Subgrid Modelling for Geophysical Flows

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Wave-train of storms near Iceland on 31 Jan 1992

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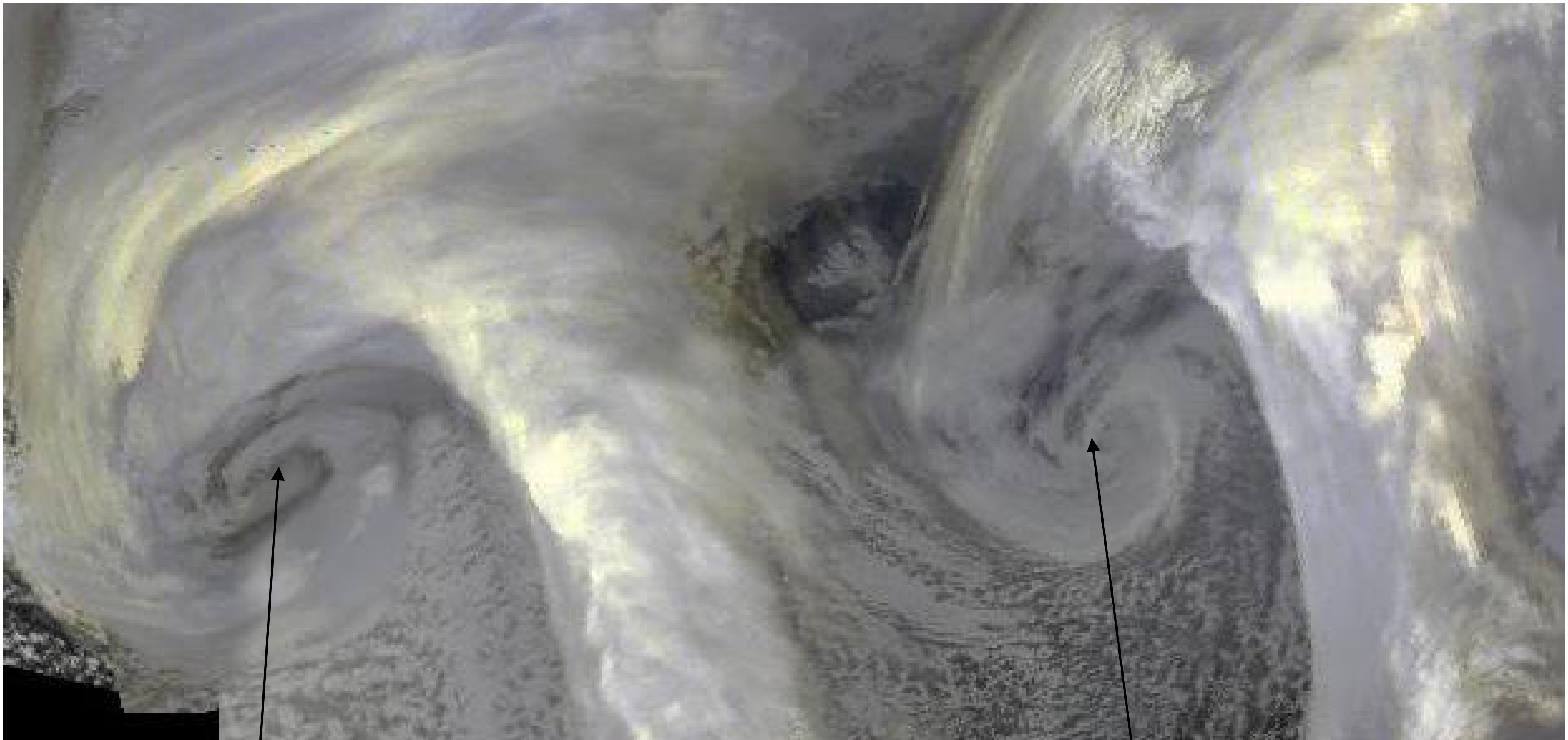

Low Pressure System

Low Pressure System

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Focus

- Systematic means of deriving subgrid effects
- Subgrid model applied without tuning parameters
- LES compared with higher resolution benchmark DNS
- Scaling laws of subgrid effects

Outline

- Introduction
- Barotropic Vorticity Equation
- Mathematical Approaches to Statistical Closure
- Homogeneous Closures – DIA & EDQNM
- Stochastic Model for DIA
- Applications of Closures to Subgrid Modelling
- Stochastic Modelling of Subgrid-scale Parameterizations
- Ideas from Inhomogeneous QDIA closure
- Barotropic Model Results for Atmospheric Flows
- Baroclinic Model Results for Atmospheric Flows
- Baroclinic Model Results for Oceanic Flows
- Subgrid Model with Scaling Laws
- Discussion and Conclusions

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Resolution Dependence of Energy Spectra

Frederiksen et al., 1996, Journal of the Atmospheric Sciences

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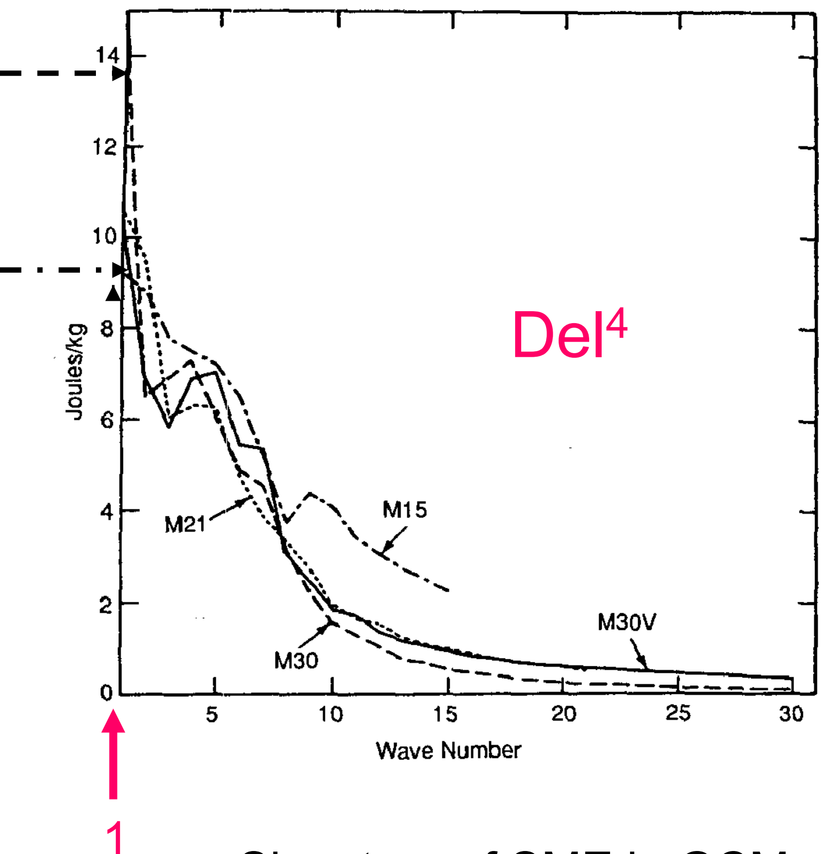
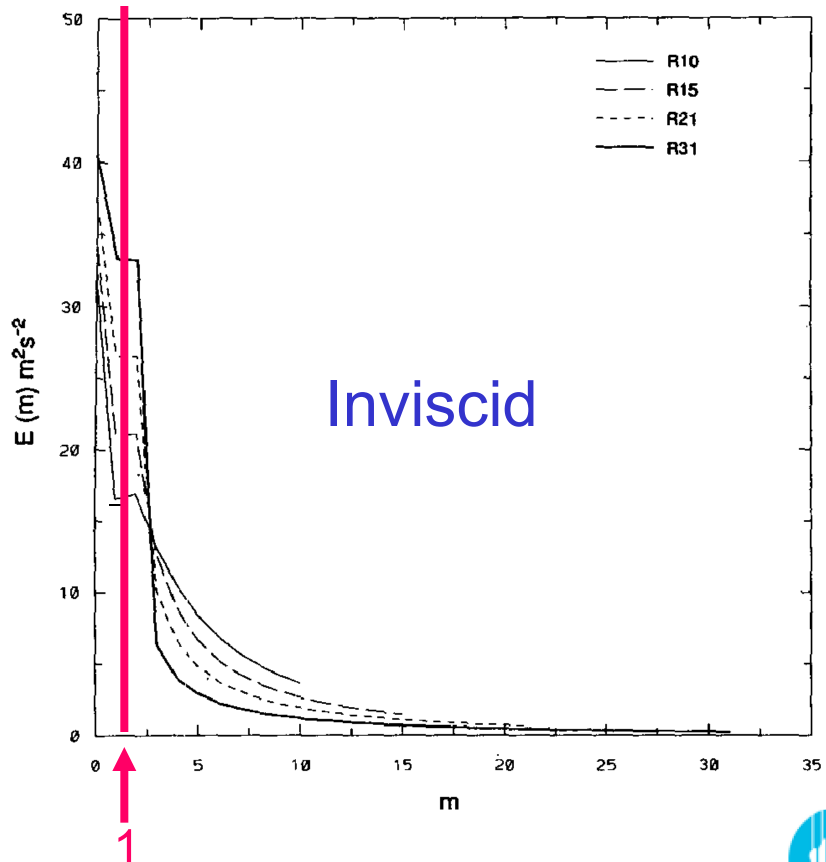
Manabe et al. 1976 - GCM



Frederiksen et al. - SME



Signature of SME in GCMs

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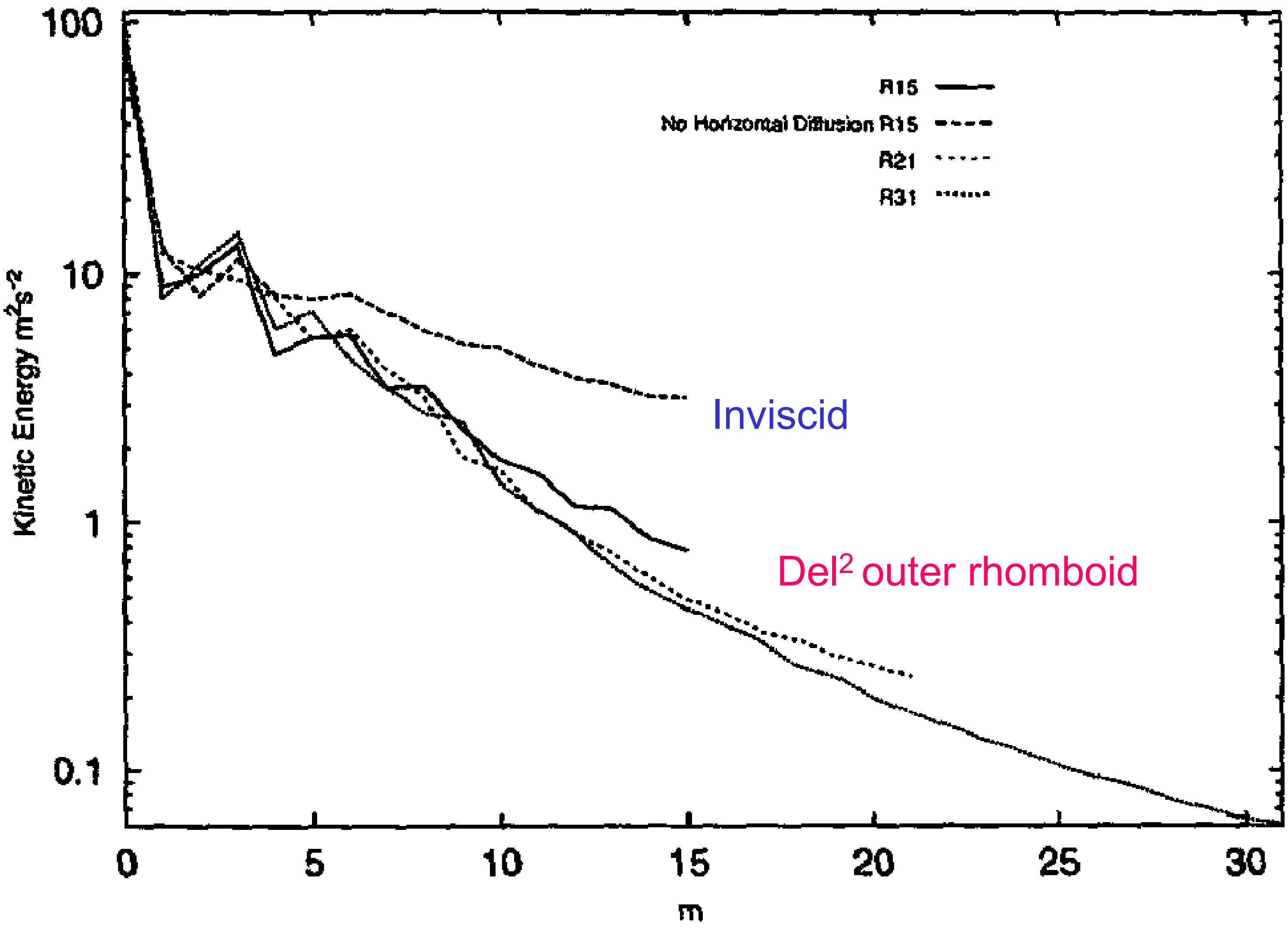
Resolution Dependence of Energy Spectra



FDK 1996

E(0)	R
91.4	15
97.5	21
104.0	31

Inviscid

77.3	15

E in m^2s^{-2}

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Barotropic Vorticity Equation – 2D Turbulence

Barotropic vorticity equation for Rossby wave turbulence

$$\frac{\partial \zeta}{\partial t} = -J(\psi, \zeta + \beta y) - D_0 \zeta + f^0.$$

Dissipation operator D_0 such as $D_0 = -\nu_0 \nabla^2$

where ν_0 is bare viscosity, f_0 is bare forcing, β is beta effect.

Jacobian

$$J(\psi, \zeta) = \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x}$$

Vorticity $\zeta = \nabla^2 \psi,$

Zonal velocity $u = -\frac{\partial \psi}{\partial y},$ Meridional velocity $v = \frac{\partial \psi}{\partial x}$

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Spectral Space Equations - Torus

Spectral representation on doubly periodic domain:

$$\zeta(\mathbf{x},t) = \sum_{\mathbf{k}} \zeta_{\mathbf{k}}(t)\exp(\mathrm{i}\mathbf{k}.\mathbf{x})$$

$$\zeta_{\mathbf{k}}(t) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d^2\mathbf{x}\,\zeta(\mathbf{x},t)\exp(-\mathrm{i}\mathbf{k}.\mathbf{x})$$

$$\left(\frac{\partial}{\partial t} + D_0(\mathbf{k})\right)\zeta_{\mathbf{k}}(t) = \sum_{\mathbf{p}}\sum_{\mathbf{q}} \left[K(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}\zeta_{-\mathbf{q}}\right] + f^0_{\mathbf{k}}$$

$$D_0(\mathbf{k}) = \nu_0 k^2 \text{ or more generally } D_0(\mathbf{k}) = \nu_0(\mathbf{k})k^2 + \mathrm{i}\omega_{\mathbf{k}}$$

$$\omega_{\mathbf{k}} = -\beta k_x / k^2,\ \ k = |\mathbf{k}|$$

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Interaction Coefficients – Planar Geometry

Here

$$k = (k_x^2 + k_y^2)^{1/2},$$

$$A(\mathbf{k},\mathbf{p},\mathbf{q}) = -(p_x q_y - p_y q_x)/p^2,$$

$$K(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{1}{2}\left[A(\mathbf{k},\mathbf{p},\mathbf{q}) + A(\mathbf{k},\mathbf{q},\mathbf{p})\right]$$
$$= \frac{1}{2}(p_x q_y - p_y q_x)(p^2 - q^2)/p^2 q^2,$$

Barotropic Vorticity Equation on the Sphere

Barotropic vorticity equation on the sphere

$$\frac{\partial \zeta}{\partial t} = -J(\psi, \zeta + By) - D_0 \zeta + f^0$$

$$B = 2\Omega \Leftrightarrow \beta$$

Spherical harmonic expansion

$$\zeta(\lambda, \mu, t) = \sum_{m=-T}^{T} \sum_{n=|m|}^{T} \zeta_{mn}(t) P_n^m(\mu) \exp(im\lambda)$$

where λ is longitude and μ is sine latitude.

Legendre functions and Equivalences

$$P_{\mathbf{k}} = P_n^m$$

$$\mathbf{k} = (m, n) = (m_k, n_k)$$

$$-\mathbf{k} = (-m, n)$$

$$k = n$$

Spectral equation on the sphere

$$\left(\frac{\partial}{\partial t} + D_0(\mathbf{k})\right)\zeta_{\mathbf{k}}(t) = \mathrm{i}\sum_{\mathbf{p}}\sum_{\mathbf{q}} \left[K(\mathbf{k}, \mathbf{p}, \mathbf{q})\zeta_{-\mathbf{p}}\zeta_{-\mathbf{q}}\right] + f_{\mathbf{k}}^0$$

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Interaction Coefficients for Spherical Geometry

Here, bare dissipation is

$$D_0(\mathbf{k}) \equiv D_0(m_k, n_k) = \nu_0(k)k(k+1) + \mathrm{i}\omega_{\mathbf{k}}$$

with

$$\omega_{\mathbf{k}} = -\frac{Bm_k}{k(k+1)}.$$

Interaction coefficients

$$K(\mathbf{k},\mathbf{p},\mathbf{q}) = \tfrac{1}{2}[A(\mathbf{k},\mathbf{p},\mathbf{q}) + A(\mathbf{k},\mathbf{q},\mathbf{p})]$$

$$A(\mathbf{k},\mathbf{p},\mathbf{q}) = -\left(\frac{1}{p(p+1)}\right)\int_{-1}^{1} d\mu P_{\mathbf{k}}\left(m_p P_{\mathbf{p}}\frac{dP_{\mathbf{q}}}{d\mu} - m_q P_{\mathbf{q}}\frac{dP_{\mathbf{p}}}{d\mu}\right)$$

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Closure theory

The Closure Problem:

$$\frac{\partial \zeta}{\partial t} \sim \lambda \zeta \zeta$$

$$\frac{\partial < \zeta >}{\partial t} \sim \lambda < \zeta \zeta >$$

$$\frac{\partial < \zeta \zeta >}{\partial t} \sim \lambda < \zeta \zeta \zeta >$$

$$\frac{\partial < \zeta \zeta \zeta >}{\partial t} \sim \lambda < \zeta \zeta \zeta \zeta >$$

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Mathematical Approaches to Closure Equations

- Summation of Feynman diagrams
- Functional approaches
- Path integral approaches
- Heuristic renormalization

Covariance, Response Function and Three-point Cumulant

$$\zeta_{\mathbf{k}} = <\zeta_{\mathbf{k}}> + \hat{\zeta}_{\mathbf{k}}.$$

Homogeneous turbulence with $<\zeta>=0$

$$C_{\mathbf{k},-\mathbf{l}}(t,t') = <\hat{\zeta}_{\mathbf{k}}(t)\hat{\zeta}_{-\mathbf{l}}(t')>;$$

$$R_{\mathbf{k},\mathbf{l}}(t,t') = <\delta\hat{\zeta}_{\mathbf{k}}(t)/\delta\hat{f}^{0}_{\mathbf{l}}(t')>,$$

$$\left\langle \hat{\zeta}_{-\mathbf{l}}(t)\hat{\zeta}_{(\mathbf{l}-\mathbf{k})}(t)\hat{\zeta}_{\mathbf{k}}(t') \right\rangle$$

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Two - point cumulant equation

$$\left(\frac{\partial}{\partial t}+D_0(\mathbf{k})\right)C_{\mathbf{k}}(t,t')$$

$$=\int_{t_0}^{t'} ds\,[S_{\mathbf{k}}(t,s)+F_{\mathbf{k}}^0(t,s)]R_{-\mathbf{k}}(t',s)-\int_{t_0}^{t} ds\,\eta_{\mathbf{k}}(t,s)C_{-\mathbf{k}}(t',s).$$

Nonlinear damping :

$$\eta_{\mathbf{k}}(t,s)=-4\sum_{\mathbf{p}}\sum_{\mathbf{q}}K(\mathbf{k},\mathbf{p},\mathbf{q})K(-\mathbf{p},-\mathbf{q},-\mathbf{k})R_{-\mathbf{p}}(t,s)C_{-\mathbf{q}}(t,s).$$

Nonlinear noise :

$$S_{\mathbf{k}}(t,s)=2\sum_{\mathbf{p}}\sum_{\mathbf{q}}K(\mathbf{k},\mathbf{p},\mathbf{q})K(-\mathbf{k},-\mathbf{p},-\mathbf{q})C_{-\mathbf{p}}(t,s)C_{-\mathbf{q}}(t,s).$$

Bare noise : $F_{\mathbf{k}}^0(t,s)=\left\langle \hat{f}_{\mathbf{k}}^0(t)\hat{f}_{-\mathbf{k}}^{\mathbf{0}}(s)\right\rangle.$

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DIA for Homogeneous Turbulence

Single - time two - point cumulant equation

$$\left(\frac{\partial}{\partial t} + \operatorname{Re} D_0(\mathbf{k})\right) C_{\mathbf{k}}(t,t)$$

$$= \operatorname{Re}\int_{t_0}^{t} ds\, [S_{\mathbf{k}}(t,s) + F_{\mathbf{k}}^{0}(t,s)] R_{-\mathbf{k}}(t,s) - \operatorname{Re}\int_{t_0}^{t} ds\, \eta_{\mathbf{k}}(t,s) C_{-\mathbf{k}}(t',s).$$

since

$$\frac{\partial C_{\mathbf{k}}(t,t)}{\partial t} = \lim t' \to t\ \{\frac{\partial C_{\mathbf{k}}(t,t')}{\partial t} + \frac{\partial C(t,t')}{\partial t'}\}$$

Response function equation

$$\left(\frac{\partial}{\partial t} + D_0(\mathbf{k})\right) R_{\mathbf{k}}(t,t') = -\int_{t'}^{t} ds\, \eta_{\mathbf{k}}(t,s) R_{\mathbf{k}}(s,t').$$

Markovian EDQNM closure from DIA

EDQNM closure from DIA closure

Specialize to isotropic case : $\mathbf{k} \to k$

FTD : $C_k(t,t')\Theta(t-t') = R_k(t,t')C_k(t,t)$

Exponential decay of response function :

$$R_k(t,t') = \exp[-\mu_k(t-t')]$$

Eddy damping : $\mu_k = \gamma[k(k+1)C_k(t,t)]^{1/2} + D_0(k),$

where $\gamma \approx 0.6.$

Bare dissipation : $D_0(k) = \nu_0(k)k(k+1)$

Cumulant equation :

$$\frac{\partial C_k(t,t)}{\partial t} + 2D_0(k)C_k(t,t) - F_o(k) = -2\eta_k(t)C_k(t,t) + 2S_k(t).$$

Nonlinear Damping and Nonlinear Noise

Specialize to white noise:

$$2\int_{t_0}^{t} ds\, F_k^0(t,s) R_{-\mathbf{k}}(t',s) \rightarrow F^0(k,t)$$

Nonlinear damping:

$$\eta_k(t) = -4\sum_{\mathbf{p}}\sum_{\mathbf{q}} K(\mathbf{k},\mathbf{p},\mathbf{q})K(-\mathbf{p},-\mathbf{q},-\mathbf{k})\theta_{kpq}C_q(t,t)$$

Nonlinear noise:

$$S_k(t) = 2\sum_{\mathbf{p}}\sum_{\mathbf{q}} K(\mathbf{k},\mathbf{p},\mathbf{q})K(-\mathbf{k},-\mathbf{p},-\mathbf{q})\theta_{kpq}C_p(t,t)C_q(t,t)$$

Triad relaxation time:

$$\theta_{kpq}(t) = \int_{t_0}^{t} ds\, R_k(t,s)R_p(t,s)R_q(t,s) = \frac{1-\exp[-(\mu_k+\mu_p+\mu_q)(t-t_0)]}{\mu_k+\mu_p+\mu_q}$$

with

$$\theta_{kpq}(\infty) = (\mu_k+\mu_p+\mu_q)^{-1}$$

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Large Eddy Simulations

Subgrid Modelling with EDQNM closure

Full resolution :

$$\mathsf{T}(T) = \{\mathbf{p},\mathbf{q} \mid -T \le m_p \le T, |m_p| \le p \le T; -T \le m_q \le T, |m_q| \le p \le T\}$$

Reduced resolution $T_R < T$

$$\mathsf{R} = \mathsf{T}(T_R)$$

Subgrid scales :

$$\mathsf{S} = \mathsf{T} - \mathsf{R}$$

EDQNM cumulant equation for resolved scales :

$$\frac{\partial C_k(t,t)}{\partial t} + 2[D_0(k) + \eta_k^{\mathsf{S}}(t)]C_k(t,t) - [F_o(k) + 2S_k^{\mathsf{S}}(t)] = -2\eta_k^{\mathsf{R}}(t)C_k(t,t) + 2S_k^{\mathsf{R}}(t).$$

Eddy Viscosity, Dissipation & Stochastic Backscatter

Eddy drain viscosity and drain dissipation

$$D_d(k) = \eta_k^{\mathsf{S}} = \nu_d(k)k(k+1); \quad \nu_d(k) = [k(k+1)]^{-1}\eta_k^{\mathsf{S}}$$

Renormalized viscosity and dissipation

$$D_r(k) = D_0(k) + D_d(k); \qquad \nu_r(k) = \nu_0(k) + \nu_d(k)$$

Stochastic backscatter

$$F_b(k) = 2S_k^{\mathsf{S}}$$

Renormalized noise

$$F_r(k) = F_0(k) + F_b(k)$$

Eddy backscatter viscosity and backscatter dissipation

$$D_b(k) = -[C_k(t,t)]^{-1}S_k^{\mathsf{S}}; \qquad \nu_b(k) = -[k(k+1)C_k(t,t)]^{-1}S_k^{\mathsf{S}}$$

Net eddy viscosity and net dissipation

$$D_n(k) = D_d(k) + D_b(k); \qquad \nu_n(k) = \nu_d(k) + \nu_b(k)$$

Renormalized net viscosity and renormalized net disipation

$$D_{rn}(k) = D_0(k) + D_n(k); \qquad \nu_{rn}(k) = \nu_0(k) + \nu_n(k)$$

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EDQNM with Renormalized Dissipation and Noise

Thus we can rewrite EDQNM cumulant equation for resolved scales

$$\frac{\partial C_k(t,t)}{\partial t} + 2[D_0(k) + \eta_k^{\mathrm{S}}(t)]C_k(t,t) - [F_o(k) + 2S_k^{\mathrm{S}}(t)] = -2\eta_k^{\mathrm{R}}(t)C_k(t,t) + 2S_k^{\mathrm{R}}(t)$$

as

$$\frac{\partial C_k(t,t)}{\partial t} + 2D_r(k)C_k(t,t) - F_r(k) = -2\eta_k^{\mathrm{R}}(t)C_k(t,t) + 2S_k^{\mathrm{R}}(t)$$

This is in the same form as the full resolution EDQNM equation

$$\frac{\partial C_k(t,t)}{\partial t} + 2D_0(k)C_k(t,t) - F_o(k) = -2\eta_k(t)C_k(t,t) + 2S_k(t)$$

with

$$D_0 \to D_r;\; F_0 \to F_r;\; \eta_k \to \eta_k^{\mathrm{R}};\; S_k \to S_k^{\mathrm{R}}$$

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DIA Eddy Viscosities and Stochastic Backscatter

DIA Eddy Viscosities and Stochastic Backscatter

DIA eddy drain viscosity

$$\nu_d(k) = [k(k+1)C_k(t,t)]^{-1} \int_{t_0}^{t} ds\, \eta_k^{\mathsf{S}}(t,s) C_k(t,s)$$

DIA stochastic backscatter

$$F_b(k) = 2\int_{t_0}^{t} ds\, S_k^{\mathsf{S}}(t,s) R_k(t,s)$$

Eddy backscatter viscosity

$$\nu_b(k) = -[k(k+1)C_k(t,t)]^{-1} \int_{t_0}^{t} ds\, S_k^{\mathsf{S}}(t,s) R_k(t,s)$$

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LES with rotation, kinetic energy m-spectra

Frederiksen and Davies 1997, Journal of the Atmospheric Sciences

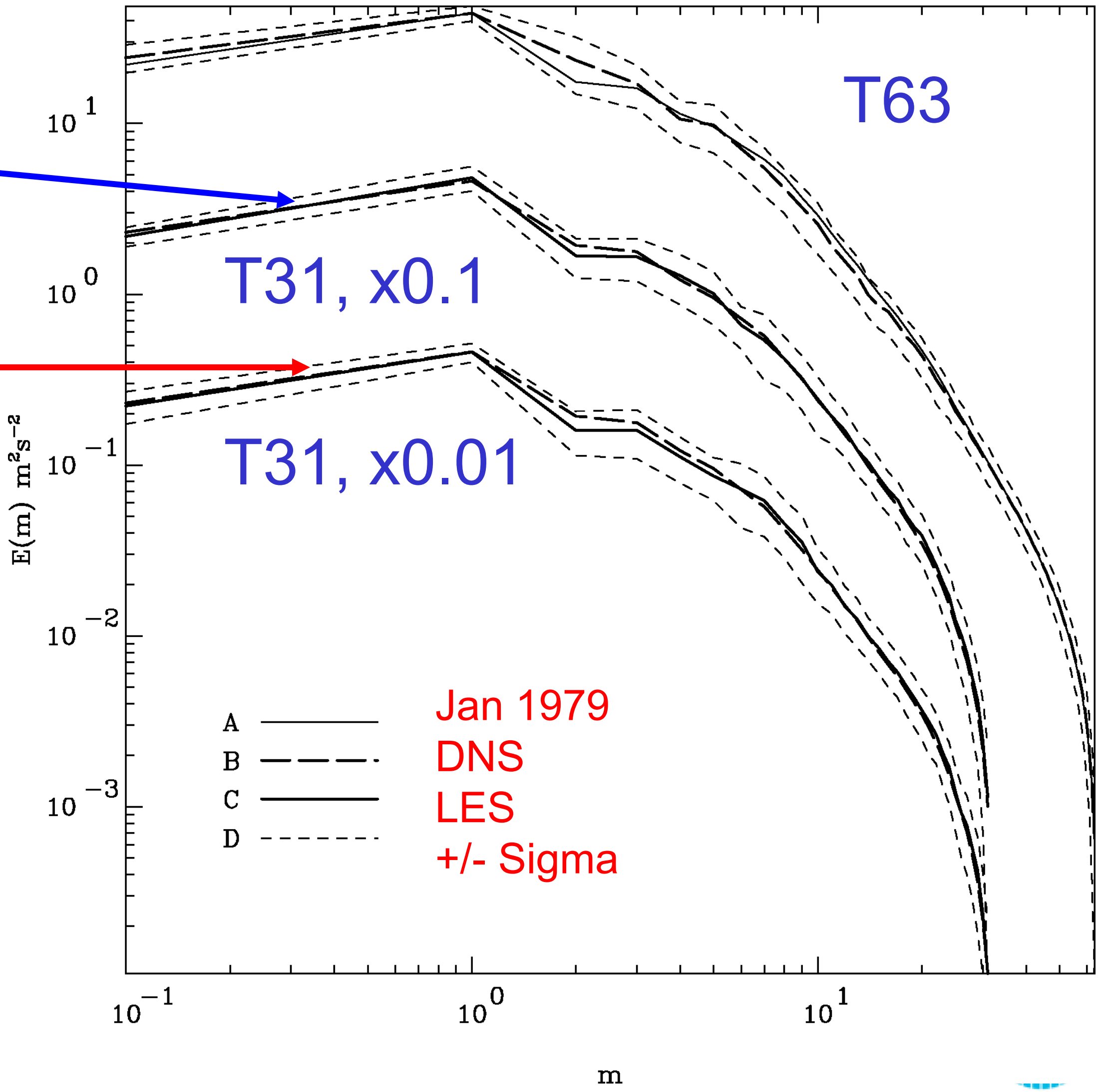
Closure based

$$\nu_r(n) = \nu_0(n) + \nu_d(n),$$

$$F_r(n) = F_0(n) + F_b(n) \Rightarrow f^r_{\mathbf{k}}$$

$$\nu_{rn}(n) = \nu_0(n) + \nu_d(n) + \nu_b(n),$$

$$F_0(n) \Rightarrow f^0_{\mathbf{k}}$$



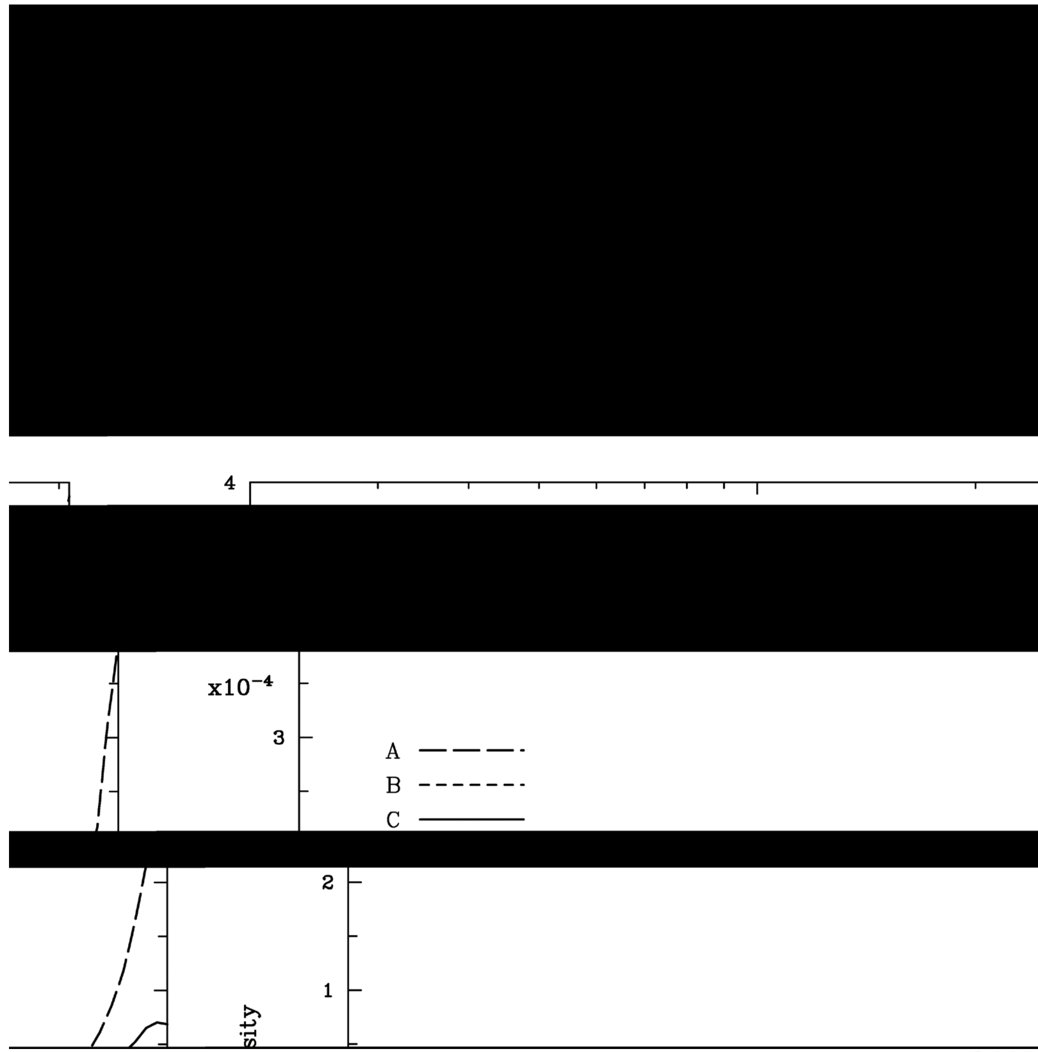
Eddy viscosities

Closure based

$$A \Rightarrow \nu_d(n)$$

$$C \Rightarrow \nu_n(n)$$

$$B \Rightarrow \nu_b(n)$$



Net dissipation

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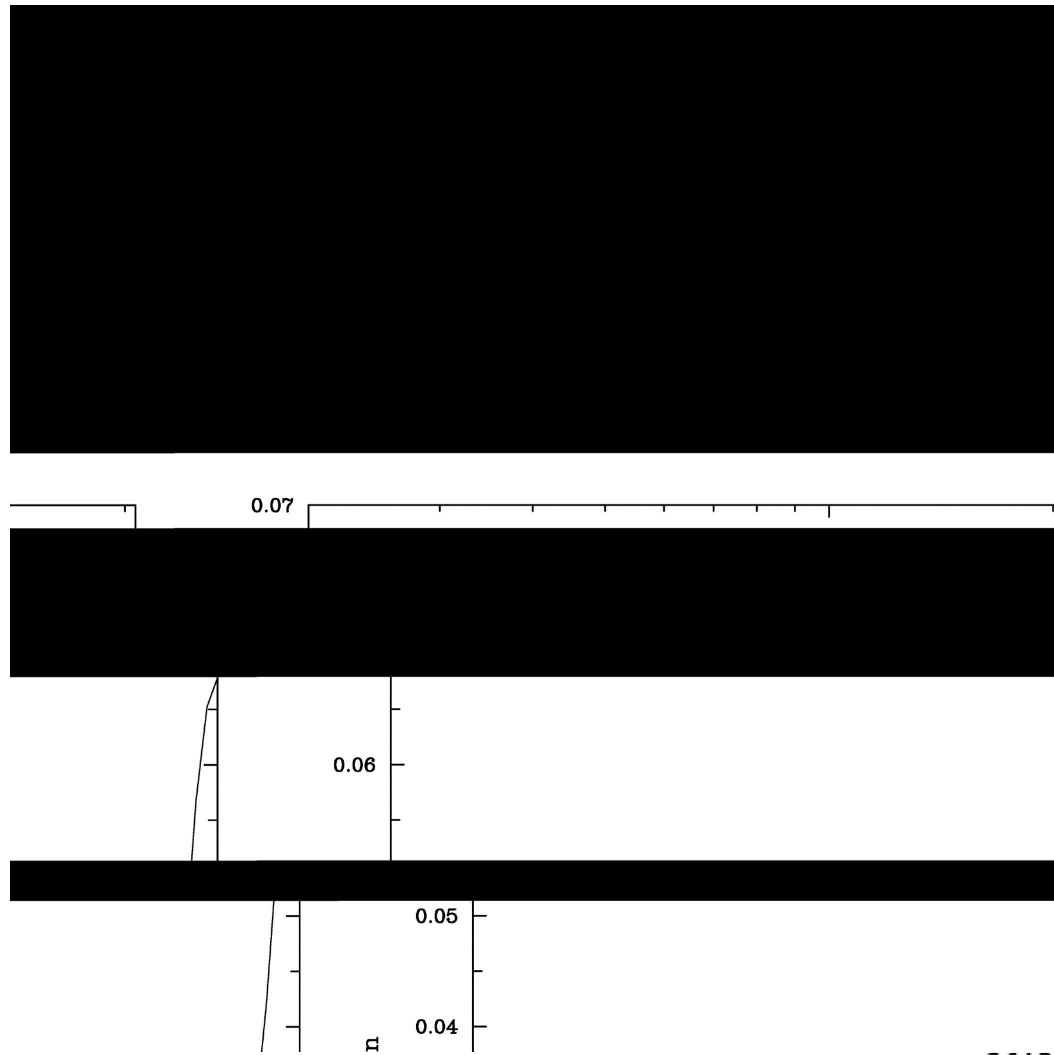
Closure based



Langevin Equation for DIA

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$$\left(\frac{\partial}{\partial t} + D_0(\mathbf{k})\right)\tilde{\zeta}_{\mathbf{k}}(t) = -\int_{t_o}^{t} ds\, \eta_{\mathbf{k}}(t,s)\tilde{\zeta}_{\mathbf{k}}(s) + \hat{f}^{o}_{\mathbf{k}}(t) + f^{S}_{\mathbf{k}}(t)$$

Nonlinear damping :

$$\eta_{\mathbf{k}}(t,s) = -4\sum_{\mathbf{p}}\sum_{\mathbf{q}} K(\mathbf{k},\mathbf{p},\mathbf{q})K(-\mathbf{p},-\mathbf{q},-\mathbf{k})R_{-\mathbf{p}}(t,s)C_{-\mathbf{q}}(t,s).$$

Nonlinear stochastic noise :

$$f^{S}_{\mathbf{k}}(t) = \sqrt{2}\sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k}+\mathbf{p}+\mathbf{q})K(\mathbf{k},\mathbf{p},\mathbf{q})\rho^{(1)}_{-\mathbf{p}}(t)\rho^{(2)}_{-\mathbf{q}}(t),$$

$$< \rho^{(i)}_{\mathbf{k}}(t)\rho^{(j)}_{-\mathbf{l}}(t') >= \delta_{ij}\delta_{\mathbf{kl}}C_{\mathbf{k}}(t,t')$$

$$< \tilde{\zeta}_{\mathbf{k}}(t)\tilde{\zeta}_{-\mathbf{k}}(t') >= C_{\mathbf{k}}(t,t').$$

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Direct Stochastic Modelling Approach

Subgrid Modelling based on Direct Numerical Simulation

RECALL :

Full resolution :

$$\mathsf{T}(T) = \{\mathbf{p},\mathbf{q} \mid -T \le m_p \le T, |m_p| \le p \le T; -T \le m_q \le T, |m_q| \le p \le T\}$$

Reduced resolution $T_R < T$

$$\mathsf{R} = \mathsf{T}(T_R)$$

Subgrid scales :

$$\mathsf{S} = \mathsf{T} - \mathsf{R}$$

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Direct Stochastic Modelling Approach

Frederiksen and Kepert 2006, Journal of the Atmospheric Sciences

$$\mathbf{q} = (\ldots, q_{\mathbf{k}}^{j}, \ldots)^{T} \Leftarrow (\ldots, \zeta_{\mathbf{k}}, \ldots)^{T}$$

$$\frac{\partial \mathbf{q}}{\partial t} = \left(\frac{\partial \mathbf{q}}{\partial t}\right)_{\mathrm{R}} + \left(\frac{\partial \mathbf{q}}{\partial t}\right)_{\mathrm{S}}$$

$$\mathbf{q} = \overline{\mathbf{q}} + \hat{\mathbf{q}}$$

$$\left(\frac{\partial \mathbf{q}(t)}{\partial t}\right)_{\mathrm{S}} = \overline{\mathbf{f}}_{\mathbf{e}} + \left(\frac{\partial \hat{\mathbf{q}}}{\partial t}\right)_{\mathrm{S}}$$

$$\overline{\mathbf{f}}_{\mathbf{e}} = \left(\frac{\partial \overline{\mathbf{q}}(t)}{\partial t}\right)_{\mathrm{S}}$$

$$\left(\frac{\partial \hat{\mathbf{q}}(t)}{\partial t}\right)_{\mathrm{S}} = -\mathbf{D}_{\mathbf{d}}\hat{\mathbf{q}}(t) + \hat{\mathbf{f}}_{\mathbf{b}}(t)$$

Direct Stochastic Modelling Approach

The drain dissipation includes memory effects as in the DIA and QDIA :

$$\mathbf{D_d} = -\left[\left\langle \int_{t_0}^{t} ds \left(\frac{\partial \hat{\mathbf{q}}(s)}{\partial t}\right)_{\mathbf{S}} \hat{\mathbf{q}}^{+}(t_0) \right\rangle\right]\left[\left\langle \int_{t_0}^{t} ds\, \hat{\mathbf{q}}(s)\hat{\mathbf{q}}^{+}(t_0) \right\rangle\right]^{-1}.$$

Subgrid nonlinear noise matrix :

$$\mathrm{F}_{\mathrm{b}} = \mathbf{F_b} + \mathbf{F_b}^{+}$$

where

$$\mathbf{F_b} = \left\langle \hat{\mathbf{f}}_{\mathrm{b}}(t)\hat{\mathbf{q}}^{+}(t) \right\rangle.$$

Direct Stochastic Modelling Approach

The Lyapunov or balance equation :

$$\mathbf{T} = \left\langle \left(\frac{\partial \hat{\mathbf{q}}(t)}{\partial t} \right)_{\mathrm{S}} \hat{\mathbf{q}}^{+}(t) \right\rangle + \left\langle \hat{\mathbf{q}}(t) \left(\frac{\partial \hat{\mathbf{q}}(t)}{\partial t} \right)_{\mathrm{S}}^{+} \right\rangle$$

$$= -\mathbf{D}_{\mathbf{d}} \left\langle \hat{\mathbf{q}}(t) \hat{\mathbf{q}}^{+}(t) \right\rangle - \left\langle \hat{\mathbf{q}}(t) \hat{\mathbf{q}}^{+}(t) \right\rangle \mathbf{D}_{\mathbf{d}}^{+} + \mathbf{F}_{\mathbf{b}}(t).$$

Yields $\mathbf{F}_{\mathbf{b}}(t)$ once $\mathbf{D}_{\mathbf{d}}$ has been calculated.

$\mathbf{F}_{\mathbf{b}}$ is calculated as coloured noise but it may be sufficient to represent by white noise in LES :

$$\left\langle \mathbf{f}_{\mathbf{b}}(t) \mathbf{f}_{\mathbf{b}}^{+}(t') \right\rangle = \mathbf{F}_{\mathbf{b}} \delta(t - t').$$

Deterministic Parameterization

The backscatter dissipation matrix is defined by :

$$\mathbf{D}_{\mathbf{b}} = -\mathbf{F}_{\mathbf{b}}\left[\left\langle \hat{\mathbf{q}}(t)\hat{\mathbf{q}}^{+}(t)\right\rangle\right]^{-1}.$$

The net dissipation matrix is defined by

$$\mathbf{D}_{\mathbf{n}} = \mathbf{D}_{\mathbf{d}} + \mathbf{D}_{\mathbf{b}}.$$

Thus,

$$\left(\frac{\partial \hat{\mathbf{q}}(t)}{\partial t}\right)_{S} = -\mathbf{D}_{\mathbf{n}}\hat{\mathbf{q}}(t)$$

and

$$\mathbf{D}_{\mathbf{n}} = -\left[\left\langle \left(\frac{\partial \hat{\mathbf{q}}(t)}{\partial t}\right)_{S} \hat{\mathbf{q}}^{+}(t)\right\rangle\right]\left[\left\langle \hat{\mathbf{q}}(t)\hat{\mathbf{q}}^{+}(t)\right\rangle\right]^{-1}.$$

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Ideas from QDIA

Frederiksen 1999 Journal of the Atmospheric Sciences

QDIA inhomogeneous closure suggests quasi - diagonal subgrid - scale terms :

$$D_{\bullet}^{ij}(\mathbf{k},\mathbf{l}) = D_{\bullet}^{ij}(\mathbf{k})\delta_{\mathbf{k},\mathbf{l}}, \quad \bullet = \mathbf{b}, \mathbf{d} \text{ or } \mathbf{n}$$

$$F_{\bullet}^{ij}(\mathbf{k},\mathbf{l}) = F_{\bullet}^{ij}(\mathbf{k})\delta_{\mathbf{k},\mathbf{l}}.$$

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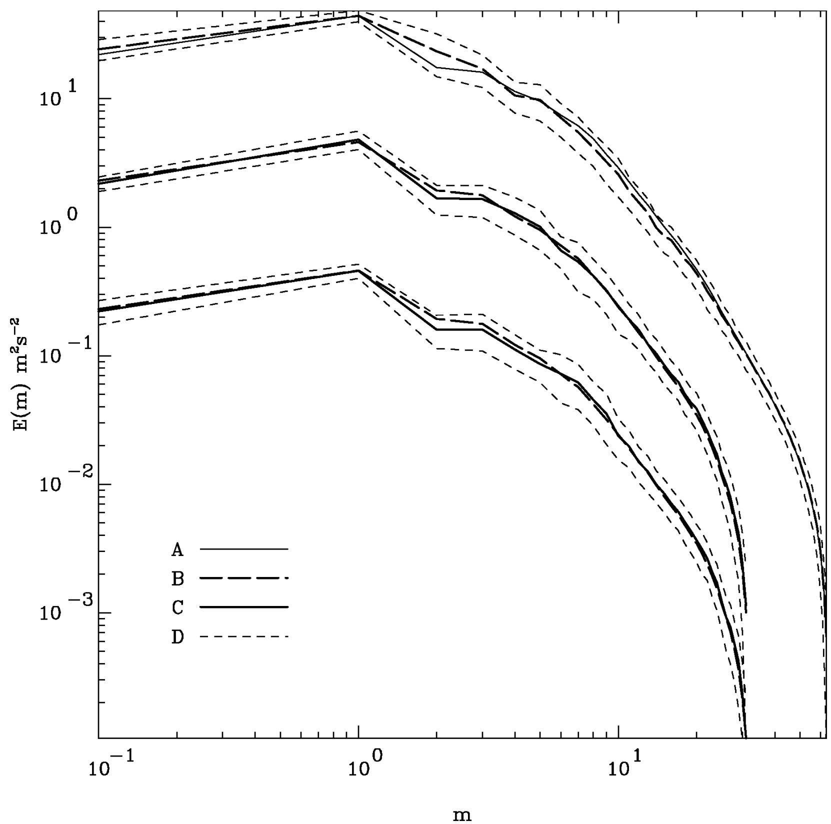
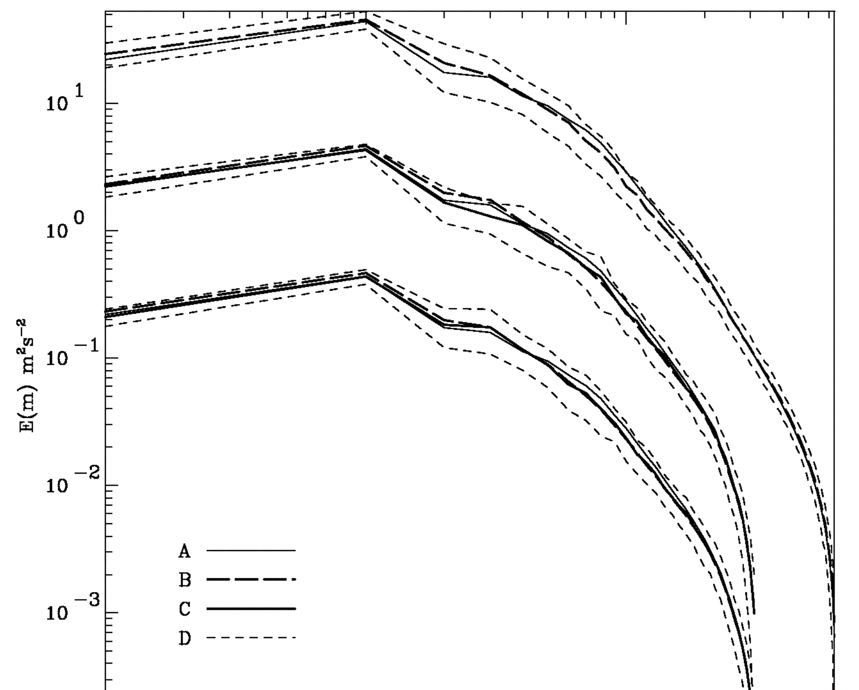
Comparison of Closure and Stochastic Modelling for Barotropic 2D Flows

Frederiksen and Kepert 2006, Journal of the Atmospheric Sciences

Essentially the same results from closure and stochastic model



Closure Results



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Comparison of Closure and Stochastic Modelling Eddy Viscosities

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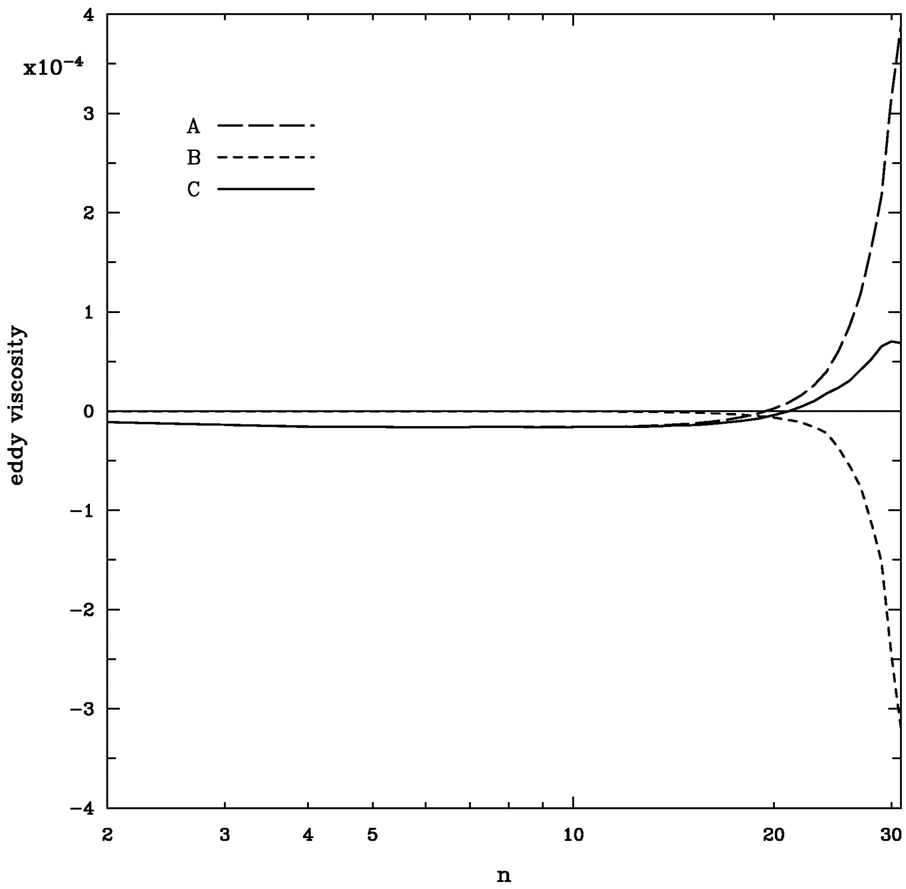
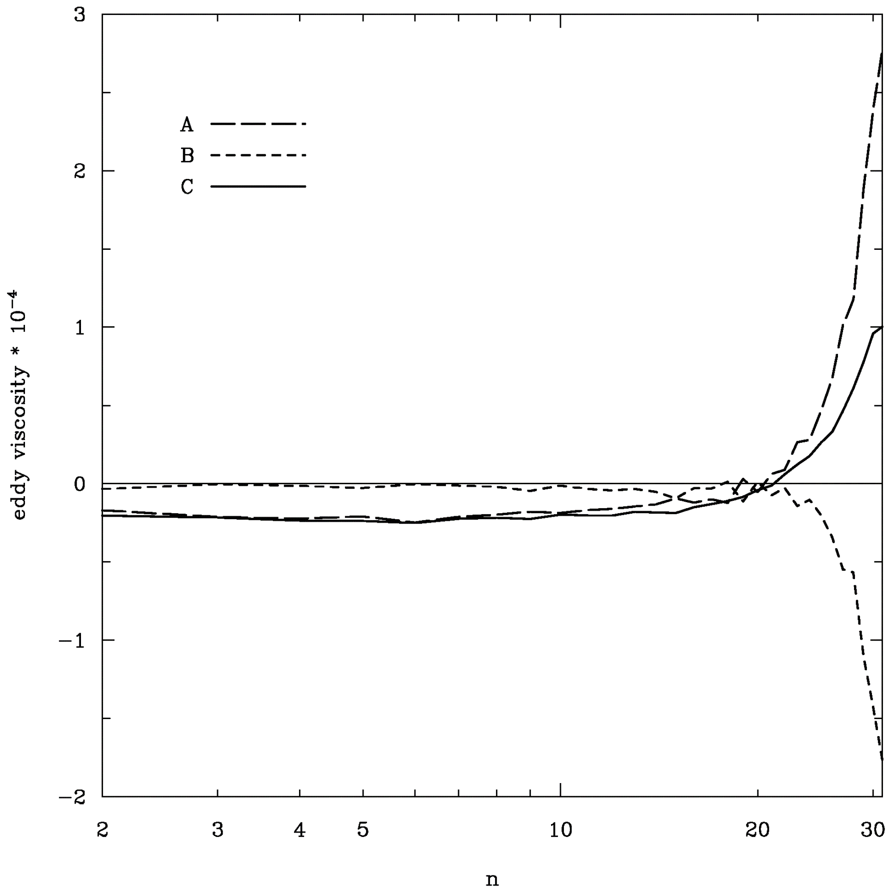
$$\mathrm{A} \Rightarrow \nu_d(n) \qquad \mathrm{B} \Rightarrow \nu_b(n) \qquad C \Rightarrow \nu_n(n)$$

Closure



Stochastic Model



QG Equations in terms of Vorticity and Temperature

$$\frac{\partial \zeta^j}{\partial t} = -J(\psi^j, \zeta^j) - B\frac{\partial \psi}{\partial \lambda} - (-1)^j \mathrm{f_C}\, w/p_m - \sum_l D_\zeta^{jl}\zeta^l - \alpha_\zeta^j \zeta^j + \kappa_\zeta^j(\tilde{\zeta}^j - \zeta^j),$$

$$\frac{\partial \theta}{\partial t} = -J(\psi^0, \theta) + \bar{\sigma}\, w/p_m - D_\theta \theta - \alpha_\theta \theta + \kappa_\theta(\tilde{\theta} - \theta)$$

Here, ψ^j are the streamfunctions at the upper ($j = 1$) and lower ($j = 2$) levels, $\zeta^j = \nabla^2 \psi^j$ is the vorticity, θ is the mean potential temperature, D_ζ^{jl} and D_θ are dissipation operators, α^j are coefficients and $\tilde{\zeta}^j$ and $\tilde{\theta}$ are mean states towards which the fields are relaxed with relaxation coefficients κ_ζ^j and κ_θ.

Also, $B = 2\Omega$, $\mathrm{f_C} = 2\Omega\mu_C$, μ_C is the sine of a typical mid-latitude, $w = p_m \nabla^2 \chi$ is the "vertical velocity" at 500 hPa, χ is the lower level velocity potential, $p_m = 500$ hPa, $\bar{\sigma}$ is the static stability.

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Jacobian & Thermal Wind

$$J(\psi,\zeta) = \frac{\partial \psi}{\partial \lambda}\frac{\partial \zeta}{\partial \mu} - \frac{\partial \psi}{\partial \mu}\frac{\partial \zeta}{\partial \lambda}$$

$$\theta = \frac{\mathrm{f_C}}{2bc_p}(\psi^1 - \psi^2) = \frac{\bar{\sigma}}{\mathrm{f_C}} F_L (\psi^1 - \psi^2)$$

$$F_L = \mathrm{f_C^2} \big/ 2bc_p\bar{\sigma} \;\; \text{and}\; b = 0.124$$

λ is longitude, μ is sine latitude

QG Potential Vorticity Equation

$$\frac{\partial q^j}{\partial t} = -J(\psi^j, q^j) - B\frac{\partial \psi}{\partial \lambda} - \sum_l D_0^{jl} q^l - \alpha^j q^j + \kappa^j(\tilde{q}^j - q^j),$$

$$q^j = \nabla^2 \psi^j + (-1)^j F_L(\psi^1 - \psi^2)$$

Here, q^j are the reduced potential vorticity fields,

at the upper ($j = 1$) and lower ($j = 2$) levels, D_0^{jl} are dissipation operators,

α^j are drag coeficients and $\tilde{q}^j$ are mean states,

towards which the fields are relaxed, with relaxation coefficients κ^j.

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Spectral QG Potential Vorticity Equation

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Spectral QG equation on the sphere

$$\frac{\partial q_{mn}^{j}(t)}{\partial t} = \mathrm{i}\sum_{m_p p}\sum_{m_q q} A_{n\ p\ q}^{m\, m_p\, m_q} \zeta_{-m_p p}^{j} q_{-m_q q}^{j} - \sum_{l} D_0^{jl}(m,n) q_{mn}^{l} - \mathrm{i}\omega_{mn}\zeta_{mn}^{j} - \alpha_{mn}^{j}\zeta_{mn}^{j}$$
$$+ \kappa_{mn}^{j}(\tilde{q}_{mn}^{j} - q_{mn}^{j}),$$

where $\omega_{mn}^{j} = -Bm[n(n+1)]^{-1}$,

or

$$\frac{\partial q_{\mathbf{k}}^{j}(t)}{\partial t} = \mathrm{i}\sum_{\mathbf{p}}\sum_{\mathbf{q}} A(\mathbf{k},\mathbf{p},\mathbf{q}) \zeta_{-\mathbf{p}}^{j} q_{-\mathbf{q}}^{j} - \sum_{l} D_0^{jl}(\mathbf{k}) q_{\mathbf{k}}^{l} - \mathrm{i}\omega_{\mathbf{k}}\zeta_{\mathbf{k}}^{j} - \alpha_{\mathbf{k}}^{j}\zeta_{\mathbf{k}}^{j}$$
$$+ \kappa_{\mathbf{k}}^{j}(\tilde{q}_{\mathbf{k}}^{j} - q_{\mathbf{k}}^{j}).$$

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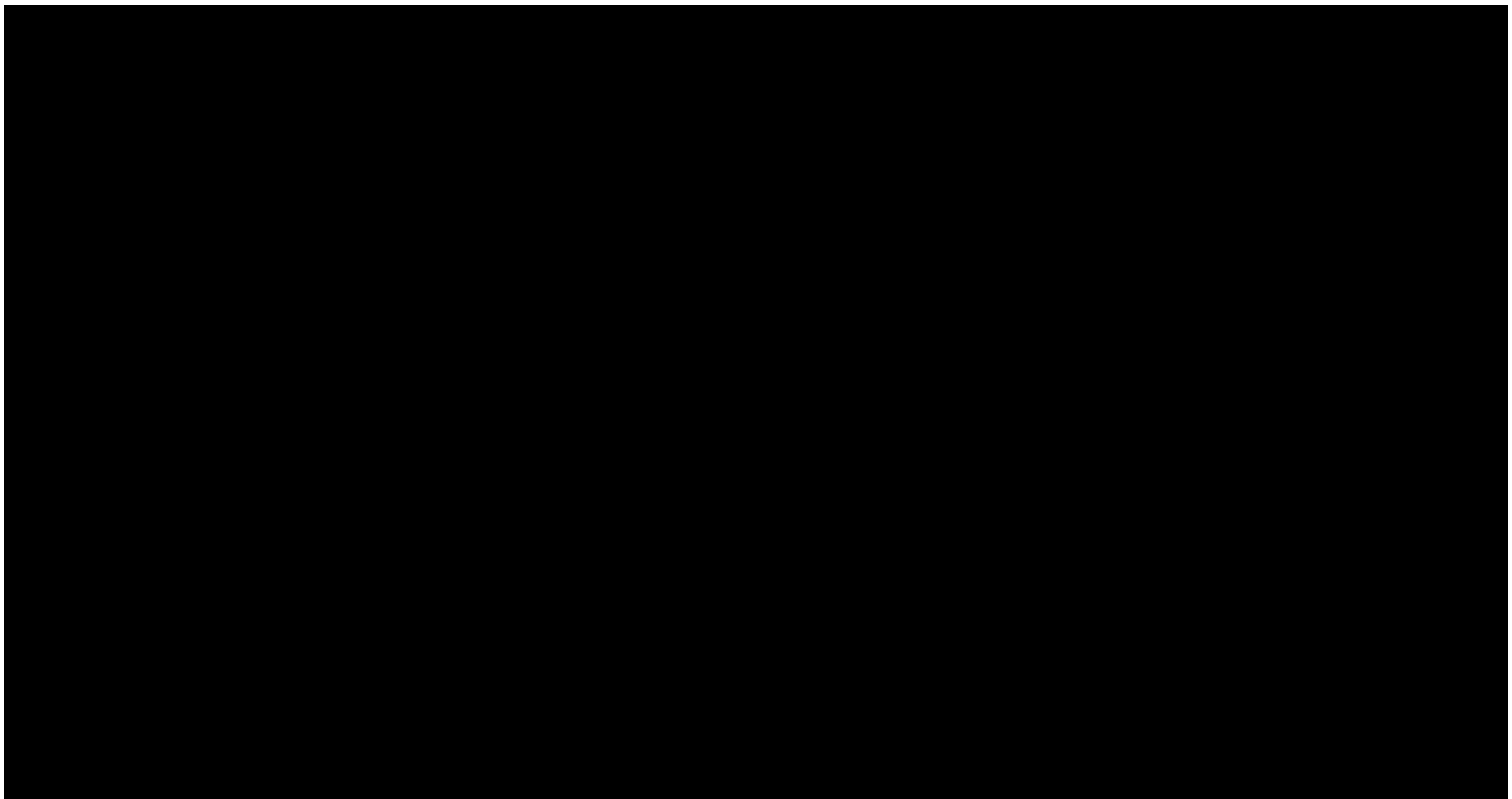
Baroclinic Atmospheric Model Results: Zonal Jets

Zidikheri and Frederiksen 2009, Journal of the Atmospheric Sciences

Longitudinally averaged (a) relaxation zonal winds, and (b) time averaged winds in T126 DNS

F_L=100



250 hPa level solid; 750 hPa dashed

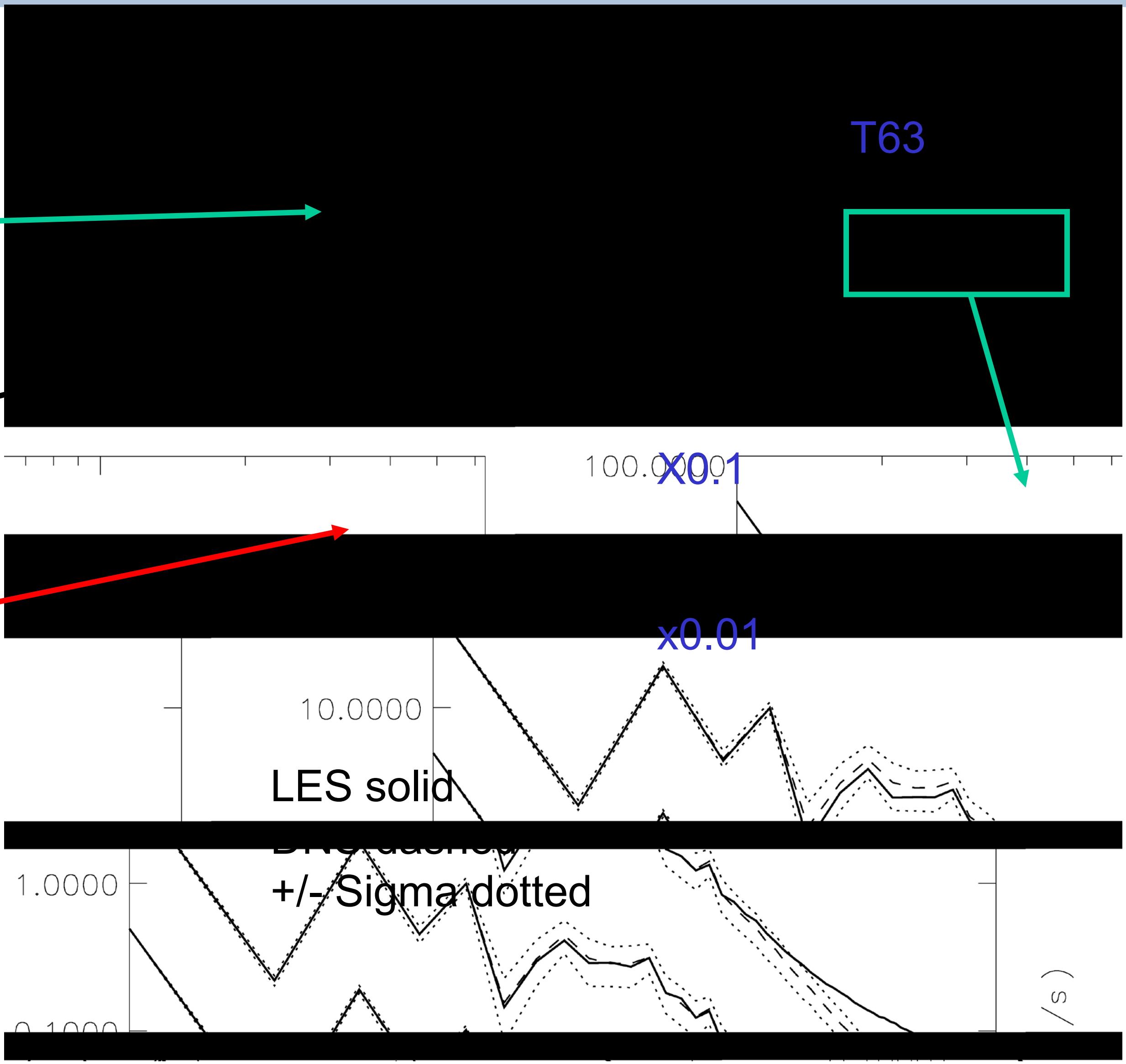
Baroclinic Atmospheric Results: LES at T63 level 1

Kinetic Energy level 1

Bare Dissipation

Renormalized Net Dissipation

Renormalized Drain Dissipation and Stochastic Backscatter



Since F_L=100 the levels decouple

Baroclinic Oceanic Results

Zidikheri and Frederiksen 2010, Phil. Transactions of the Royal Society

www.csiro.au

F_L=10,000

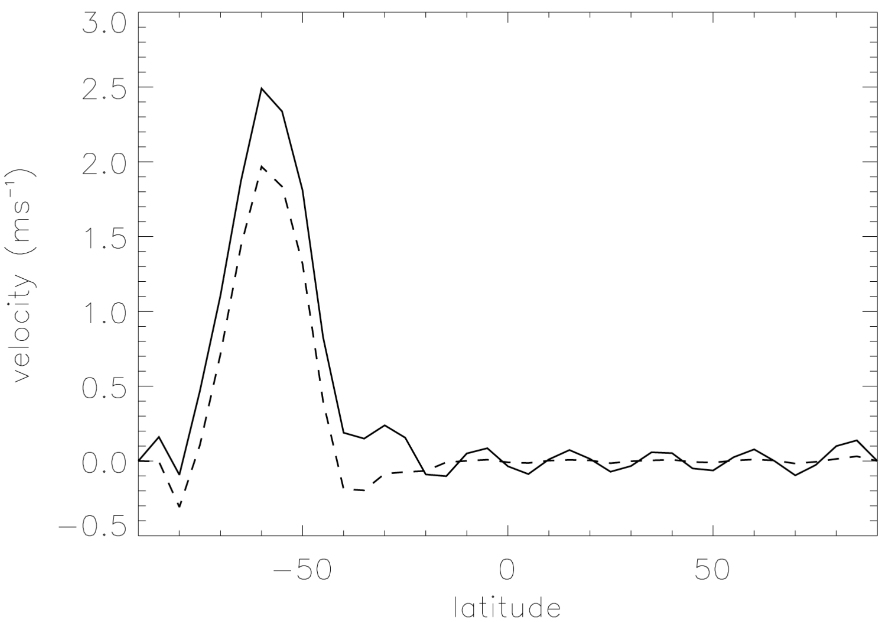


Figure : Longitudinally averaged time-averaged currents for Level 1 (upper level, solid) and Level 2 (lower level, dashed).

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Ocean: Barotropic and Baroclinic Dissipations

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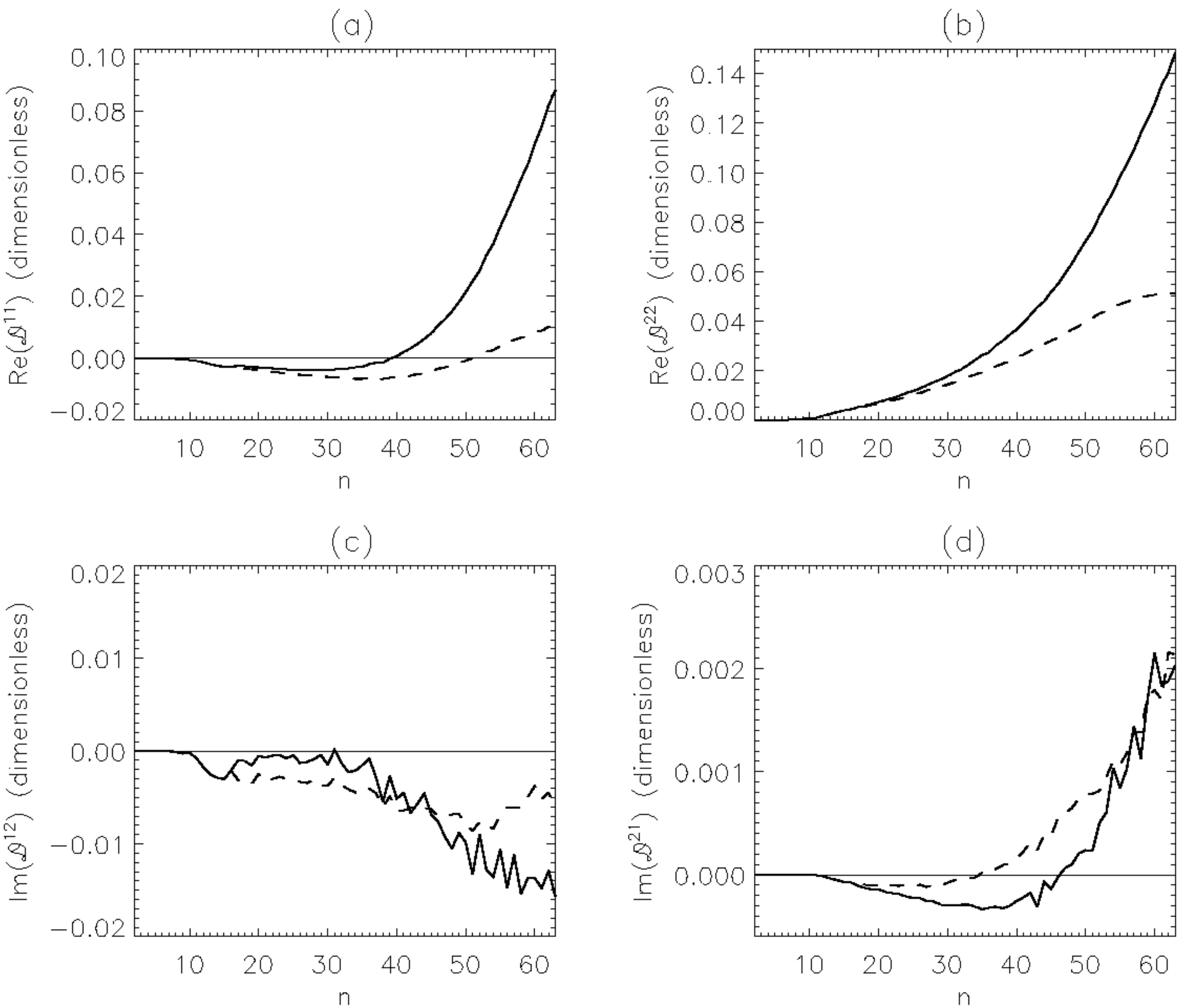

Drain-solid

Net - dashed

T252 to T63

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Ocean: Barotropic and Baroclinic Eddy Noise

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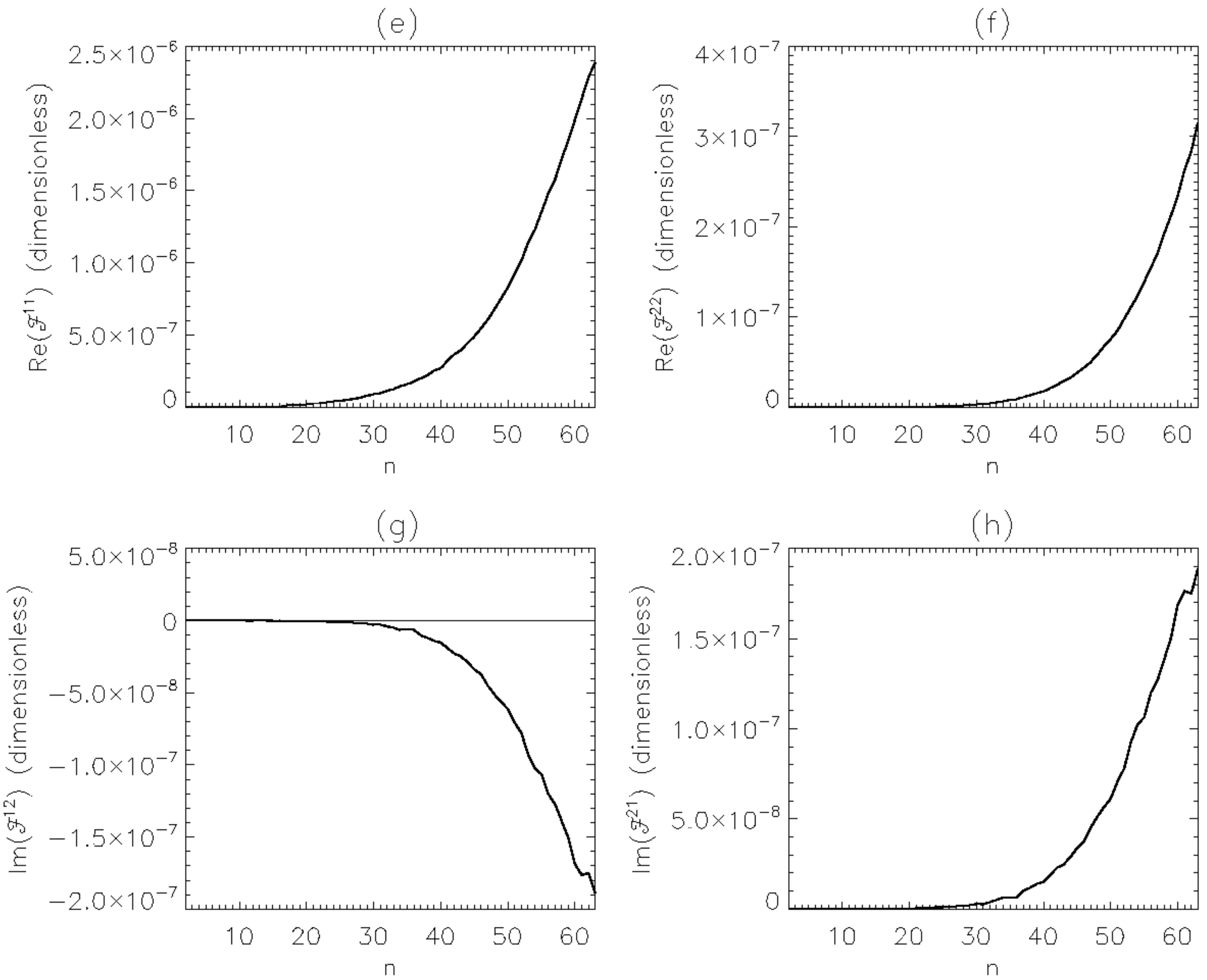

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Baroclinic Ocean LES

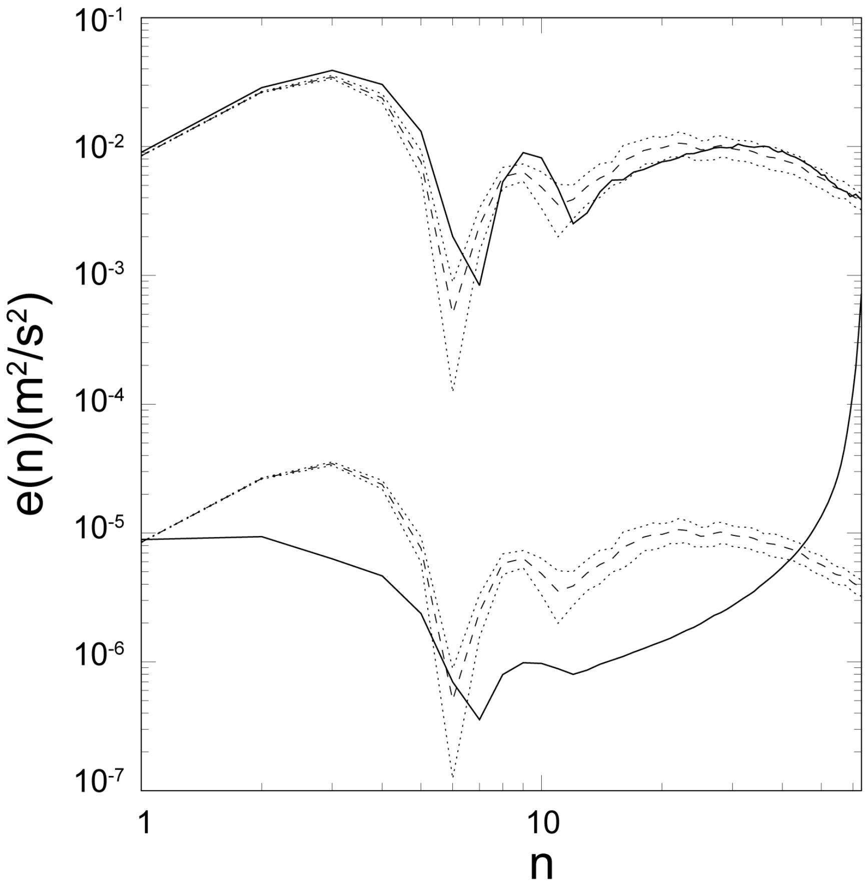

Figure : Kinetic energy spectra in m^2s^{-2} at Level 1 as functions of total wavenumber for LES at T63 (solid); for DNS at T252 truncated back to T63 (dashed); and DNS $\overline{e(n)}$ ± standard deviation (dotted). The top results are for LES with subgrid-scale parameterizations; the bottom results are for LES without subgrid-scale parameterizations scaled by 10^{-3}.

Scaling Laws for Subgrid Model - Atmosphere

Kitsios, Frederiksen & Zidikheri 2011, JAS

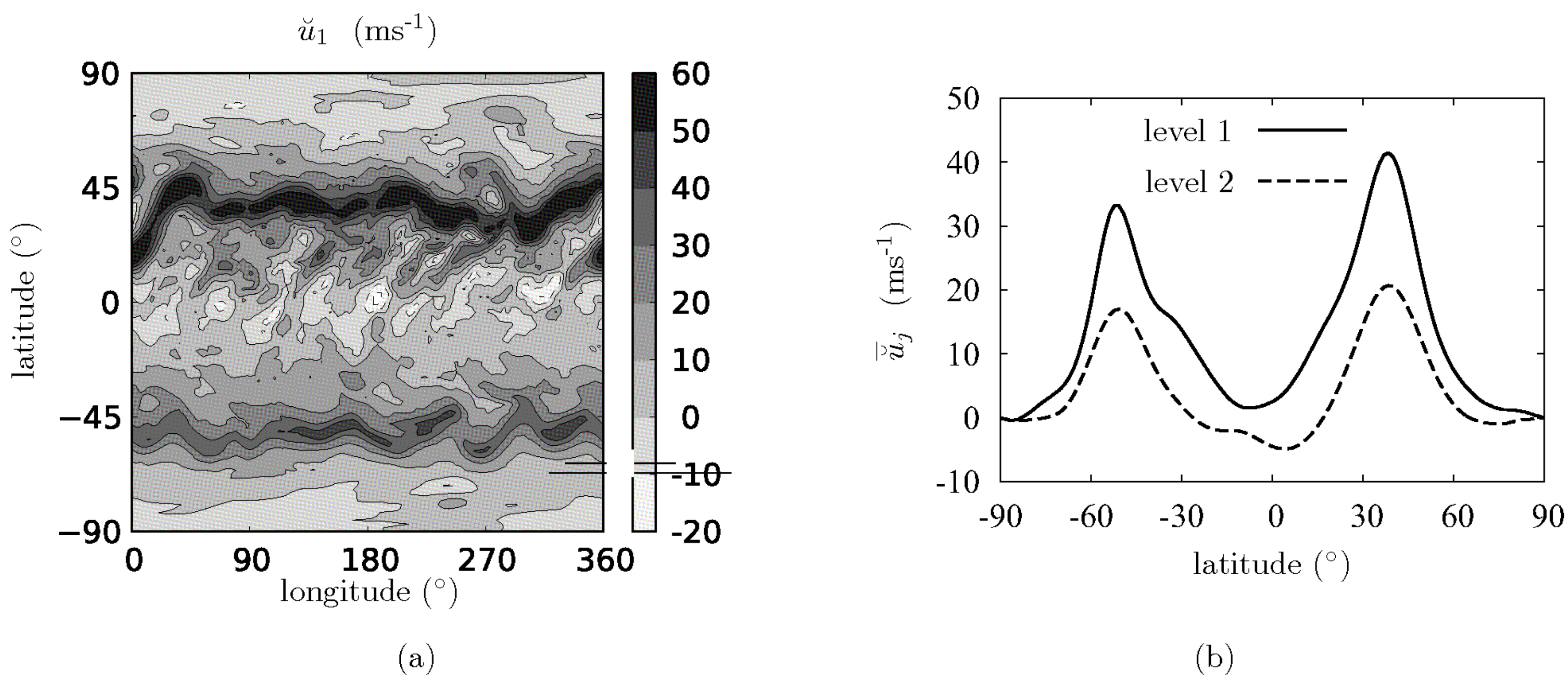

FIG. 1. Flow field from $\mathcal{A}504$ illustrated by: (a) level 1 instantaneous zonal velocity field $\breve{u}_1$; and (b) the time averaged zonal velocity $\overline{\breve{u}}_j$ at level $j = 1$ and level $j = 2$. In both figures the velocity is presented in metres per second (ms^{-1}).

Cases A and B

Case A

$\breve{\alpha}_{mn}^{1}$ correspond to 20 day e - folding time for $n \leq 15$

$\breve{\alpha}_{mn}^{2}$ correspond to 5 day e - folding time for $n \leq 15$

$e_1(n) \sim n^{-3.2}$; $e_2(n) \sim n^{-3.2}$ close to n^{-3} inertial range

Case B

$\breve{\alpha}_{mn}^{1}$ correspond to 20 day e - folding time for all n

$\breve{\alpha}_{mn}^{2}$ correspond to 5 day e - folding time for all n

$e_1(n) \sim n^{-3.5}$; $e_2(n) \sim n^{-4.1}$

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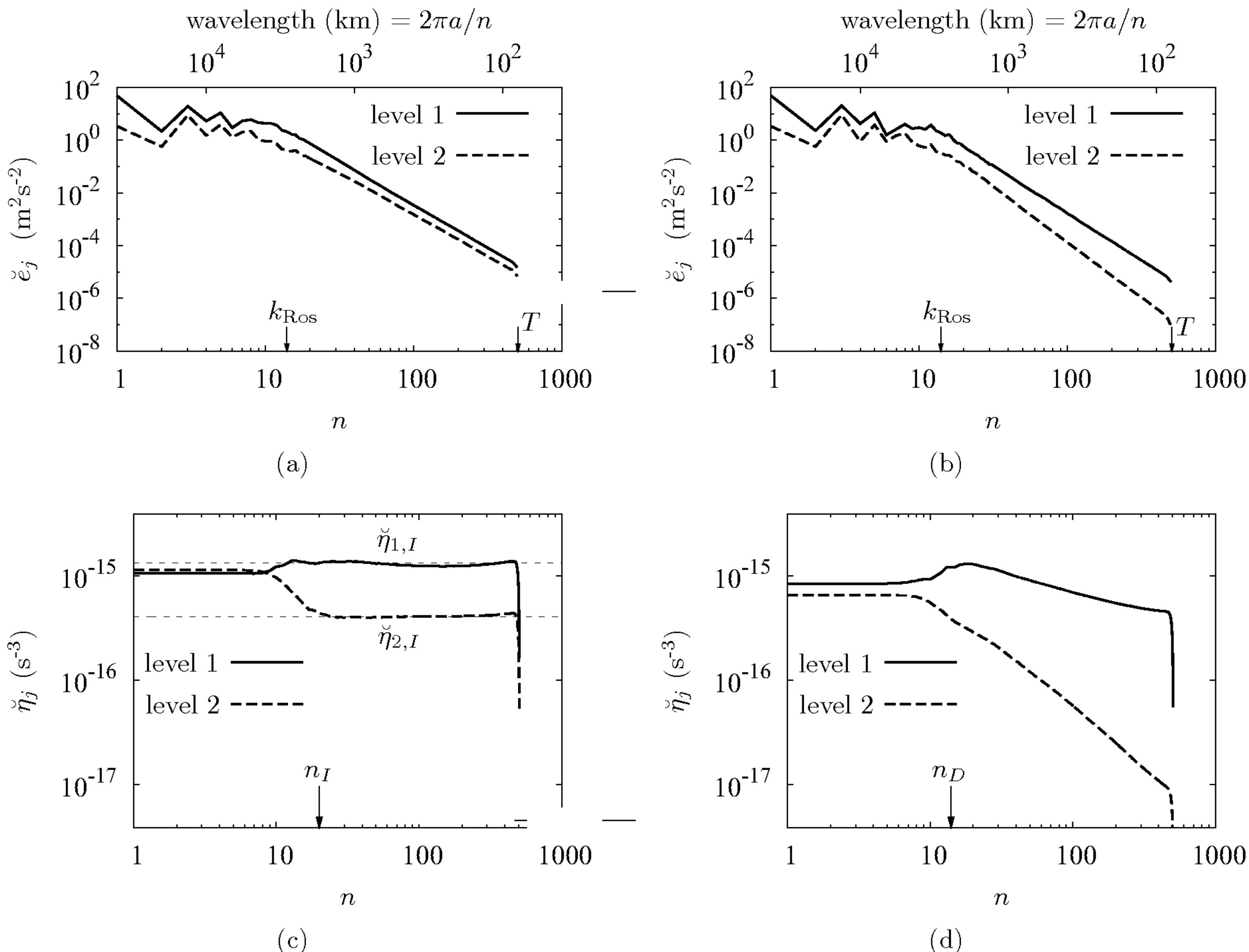
Kinetic Energy and Enstrophy Flux Spectra

A

B



(a)

(b)

(c)

(d)

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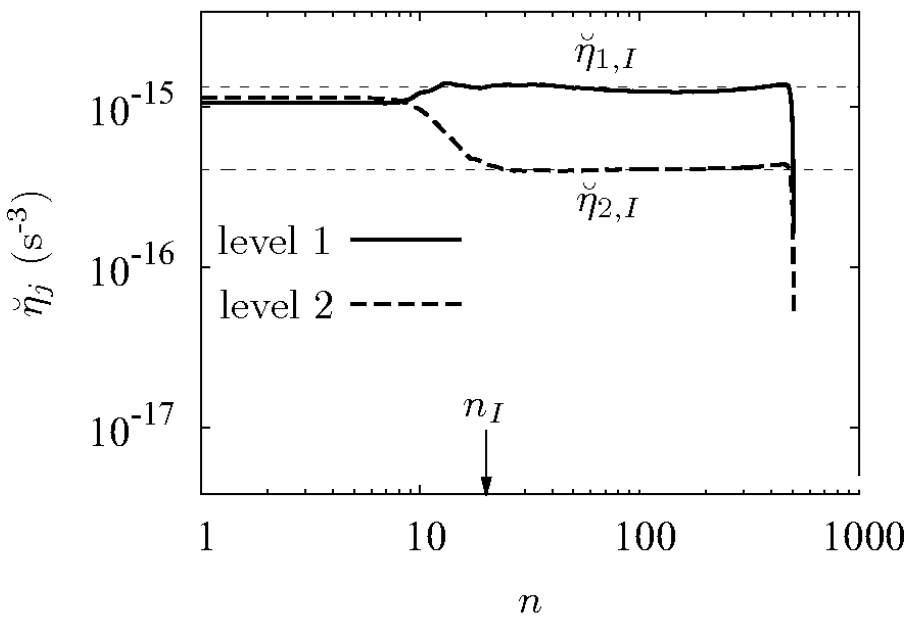
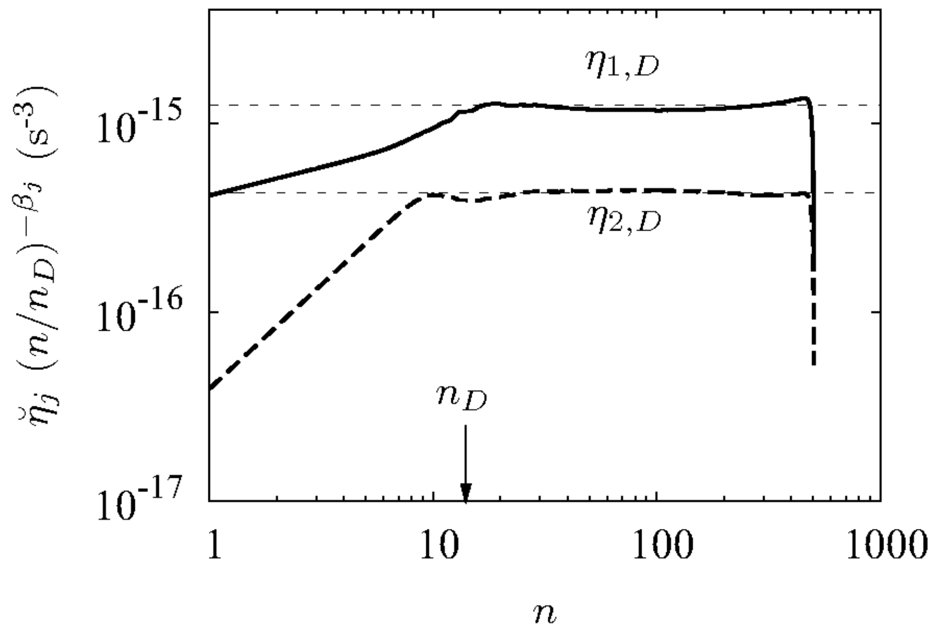
Enstrophy Flux Spectra (Case B - Rectified)

Case A



Case B



$\breve{\eta}_{j,ref} = \breve{\eta}_{j,I}$ for case A; $\breve{\eta}_{j,ref} = \breve{\eta}_{j,D}$ for case B

Drain, Backscatter and Net Eddy Viscosities

T126 to T63 Level 1

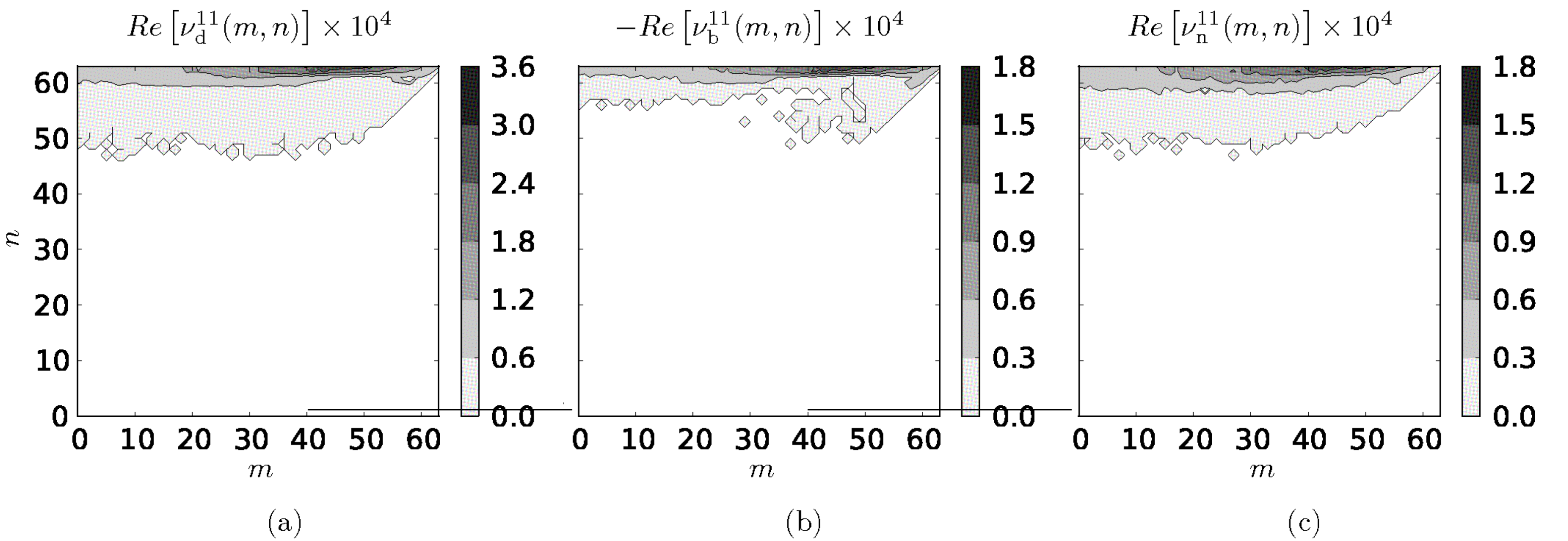


FIG. 3. Anisotropic subgrid-scale coefficients required to truncate the $\mathcal{A}126$ DNS back to an LES with $T_R = 63$: (a) $Re\left[\nu_{\mathrm{d}}^{11}(m,n)\right]$; (b) $-Re\left[\nu_{\mathrm{b}}^{11}(m,n)\right]$; and (c) $Re\left[\nu_{\mathrm{n}}^{11}(m,n)\right]$. Note the n axis in (a) is applicable to all figures.

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Net Eddy Viscosities with Changing Resolution T_R

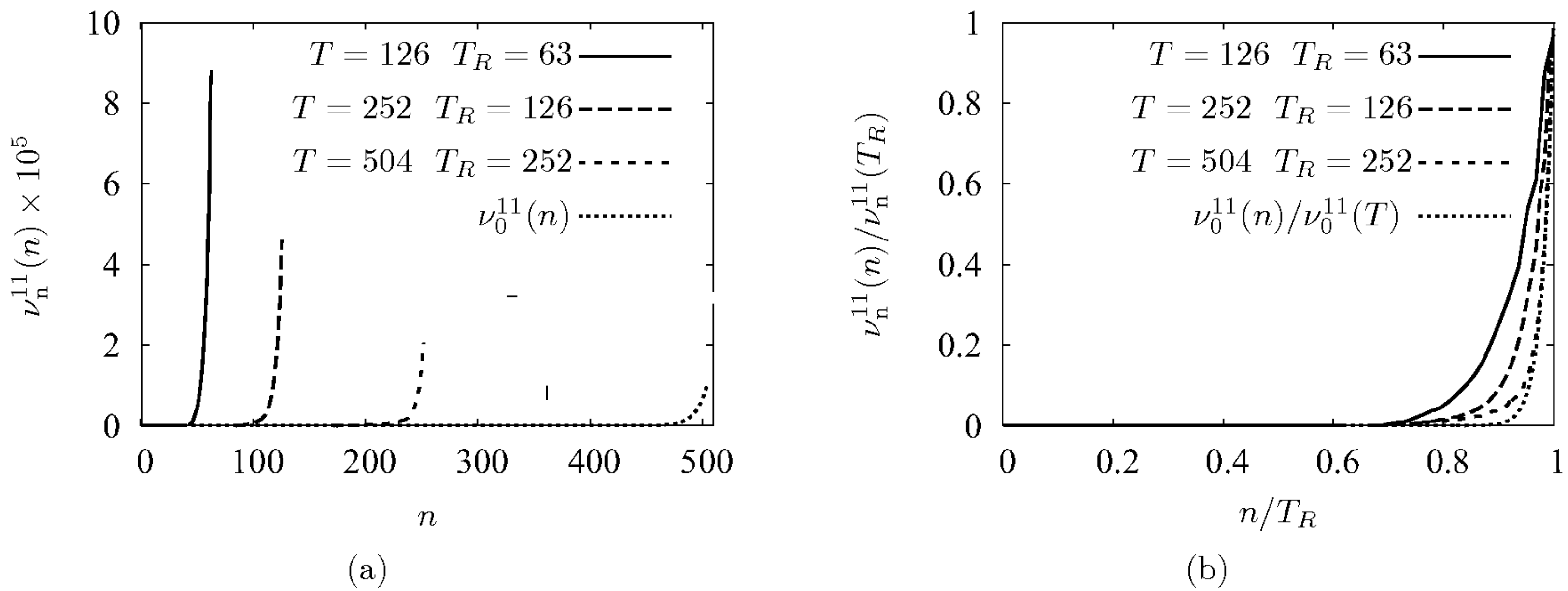

FIG. 4. Isotropic net eddy viscosity subgrid coefficients: (a) $\nu_n^{11}(n)$; and (b) scaled such that $\nu_n^{11}(n)/\nu_n^{11}(T_R)$ plotted against n/T_R. The bare eddy viscosity ν_0^{11} for the $\mathcal{A}504$ DNS is also included in the above figures.

Scaling Properties

Kraichnan (1967)

$$e_j(n) = C\eta_{j,I}^{2/3} n^{-3}$$

Leith (1971)

$$\nu^{jj}(T_R) \sim \eta_{j,I}^{1/3}$$

Scaling Properties

$$D_{\bullet}^{jj}(n) = D_{\bullet}^{jj}(T_R)\left(\frac{n}{T_R}\right)^{\rho_{\bullet}^{j}},$$

$$n(n+1) \sim n^2,\ T_R(T_R+1) \sim T_R^2$$

$$\nu_{\bullet}^{jj}(n) = \nu_{\bullet}^{jj}(T_R)\left(\frac{n}{T_R}\right)^{\rho_{\bullet}^{j}-2},\ \bullet = \text{n, d or b}$$

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Scaling Properties of Net Viscosity

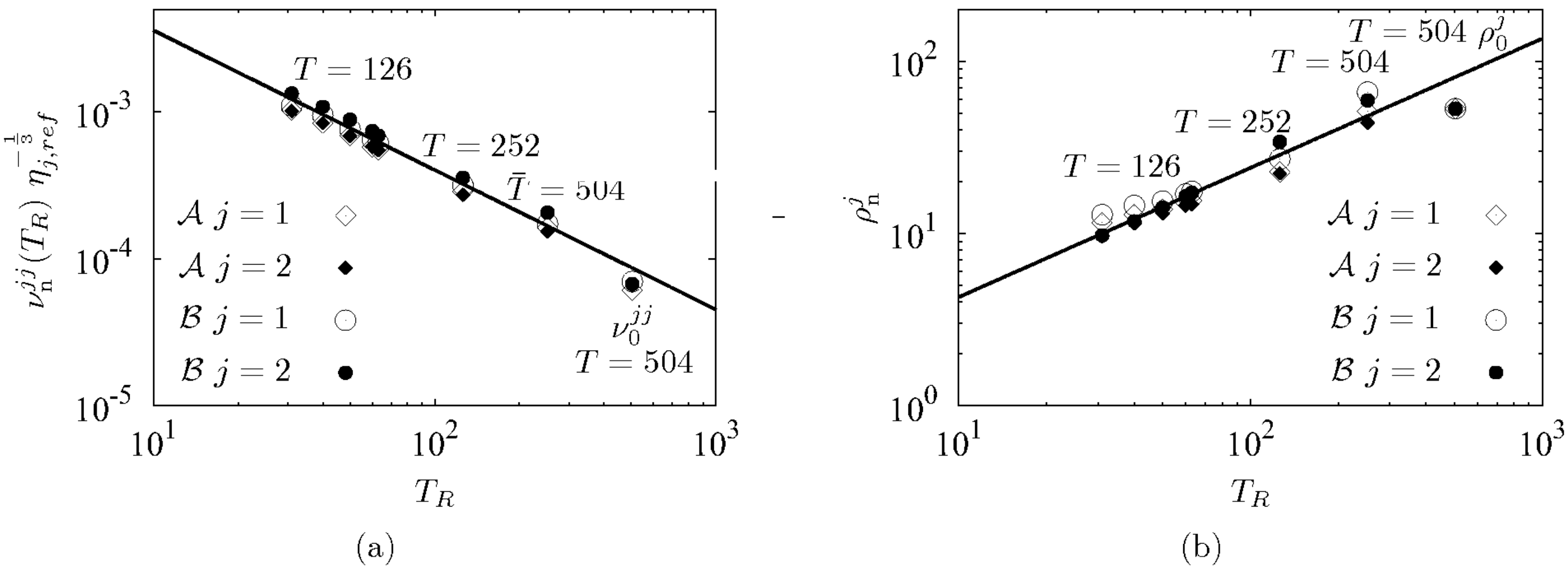

FIG. 5. The scaling of the subgrid coefficients properties: (a) $\nu_n^{jj}(T_R)$ with trend line $0.032\ \eta_{j,ref}^{1/3}\ T_R^{-1}$; and (b) ρ_n^j with trend line $0.76\ T_R^{0.75}$. The truncation wavenumber of the DNS from which theses coefficients were determined is labelled on the figures. As a point of reference, the data points at $T_R = 504$ represent the properties of the bare dissipation applied to the $\mathcal{A}504$ and $\mathcal{B}504$ DNS. The solid lines are the least squares regression lines fit to the data.

Convergence of Drain and Backscatter Viscosities

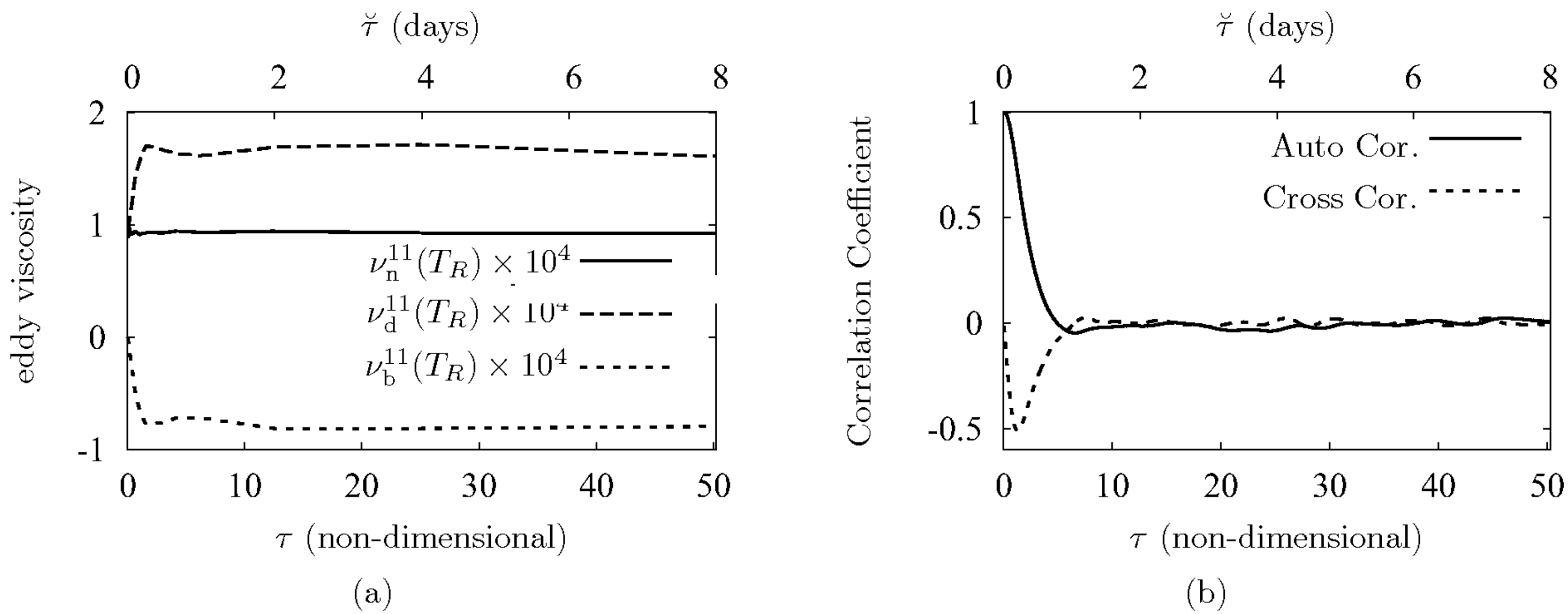

FIG. 6. For the $\mathcal{B}126$ DNS truncated back to $T_R = 63$, the impact of the integration period (τ) on the subgrid coefficients is illustrated by: (a) $\nu_{\mathrm{n}}^{11}(T_R)$, $\nu_{\mathrm{d}}^{11}(T_R)$, $\nu_{\mathrm{n}}^{11}(T_R)$ versus τ; and to illustrate what contributes to $\nu_{\mathrm{d}}^{11}(T_R)$ (b) the cross-correlation (Cross Cor.) between $\left(\widehat{q}^{S}\right)^{1}_{mn}(t+\tau)$ and $\widehat{q}^{1}_{mn}(t)$, and the auto-correlation (Auto Cor.) between $\widehat{q}^{1}_{mn}(t+\tau)$ and $\widehat{q}^{1}_{mn}(t)$, for the wavenumber pair $(m, n) = (T_R, T_R)$.

Convergence of Drain and Backscatter Viscosities

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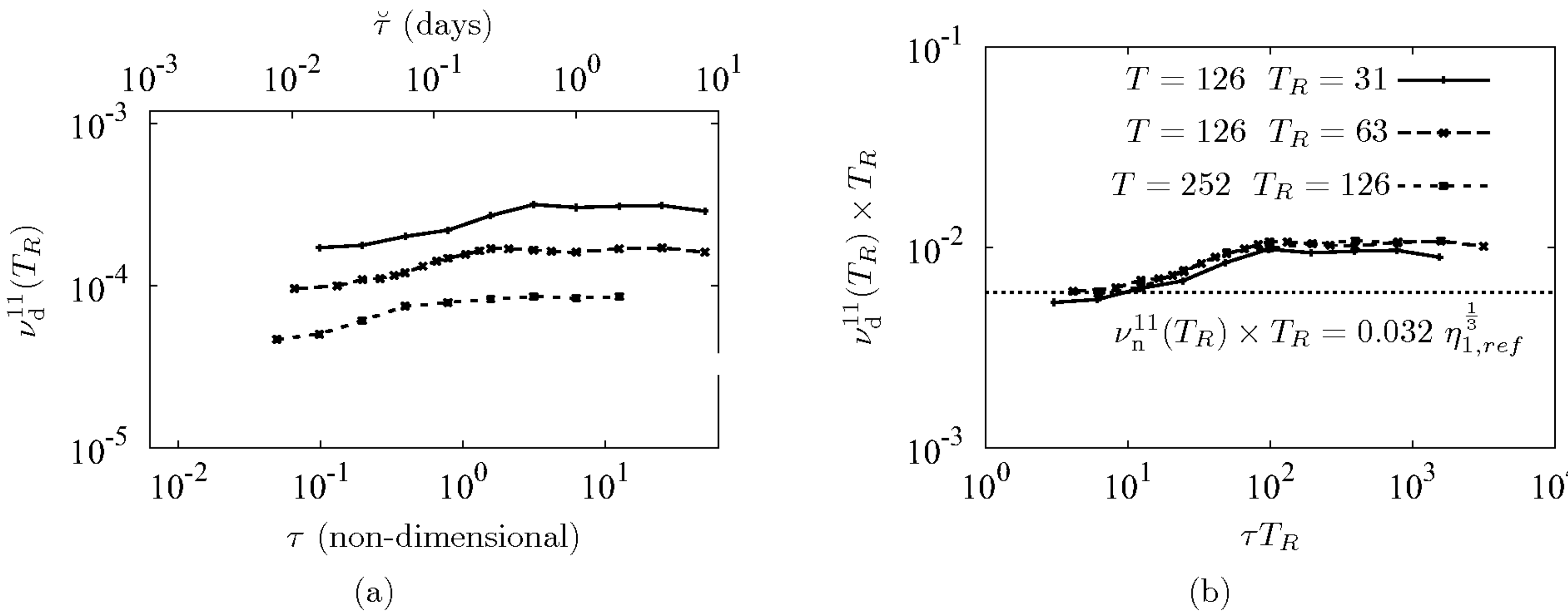

FIG. 7. For various DNS (T) and LES truncation resolutions (T_R): (a) $\nu_{\rm d}^{11}(T_R)$ versus τ; and (b) scaled such that $\nu_{\rm d}^{11}(T_R) \times T_R$ is plotted versus τT_R, with the dashed horizontal line representing $\nu_{\rm n}^{11}(T_R) \times T_R = 0.032\ \eta_{1,ref}^{1/3}$, where $\eta_{1,ref} = \eta_{1,D}$. The legend in (b) is also applicable to the data in (a).

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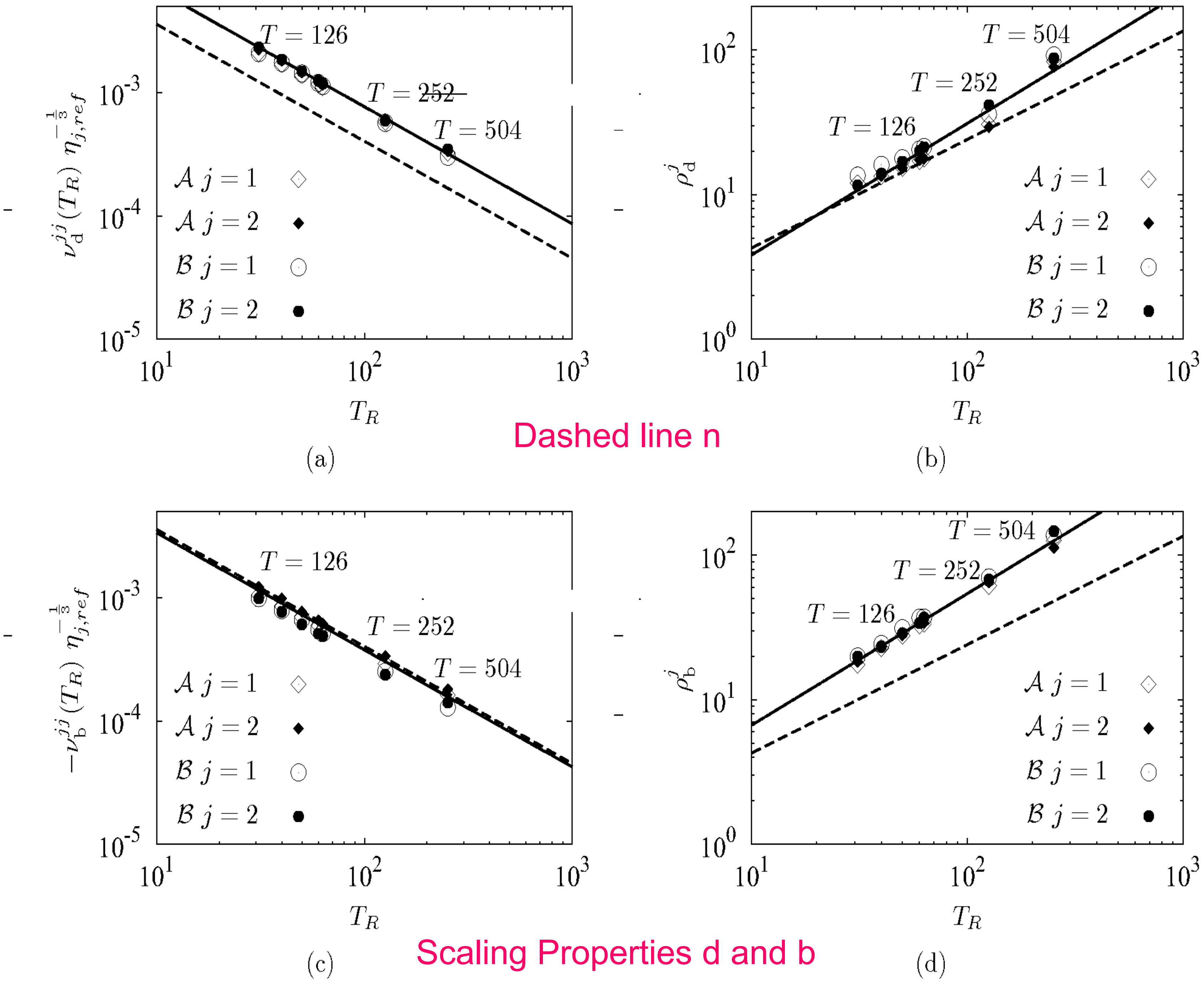

$T = 126$
$T = 252$
$T = 504$
$\mathcal{A}\ j = 1$
$\mathcal{A}\ j = 2$
$\mathcal{B}\ j = 1$
$\mathcal{B}\ j = 2$
$\nu_{\mathrm{d}}^{jj}(T_R)\ \eta_{j,ref}^{-\frac{1}{3}}$
ρ_{d}^{j}
$-\nu_{\mathrm{b}}^{jj}(T_R)\ \eta_{j,ref}^{-\frac{1}{3}}$
ρ_{b}^{j}
T_R
(a)
(b)
(c)
(d)
Dashed line n
Scaling Properties d and b

Scaling Laws $\nu_{\mathrm{d}}^{jj}(T_R) \approx 2\nu_{\mathrm{n}}^{jj}(T_R) \approx -2\nu_{b}^{jj}(T_R)$

definition	
$\nu_{\mathrm{n}}^{jj}(n) = \nu_{\mathrm{n}}^{jj}(T_R)\left(\frac{n}{T_R}\right)^{\rho_{\mathrm{n}}^{j}-2}$	$\rightarrow(26)$
$\nu_{\mathrm{d}}^{jj}(n) = \nu_{\mathrm{d}}^{jj}(T_R)\left(\frac{n}{T_R}\right)^{\rho_{\mathrm{d}}^{j}-2}$	$\rightarrow(29)$
$\nu_{\mathrm{b}}^{jj}(n) = \nu_{\mathrm{b}}^{jj}(T_R)\left(\frac{n}{T_R}\right)^{\rho_{\mathrm{b}}^{j}-2}$	$\rightarrow(32)$

$$\nu_{\mathrm{d}}^{jj}(T_R) \approx 2\nu_{\mathrm{n}}^{jj}(T_R) \approx -2\nu_{b}^{jj}(T_R)$$

value at truncation		power exponent	
$\nu_{\mathrm{n}}^{jj}(T_R) = 0.032\ \eta_{j,ref}^{\frac{1}{3}}\ T_R^{-1}$	$\rightarrow(27)$	$\rho_{\mathrm{n}}^{j} = 0.76\ T_R^{0.75}$	$\rightarrow(28)$
$\nu_{\mathrm{d}}^{jj}(T_R) = 0.063\ \eta_{j,ref}^{\frac{1}{3}}\ T_R^{-1}$	$\rightarrow(30)$	$\rho_{\mathrm{d}}^{j} = 0.47\ T_R^{0.91}$	$\rightarrow(31)$
$\nu_{\mathrm{b}}^{jj}(T_R) = -0.031\ \eta_{j,ref}^{\frac{1}{3}}\ T_R^{-1}$	$\rightarrow(33)$	$\rho_{\mathrm{b}}^{j} = 0.82\ T_R^{0.91}$	$\rightarrow(34)$

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LESs with net eddy viscosity from DNS

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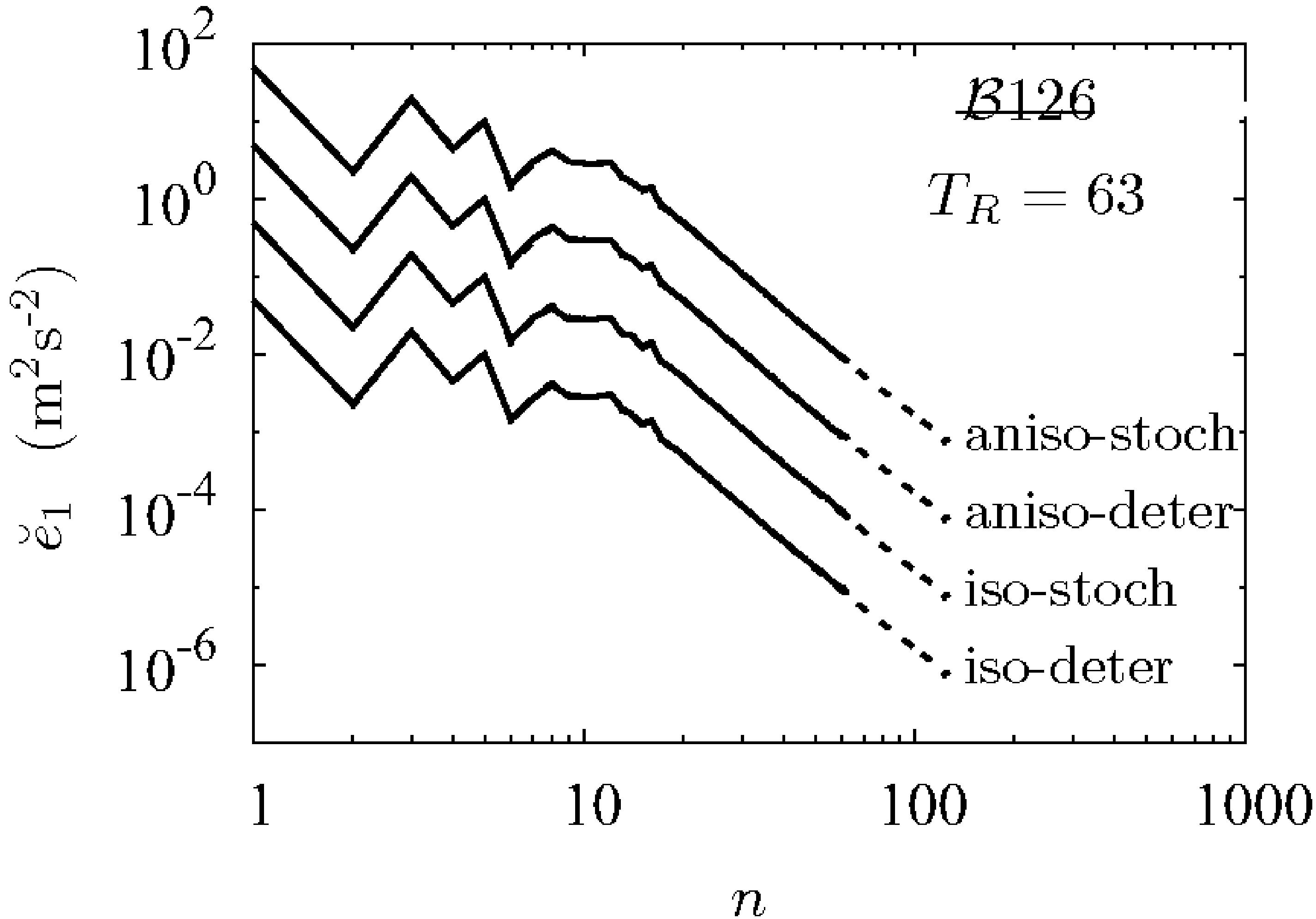
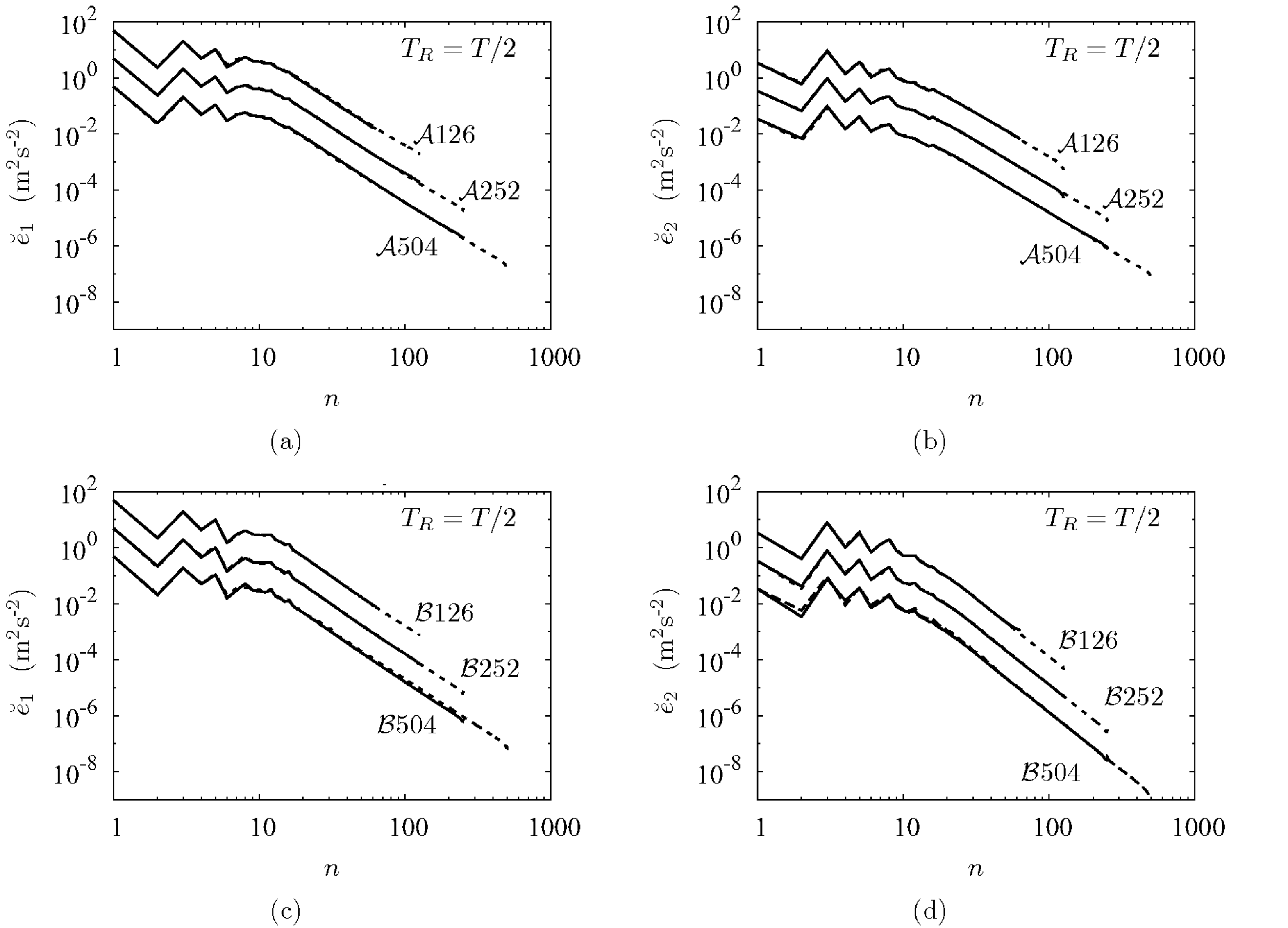


LESs with net eddy viscosity satisfying scaling law.

Nearly identical results using drain viscosity and stochastic backscatter



(a) (b) (c) (d)

Discussion

More general applications ?

From stochastic LES for q to w:

$$(q^j, \frac{\partial q^j}{\partial t}) \Rightarrow (\zeta^j, \frac{\partial \zeta^j}{\partial t}; \theta, \frac{\partial \theta}{\partial t}) \Rightarrow w$$

and particularly updrafts in vertical velocity.

Morales and Nenes, 2010 JGR Statistics of updrafts in clouds

Conclusions

- Closure based subgrid-scale parameterizations have been compared with those based on a stochastic modelling approach.
- A stochastic method for subgrid-scale modelling has been applied to barotropic and baroclinic atmospheric and oceanic turbulent flows.
- Baroclinic turbulent flows with mean jets and rotation have been considered within QG models for the atmosphere and ocean.
- The subgrid-scale parameterizations of eddy viscosity and dissipation and stochastic backscatter have a cusp at the smallest retained scales of LES.
- LES with renormalized subgrid-scale parameterizations are in excellent agreement with DNS.
- For atmospheric flows deterministic and stochastic parameterizations are equally successful in reproducing mean flows and spectra.
- For oceanic flows a stochastic parameterization is necessary to represent injection due to unresolved baroclinic instability.
- A subgrid model with scaling laws has been developed for atmospheric QG simulations.

Thank you

Recent Reviews:

J. Denier and J.S. Frederiksen, *Frontiers in Turbulence and Coherent Structures*, World Scientific Lecture Notes in Complex Systems, Vol. **6**, 490pp, (2007).

J.S. Frederiksen and T.J. O'Kane, Entropy, closures and subgrid modeling. *Entropy,* **10**, 635-683 (2008).

J.S Frederiksen, T.J. O'Kane and M.J. Zidikheri, Subgrid modelling for geophysical flows. Phil. Trans. Roy. Soc., (2011)

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- J.S. Frederiksen and A.G. Davies, Eddy viscosity and stochastic backscatter parameterizations on the sphere for atmospheric circulation models. *J. Atmos. Sci.*, **54**, 2475-2492, (1997).
- J.S. Frederiksen, Subgrid scale parameterizations of eddy-topographic force, eddy viscosity and stochastic backscatter for flow over topography. *J. Atmos. Sci*., **56**, 1481-1494 (1999).
- J.S. Frederiksen, M.R. Dix and A.G. Davies, The effects of closure-based eddy diffusion on the climate and spectra of a GCM. *Tellus* **55** A, 31-44 (2003).
- J.S. Frederiksen and T.J. O'Kane, Turbulence closures and subgrid-scale parameterizations. *Frontiers in Turbulence and Coherent Structures,* Chapter 14, 315-354, (2007).
- T.J. O'Kane and J.S. Frederiksen, Statistical dynamical subgrid-scale parameterizations for geophysical flows. *Physica Scripta,* **T132,** 014033, 11pp (2008).

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Stochastic Subgrid Modelling

- J.S Frederiksen and S.M. Kepert, Dynamical subgrid-scale parameterizations from direct numerical simulations. *J. Atmos. Sci.*, **63**, 3006-3019 (2006).
- J.S Frederiksen and M.J. Zidikheri, Stochastic subgrid modelling for atmospheric large eddy simulations, *ANZIAM J.,* **50**, C490-C504 (2008).
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- M.J. Zidikheri and J.S. Frederiksen, Stochastic modelling of unresolved eddy fluxes. *Geophys. Astrophys. Fluid Dyn.,* **104**, 323-348 (2010).
- V. Kitsios, J.S. Frederiksen and M.J. Zidikheri, Subgrid Model with Scaling Laws for Atmospheric Simulations. *J. Atmos. Sci.,* **68**, (2011), to be submitted.

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