mathbf{r}} \mathbf{u}(\mathbf{r}, t)$$

we have the relations

$$\begin{gathered} \langle u_i(\mathbf{k}, t)\rangle = 0, \\ \langle u_i(\mathbf{k}, t) u_j(\mathbf{k}', t)\rangle = u(k, |t - t'|)\sigma_{ij}(\mathbf{k})\delta(\mathbf{k} + \mathbf{k}'), \\ \sigma_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2. \end{gathered} \tag{3}$$

We assume that the mean value of the products of an even number of velocities breaks up into a sum of products of all possible pairs of mean values (Gaussian distribution). This enables us to use a diagram technique. The magnetic field at the initial instant of time $H_i^{(0)}(\mathbf{k})$ will also be assumed to be a random quantity uncorrelated with the velocity of the liquid, with

$$\langle H_i^{(0)}(\mathbf{k}) H_j^{(0)}(\mathbf{k}')\rangle = H_0(\mathbf{k})\sigma_{ij}(\mathbf{k})\delta(\mathbf{k} + \mathbf{k}'). \tag{3'}$$

The problem consists of determining the quantity

$$\langle\langle H_i(\mathbf{k}, t) H_j(\mathbf{k}', t)\rangle\rangle = H(k, t)\sigma_{ij}(\mathbf{k})\delta(\mathbf{k} + \mathbf{k}'), \tag{4}$$

where the double angle brackets denote averaging over the velocity distributions of the liquid and of the magnetic field $\mathbf{H}^{(0)}$. For H(k, t), obviously, there is the boundary condition $H(k, 0) = H_0(k)$.

In such a formulation, the problem is still very complicated. Therefore we shall consider first only a limiting case in which the characteristic time u(k, t) tends to zero, and approximate the time-dependent correlation function by a δ function

$$u(k, t) = v(k)\delta(t). \tag{5}$$

In this approximation the problem can already be solved exactly.

The statistical properties of the liquid particles, having a velocity correlator in the form (5), turn out to be very simple. We deal essentially with a continuous generalization of the discrete model of random walks, in which each displacement does not depend on the preceding one. In fact, if we consider the particle trajectory $\mathbf{r}(t)$, then the displacement $\delta\mathbf{r}(t) = \mathbf{r}(t + \delta t) - \mathbf{r}(t)$ is not correlated with $\mathbf{r}(t)$:

$$\langle \mathbf{r}(t)\delta\mathbf{r}(t)\rangle = 0, \tag{6}$$

so that $\mathbf{r}(t)$ is a random quantity with Gaussian distribution, and

$$\frac{d}{dt}\langle \mathbf{r}(t)^2\rangle = 2v_0, \qquad v_0 = \int d\mathbf{k} v(k). \tag{7}$$

The distance between two liquid particles $\boldsymbol{\rho}(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t)$ is also a random Gaussian quantity. For the rate of change of $\langle\rho^2\rangle$, by virtue of the relation

$$\langle \exp\{i\mathbf{k}\boldsymbol{\rho}(t)\}\rangle = \exp\{-{}^1/_2 k^2 \langle\rho^2\rangle\}$$

we have the equation

$$\frac{d\langle\rho^2\rangle}{dt} = 4\int d\mathbf{k} v(k)\left[1 - \exp\left\{-\frac{1}{2}k^2\langle\rho^2\rangle\right\}\right]. \tag{8}$$

The runaway velocity of two close particles is proportional to the distance between them

$$\frac{d\langle\rho^2\rangle}{dt} = 12 v_2 \langle\rho^2\rangle, \quad v_2 = \frac{1}{3!}\int d\mathbf{k}\, k^2 v(k). \tag{9}$$

At large distances, the particles move apart with a velocity governed by double the diffusion coefficient (7). Henceforth, the most important will be only the first two

*$[\mu\mathbf{H}] \equiv \mu \times \mathbf{H}$.

The Kazantsev-Kraichnan Dynamo Model

Kazantsev (1968) studied the turbulent kinematic dynamo model

$$\partial_t \mathbf{B} = \boldsymbol{\nabla}\times(\mathbf{u}^\nu \times \mathbf{B}) + \lambda \triangle \mathbf{B},$$

with Gaussian random velocity field with zero mean and covariance

$$\langle u_i^\nu(\mathbf{x},t) u_j^\nu(\mathbf{x}',t')\rangle = [D_0^\nu \delta_{ij} - S_{ij}^\nu(\mathbf{x}-\mathbf{x}')]\delta(t-t')$$

temporal white-noise and spatially rough for $r_\nu \ll r \ll L$,

$$S_{ij}^\nu(\mathbf{r}) = D_1[(1+h)\delta_{ij} - h\hat{r}_i\hat{r}_j]r^{2h}$$

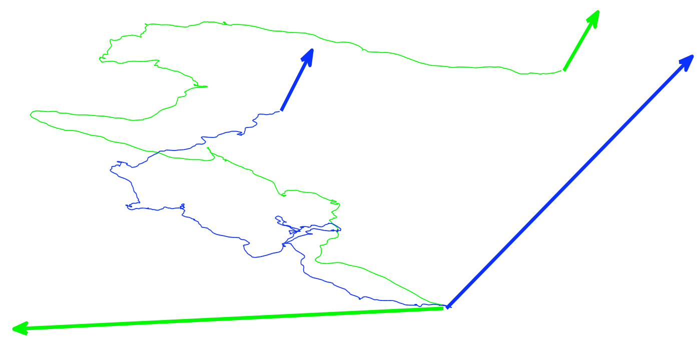
with $0 < h < 1$, but smooth for $r < r_\nu$.

Kazantsev (1968) found for $Pr = \nu/\lambda \ll 1$ that there is a critical roughness exponent $h_c = 1/2$ in 3D, with kinematic dynamo for $h > h_c$ but not for $h < h_c$.

Why? Stretching by velocity-gradients is *much larger* for $h < h_c$!

Explanation of the Kazantsev Dynamo Transition

The magnetic vectors arrive from a large volume $\propto R^3(t)$ with $R^2(t) \sim t^{1/(1-h)}$ (analogue of Richardson diffusion).



Eyink & Neto (2009) show that dynamo effect requires sufficient angular correlation of independent pairs of line-vectors advected by the same velocity realization that arrive simultaneously at the same space point. The line-vector correlation

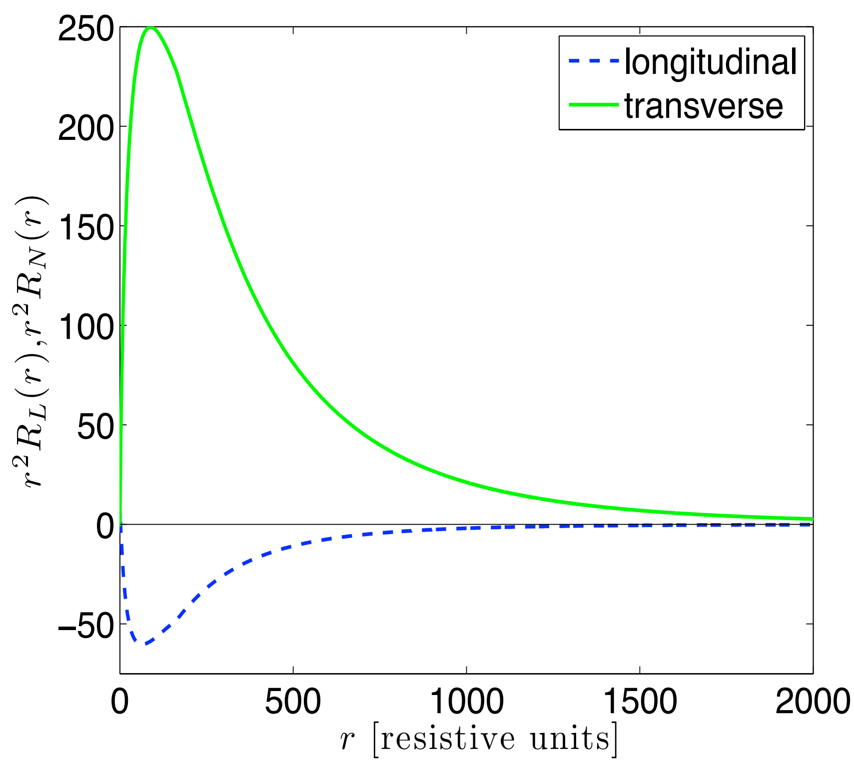
$$R_{ij}(\mathbf{r},t) = \langle \boldsymbol{\ell}_i(\mathbf{a},t) \cdot \boldsymbol{\ell}'_j(\mathbf{a}',t) \delta^3(\mathbf{x}(\mathbf{a},t) - \mathbf{x}'(\mathbf{a}',t)) \rangle$$

with

$$\boldsymbol{\ell}_i(\mathbf{a},t) = \widehat{\mathbf{e}}_i \cdot \boldsymbol{\nabla}_a \mathbf{x}(\mathbf{a},t)$$

and $\mathbf{r} = \mathbf{a}' - \mathbf{a}$, decays as a power of t for $h < h_c$ but grows $\propto e^{\gamma t}$ for $h > h_c$.

Line-Vector Correlation Functions in the Dynamo Regime



A WKB analysis shows that for $r \gg \ell_\eta$ there is a stretched-exponential correlation

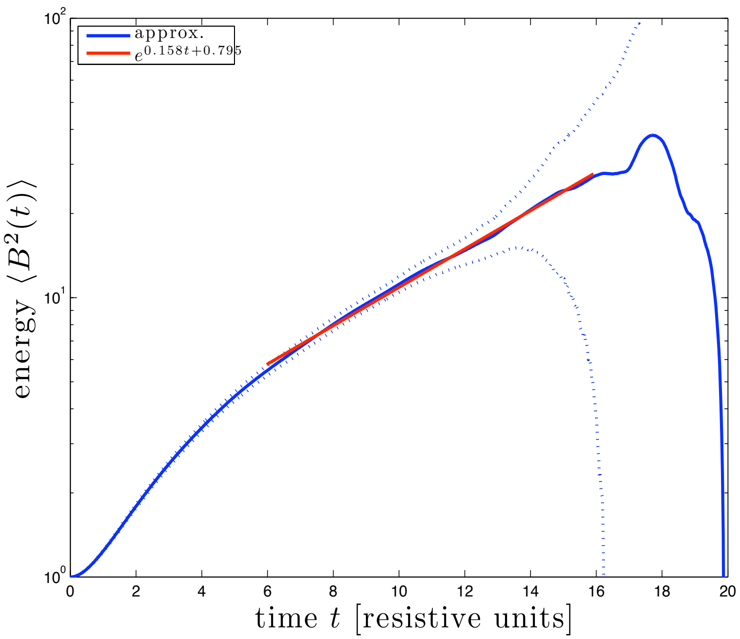
$$R_L(r,t), R_N(r,t) \sim e^{\gamma t} \exp\left(-\frac{3\sqrt{2\gamma}}{2D_1} r^{1/3}\right)$$

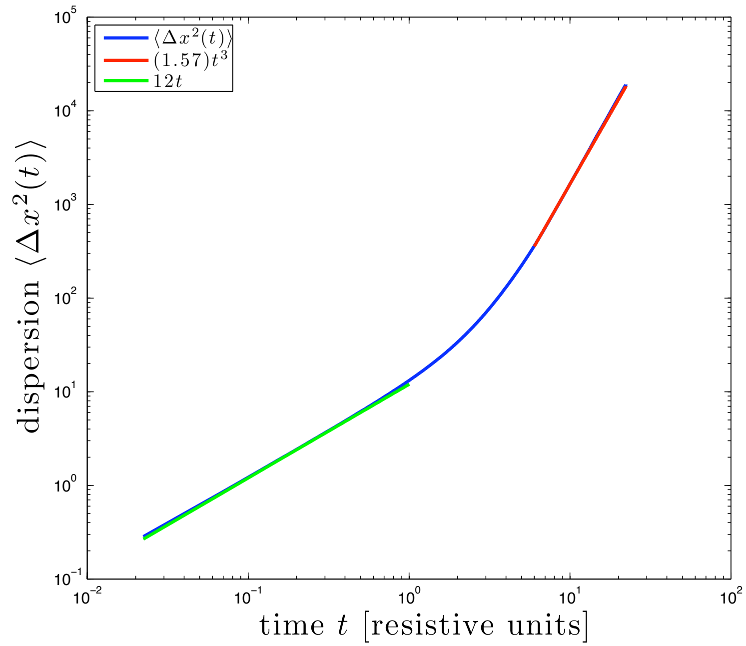
in the KK model with $h = 2/3$, where γ is the dynamo growth rate and D_1 is the amplitude of the velocity-covariance. Eyink (2010).

Line-vectors arriving from thousands of resistive-lengths apart contribute substantially to the dynamo. Notice also the "anti-dynamo effect" for initially longitudinal line-vectors.

Small-Scale Kinematic Dynamo in Hydro Turbulence ($Pr_m = 1$)

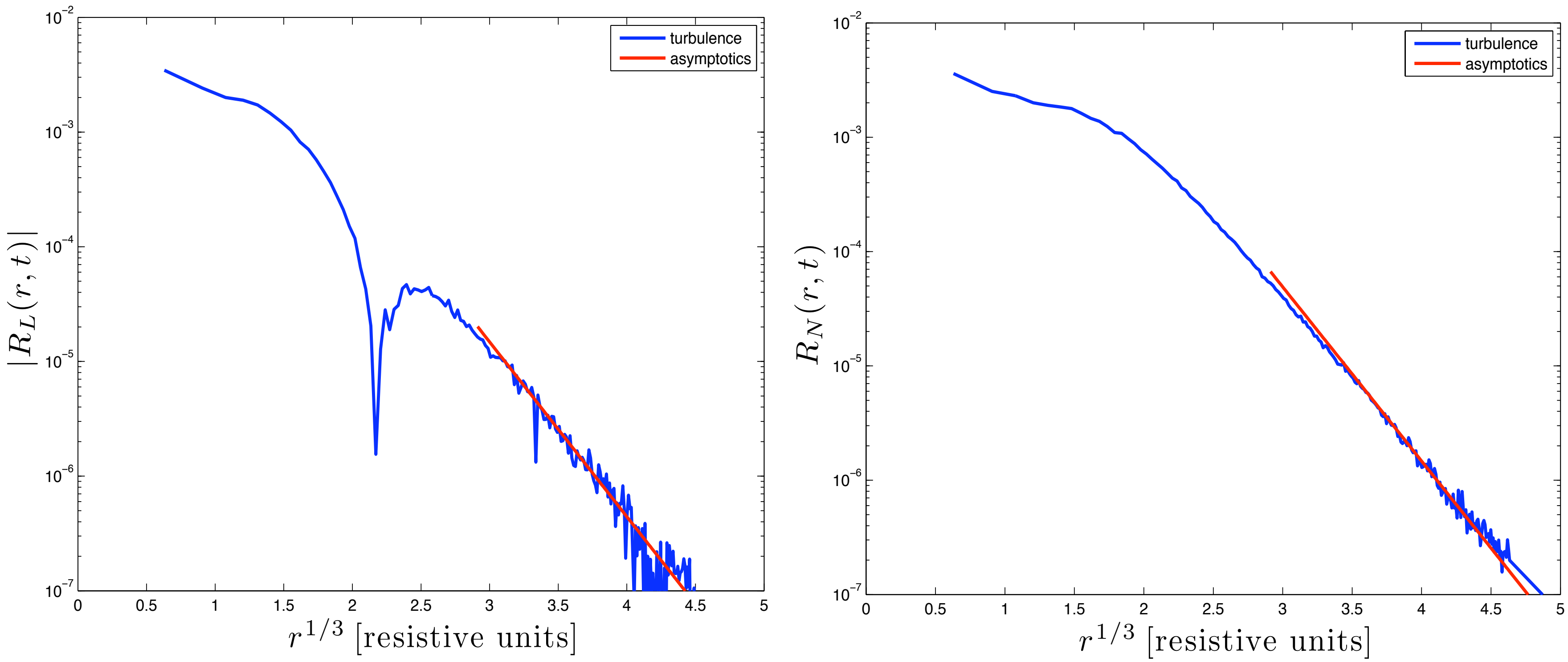
The stochastic Lundquist formula may be exploited numerically to study dynamo effect in hydrodynamic turbulence by Lagrangian tracking of fluid particles. We show results using 1024^3 hydro DNS data from the JHU Turbulence Database: `http://turbulence.pha.jhu.edu`





The asymptotic exponential growth range and Richardson t^3-range begin at the same time!

Line-Vector Correlations in Hydrodynamic Turbulence



The stretched-exponential $\exp\left(-\frac{3\sqrt{2\gamma}}{2D_1} r^{1/3}\right)$ is observed, with D_1 in the KK model chosen to give the same prefactor of the t^3-law as that found numerically in hydro turbulence.