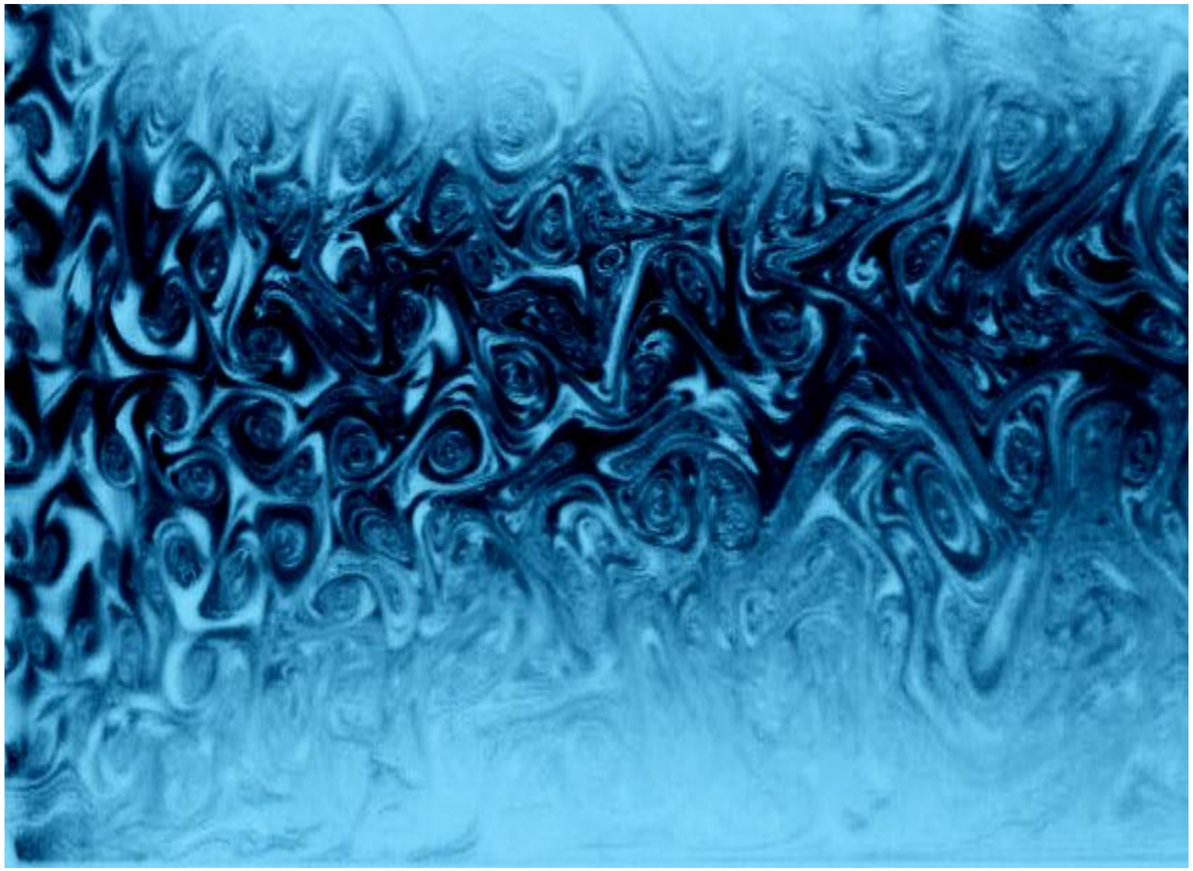
The friction factor of wall-bounded turbulence in 2D & 3D: roughness-induced criticality and the spectral link



Nigel Goldenfeld

Department of Physics

University of Illinois
at Urbana-Champaign

Nicholas Guttenberg
Tuan Tran
Carlo Zuniga Zamalloa

Gustavo Gioia
Pinaki Chakraborty

Walter Goldburg
Hamid Kellay

Supported by NSF-DMR-06-04435

Millennium Problems

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven *Prize Problems*. The Scientific Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7 million prize fund for the solution to these problems, with $1 million allocated to each. During the Millennium Meeting held on May 24, 2000 at the Collège de France, Timothy Gowers presented a lecture entitled *The Importance of Mathematics*, aimed for the general public, while John Tate and Michael Atiyah spoke on the problems. The CMI invited specialists to formulate each problem.

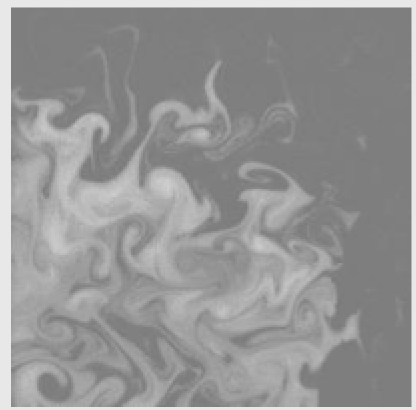
- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills Theory

- Rules
- Millennium Meeting Videos

Navier-Stokes Equation

Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

- The Millennium Problems
- Official Problem Description — Charles Fefferman
- Lecture by Luis Cafarelli (video)



Millennium Problems

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven *Prize Problems*. The Scientific Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7 million prize fund for the solution to these problems, with $1 million allocated to each. During the Millennium Meeting held on May 24, 2000 at the Collège de France, Timothy Gowers presented a lecture entitled *The Importance of Mathematics*, aimed for the general public, while John Tate and Michael Atiyah spoke on the problems. The CMI invited specialists to formulate each problem.

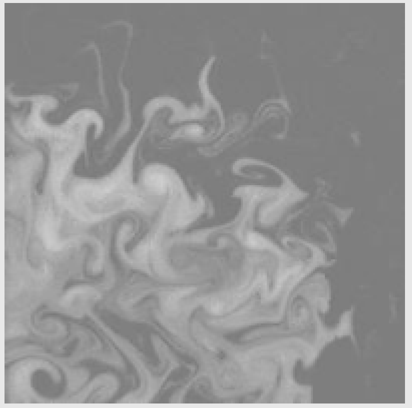
- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills Theory

- Rules
- Millennium Meeting Videos

Navier-Stokes Equation

Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

- The Millennium Problems
- Official Problem Description — Charles Fefferman
- Lecture by Luis Cafarelli (video)



Navier-Stokes Equation

Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

A fundamental problem in analysis is to decide whether such smooth, physically reasonable solutions exist for the Navier–Stokes equations. To give reasonable leeway to solvers while retaining the heart of the problem, we ask for a proof of one of the following four statements.

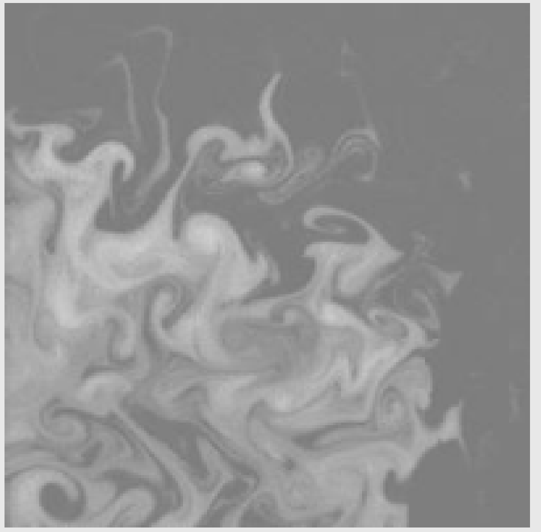
(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6), (7).

(B) Existence and smoothness of Navier–Stokes solutions in $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (8); we take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (10), (11).

(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (4), (5), for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (8), (9), for which there exist no solutions (p,u) of (1), (2), (3), (10), (11) on $\mathbb{R}^3 \times [0,\infty)$.

- The Millennium Problems
- Official Problem Description — Charles Fefferman
- Lecture by Luis Cafarelli (video)



Turbulent structure at many scales

Soap film experiment

M. A. Rutgers, X-l. Wu, and W. I. Goldburg.
"The Onset of 2-D Grid Generated Turbulence in Flowing Soap Films," Phys. Fluids 8, S7, (Sep. 1996).

Turbulent flows are multiscale

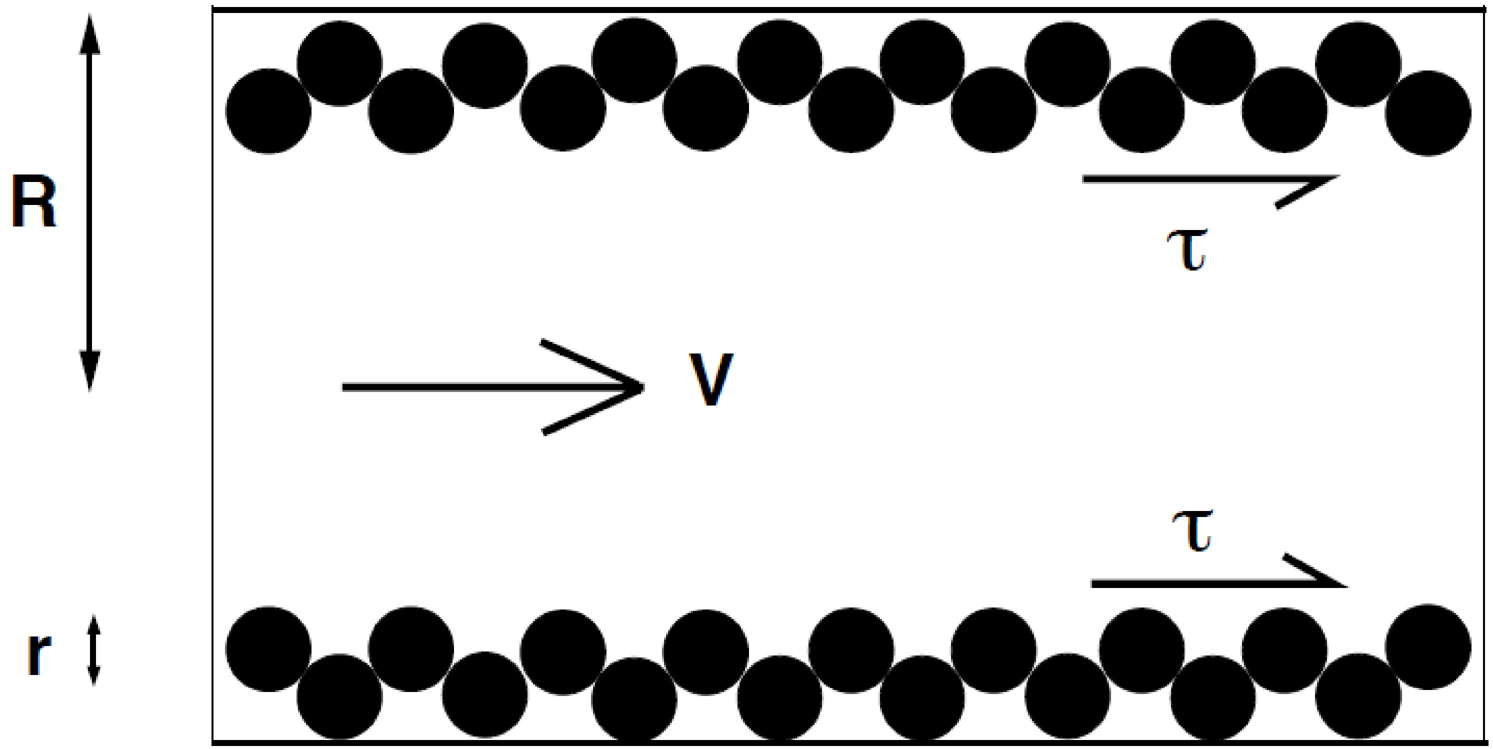
Structure at many scales …

- Yet, Prandtl's theory (1921) of wall-bounded turbulence does not represent this structure
- Thus it cannot influence the friction factor and velocity profile, in this theory!
- Is this really plausible?

Soap film experiment

M. A. Rutgers, X-l. Wu, and W. I. Goldburg. "The Onset of 2-D Grid Generated Turbulence in Flowing Soap Films," Phys. Fluids 8, S7, (Sep. 1996).

Pipe flow



$$\mathrm{Re} \equiv \frac{\mathrm{VR}}{\nu} \qquad \mathrm{Roughness} \equiv \frac{\mathrm{r}}{\mathrm{R}} \qquad \mathrm{Friction\ Factor,\ f} \equiv \frac{\tau}{\rho \mathrm{V}^2}$$

Re = Reynolds number **ν = viscosity/density = kinematic viscosity**

Nikuradse's pipe experiment (1933) to measure the friction factor *f*

$$f = \Delta P / l \rho U^2$$

Pipe diameter is 25-100 mm
Pipe length is ~ 70 diameters

Monodisperse sand grains
0.8mm glued to sides of pipe

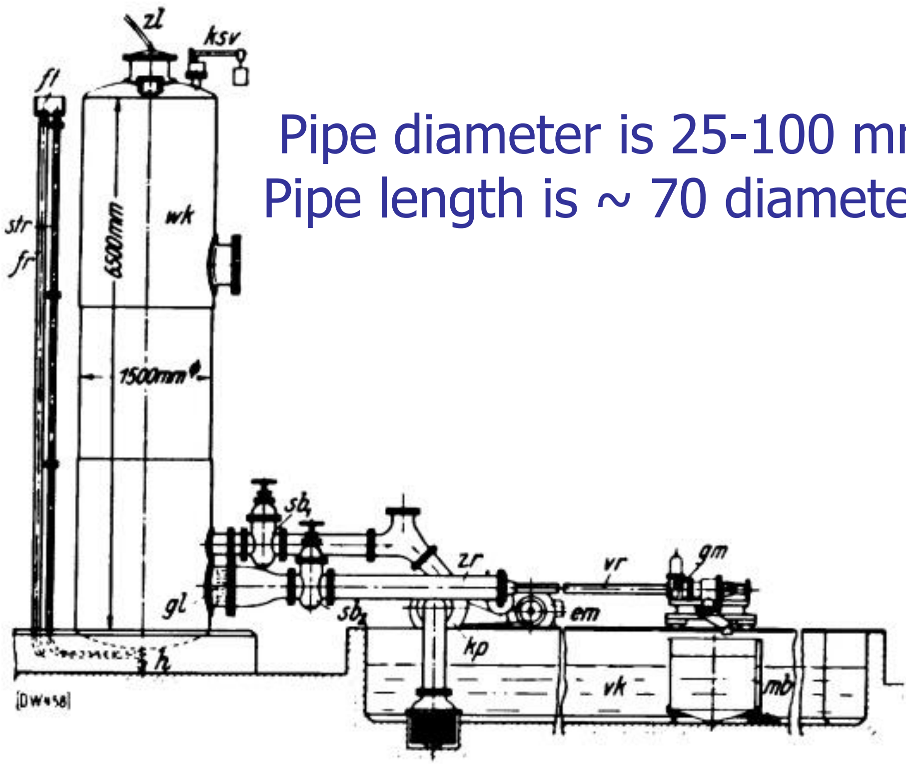


Figure 3.- Test apparatus.

em = electric motor
kp = centrifugal pump
vk = supply canal
wk = water tank
vr = test pipe
zl = supply line
str = vertical pipe
fr = overflow pipe
ft = trap
h = outlet valve
zr = feed line
mb = measuring tank
gm = velocity measuring device
ksv = safety valve on water tank
sb_1 = gate valve between wk and kp
sb_2 = gate valve between wk and zr
gl = baffles for equalizing flow

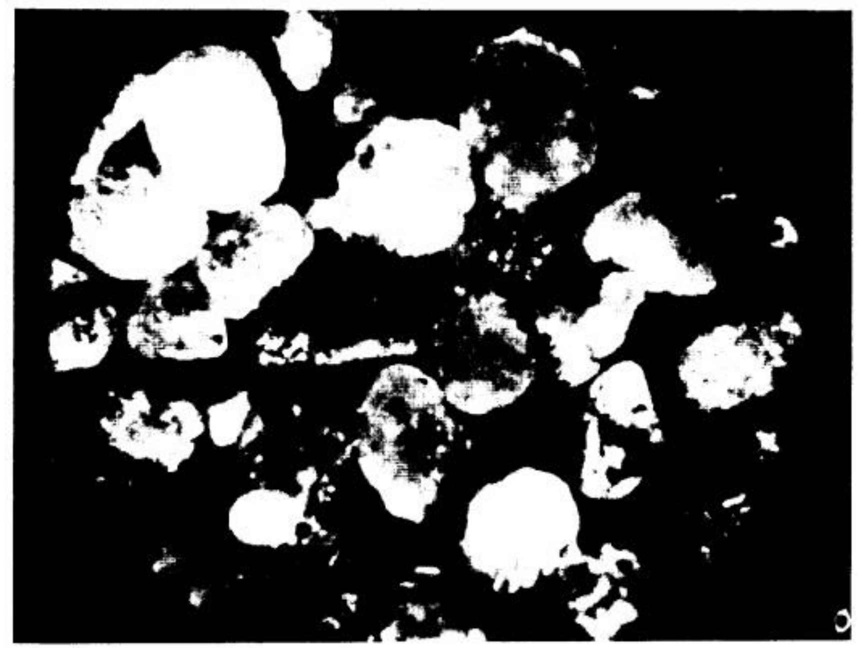


Figure 4.- Microphotograph of sand grains which produce uniform roughness. (Magnified about 20 times.)

The main message of my talk:

- **Multiscale structure of turbulence reminiscent of critical phenomena at phase transitions …**
  - **Pipe flow turbulence in smooth and rough pipes behaves as if governed by a non-equilibrium critical point**

- **Spectral connection.**
  - **Usually we talk about either spectral properties or large-scale flow properties, such as the friction factor or mean velocity profile (MVP)**
    - **Standard theory for wall-bounded turbulent flows does not connect these**
  - **We show that these are directly linked. The friction factor and MVP depend upon the functional form of the energy spectrum.**

- **We can predict how f should behave in 2D where there are two types of cascade**
  - **We observe agreement with DNS and soap film experiments**

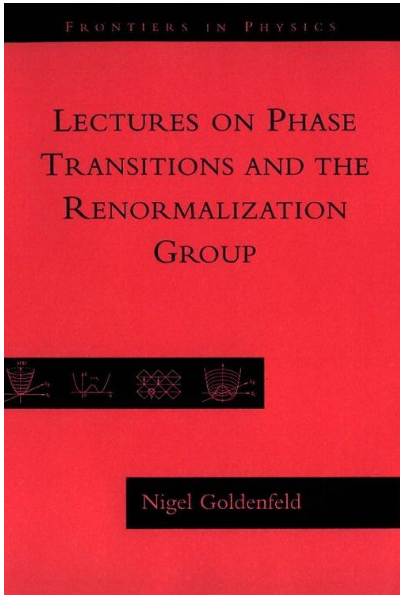
- **Prandtl theory cannot make these predictions … and therefore is incomplete. It does not have a way to represent the nature of the turbulent state.**

Outline

- Introduction
  - Critical phenomena, large fluctuations, data collapse
- Analogies between turbulence and critical phenomena
  - Beyond power law scaling, data collapse
- Turbulent pipe flow and criticality in 3D
  - Nikuradse data
  - Friction factor depends on spectral structure
- Turbulent pipe flow and simulations in 2D
  - In 2D there are two cascades and different forms of spectrum → new predictions for friction factor
  - Experimental results in 2D soap films
- Preliminary data on spectral connection in 3D turbulence
- Mean velocity profile from the spectral link

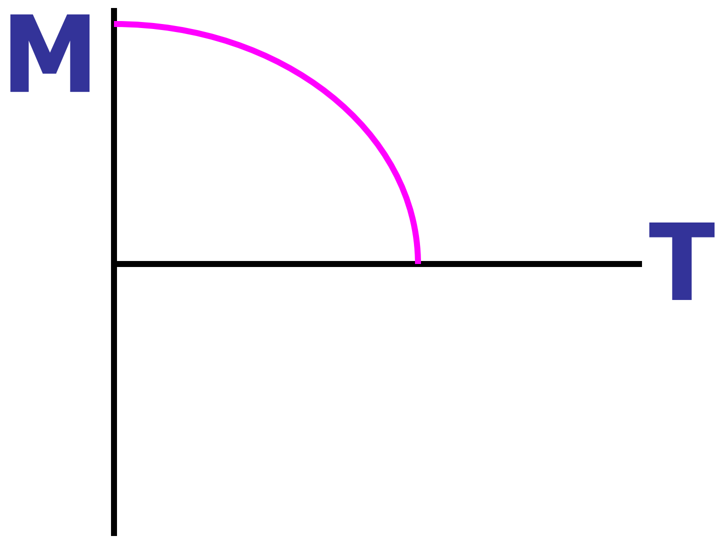
Turbulence as a critical phenomenon

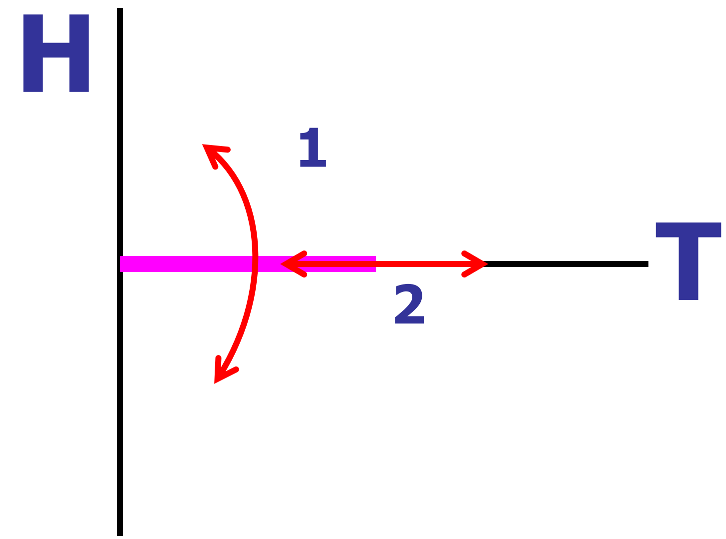
Is turbulence a critical phenomenon?



- **Common features**
  - Strong fluctuations
  - Power law correlations
- **Critical phenomena now solved**
  - Widom discovered "data collapse" (1963)
  - Kadanoff explained data collapse from coarse-graining (1966)
  - Wilson systemised and extended Kadanoff's theory (1971)
- **Turbulence still unsolved**
  - Can we repeat the pattern of discovery exemplified by critical phenomena?

Critical phenomena in magnets





$M \sim M_0[|T - T_c|/T_c]^\beta$ for $H = 0$ as $T \to T_c$

Critical isotherm: $M \sim H^{1/\delta}$ for $T = T_c$

- Widom (1963) pointed out that both these results followed from a *similarity formula*:

$$M(t,h) = |t|^\beta f_M(h/t^\Delta)$$

where $t \equiv (T - T_c)/T_c$ for some choice of exponent Δ and scaling function $f_M(x)$

Critical phenomena in magnets

$$M(t,h) = |t|^\beta f_M(h/t^\Delta)$$

where $t \equiv (T-T_c)/T_c$ for some choice of exponent Δ and scaling function $f_M(x)$

- **To determine the properties of the scaling function and unknown exponent, we require:**
  - **$f_M(z)$ = const. for z = 0**
    - **This gives the correct behaviour of the magnetization at zero field, for $T < T_c$**
  - **For large values of z, i.e. non-zero h, and t →0, we need the t dependence to cancel out.**

    Thus $f_M(z) \sim z^{1/\delta}, z \to \infty$.

    Calculate Δ: t dependence will only cancel out if $\beta - \Delta/\delta = 0$

$$M = |t|^\beta f_M(h/|t|^{\beta\delta})$$

- **This data collapse formula connects the scaling of correlations with the thermodynamics of the critical point**

Critical phenomena in magnets

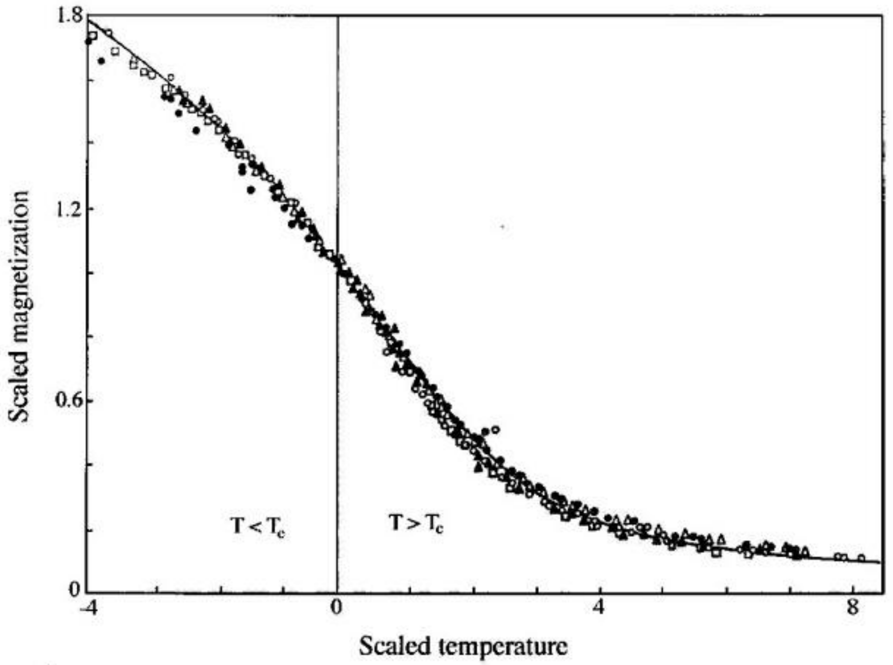


FIG. 1. Experimental *MHT* data on five different magnetic materials plotted in scaled form. The five materials are $CrBr_3$, EuO, Ni, YIG, and Pd_3Fe. None of these materials is an idealized ferromagnet: $CrBr_3$ has considerable lattice anisotropy, EuO has significant second-neighbor interactions. Ni is an itinerant-electron ferromagnet, YIG is a ferrimagnet, and Pd_3Fe is a ferromagnetic alloy. Nonetheless, the data for all materials collapse onto a single scaling function, which is that calculated for the $d=3$ Heisenberg model [after Milošević and Stanley (1976)].

- **M(H,T) ostensibly a function of two variables**
- **Plotted in appropriate scaling variables get ONE universal curve**
- **Scaling variables involve critical exponents**

Stanley (1999)

Scale invariance in turbulence

- Eddies spin off other eddies in a Hamiltonian process.
  - Does not involve friction!
  - Hypothesis due to Richardson, Kolmogorov, …
- Implication: viscosity will not enter into the equations

Scale invariance in turbulence

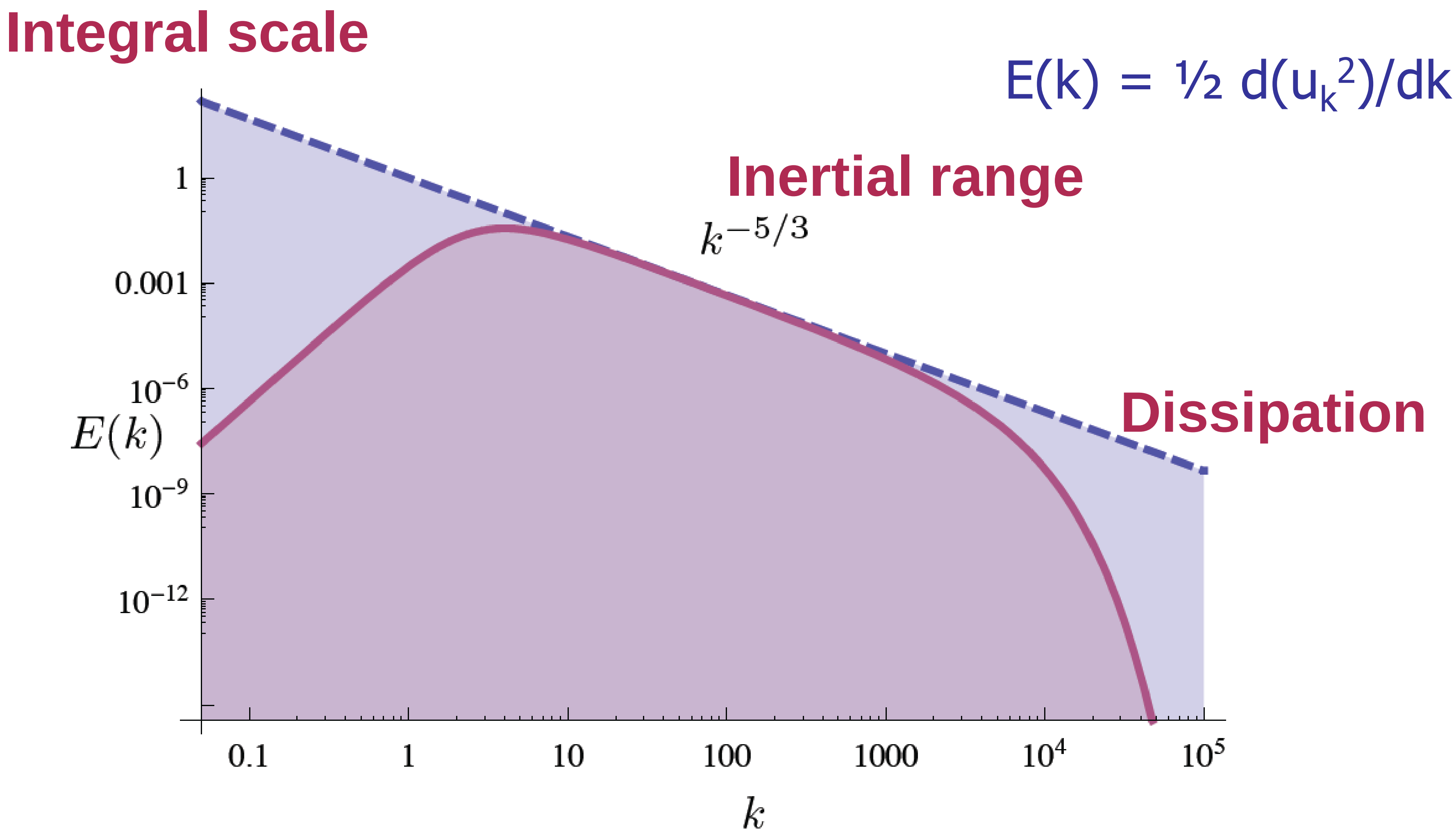
- **Compute E(k), turbulent kinetic energy in wave number range k to k+dk**
  - **E(k) depends on k**
  - **E(k) will depend on the rate at which energy is transferred between scales: ε**
- **Dimensional analysis:**
  - **$E(k) \sim \varepsilon^{2/3} k^{-5/3}$**

Scale invariance in turbulence

A.N. Kolmogorov

- **Compute E(k), turbulent kinetic energy in wave number range k to k+dk**
  - E(k) depends on k
  - E(k) will depend on the rate at which energy is transferred between scales: ε
- **Dimensional analysis:**
  - $E(k) \sim \varepsilon^{2/3} k^{-5/3}$

The energy spectrum



What are the analogues of data collapse for turbulence?

Power law scaling is not enough!

Critical phenomena and turbulence

PHYSICAL REVIEW E VOLUME 50, NUMBER 6 DECEMBER 1994

Analogies between scaling in turbulence, field theory, and critical phenomena

Gregory Eyink and Nigel Goldenfeld
Physics Department and Beckman Institute, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080
(Received 16 June 1994)

We discuss two distinct analogies between turbulence and field theory. In one analog, the field theory has an infrared attractive renormalization-group fixed point and corresponds to critical phenomena. In the other analog, the field theory has an ultraviolet attractive fixed point, as in quantum chromodynamics.

$E(k) \sim k^{-5/3}$ Energy spectrum

$G(k) \sim k^{-2}$ Spin correlations

Turbulence	Critical phenomena
space separation r	wave number k
viscosity ν	temperature variable $\tau-\tau_c$
energetic length scale L	uv cutoff Λ (or inverse lattice spacing a^{-1})
mean dissipation $\bar{\varepsilon}$	stiffness constant K
dissipation wave number $k_d \equiv \eta_d^{-1}$	correlation length ξ
velocity correlation function $S_2(\mathbf{r})=\langle[\mathbf{v}(\mathbf{r}'+\mathbf{r})-\mathbf{v}(\mathbf{r}')]^2\rangle$	spin correlation function $C(\mathbf{k})=\sum_{\mathbf{r}\in a\mathbb{Z}^d} a^d e^{i\mathbf{k}\mathbf{r}}\langle\sigma(\mathbf{r})\sigma(\mathbf{0})\rangle$
intermittency exponent μ	correlation exponent η_c

Critical phenomena and turbulence

	Critical phenomena	Turbulence
Correlations	$G(k) \sim k^{-2}$	$E(k) \sim k^{-5/3}$
Large scale thermodynamics	$M(h,t) = \lvert t\rvert^{\beta} f_M(h t^{-\beta\delta})$	?

What is analogue of critical point data in turbulence?

- **Need analogues of the *two* scaling limits $T \rightarrow T_c$ and $H \rightarrow 0$**
- **Experimental data on a real flow**
  - Systematic in same geometry over many decades of Re
  - Systematic variation over the other parameter
- **The other parameter**
  - Should couple in some way to the turbulent state
  - Key idea: boundary roughness can play this role

Nikuradse's pipe experiment (1933) to measure the friction factor *f*

$$f = \Delta P / l \rho U^2$$

Pipe diameter is 25-100 mm
Pipe length is ~ 70 diameters

Monodisperse sand grains
0.8mm glued to sides of pipe

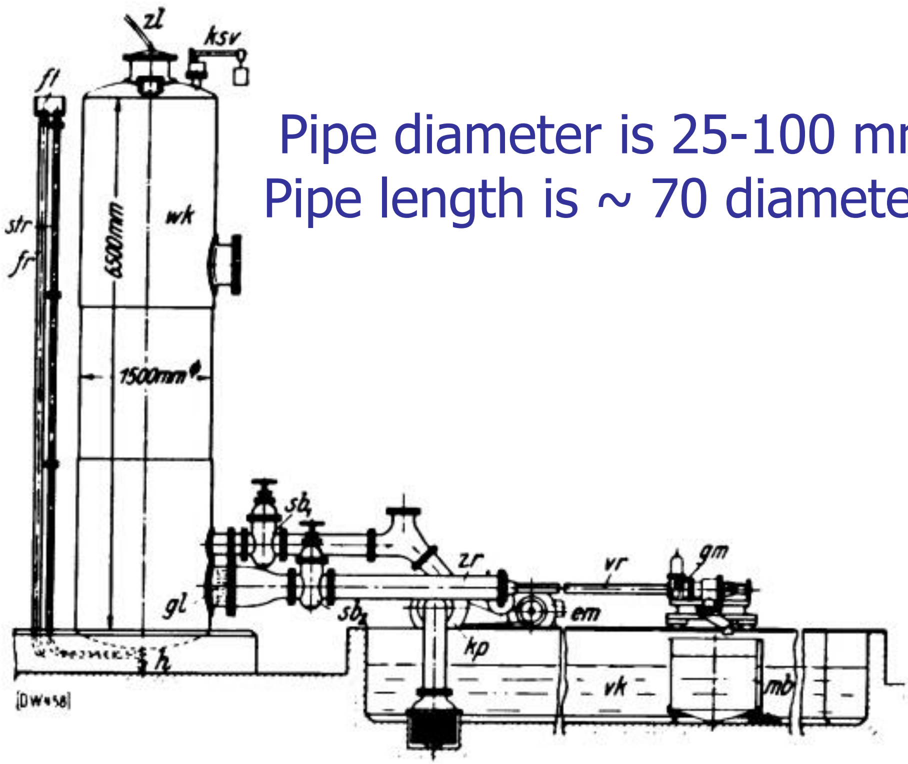


Figure 3.- Test apparatus.

em = electric motor
kp = centrifugal pump
vk = supply canal
wk = water tank
vr = test pipe
zl = supply line
str = vertical pipe
fr = overflow pipe
ft = trap
h = outlet valve
zr = feed line
mb = measuring tank
gm = velocity measuring device
ksv = safety valve on water tank
sb_1 = gate valve between wk and kp
sb_2 = gate valve between wk and zr
gl = baffles for equalizing flow

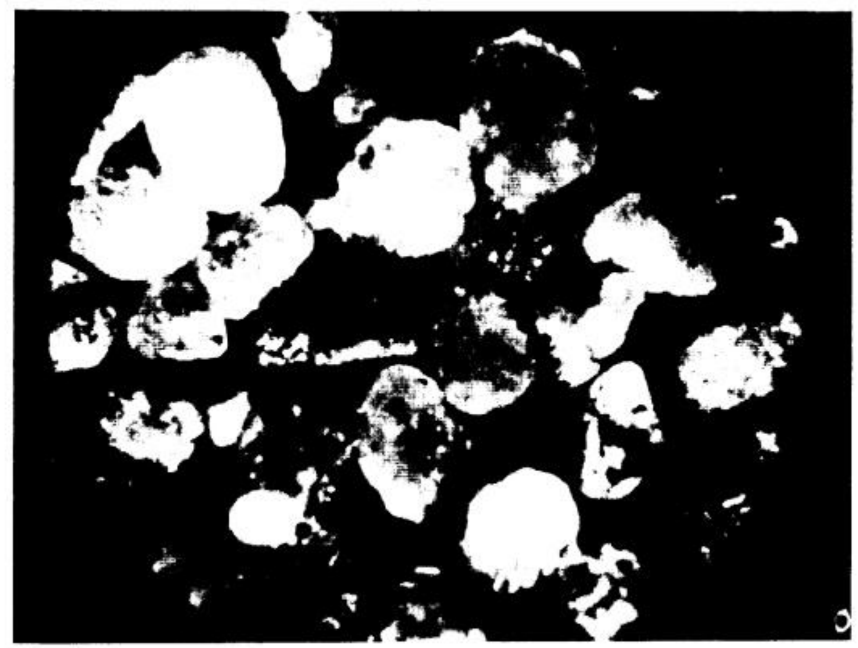


Figure 4.- Microphotograph of sand grains which produce uniform roughness. (Magnified about 20 times.)

Friction factor in turbulent rough pipes

Laminar

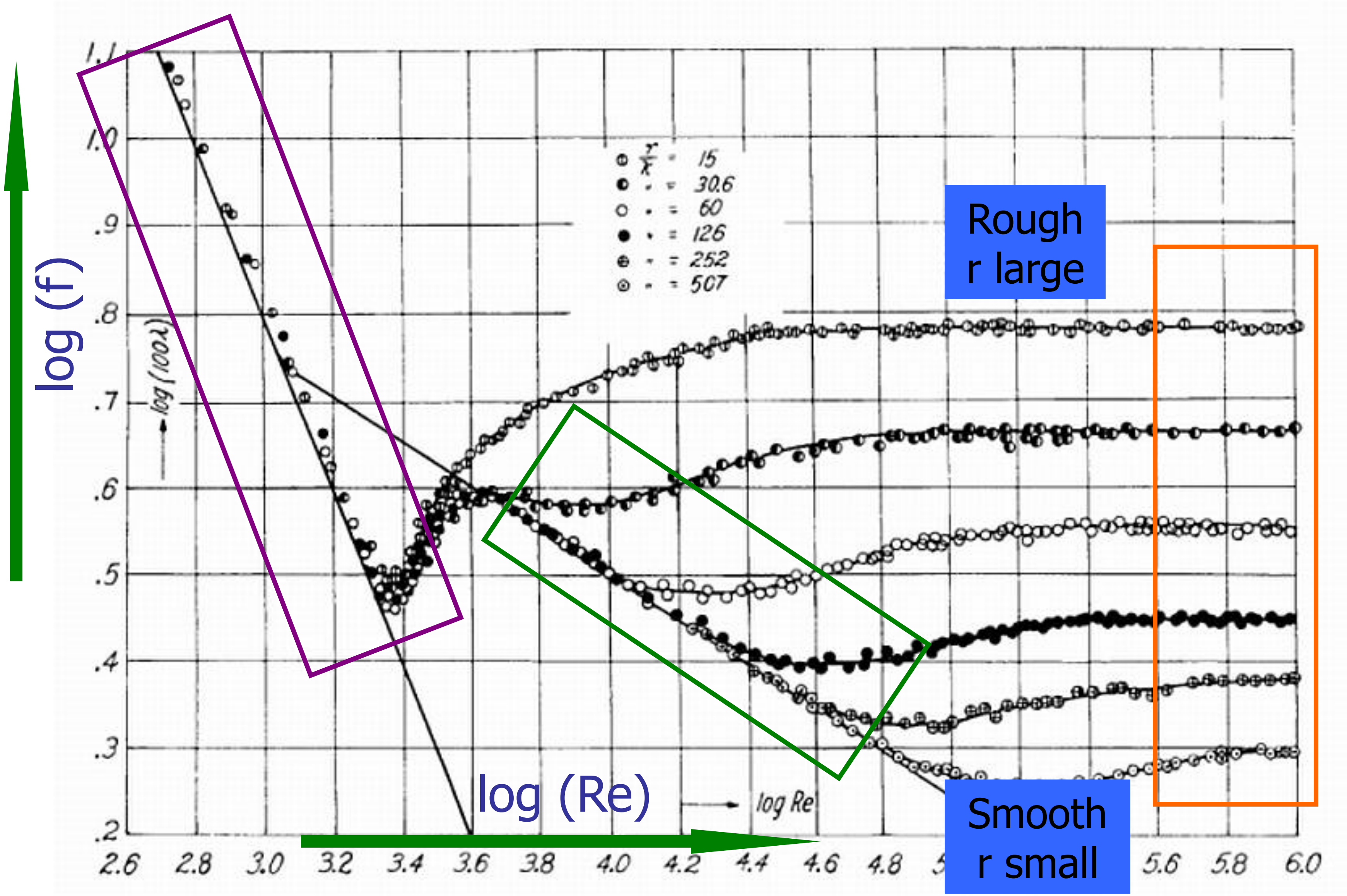
$f \sim 12/Re$

Blasius

$f \sim Re^{-1/4}$

Strickler

$f \sim (r/D)^{1/3}$



Strickler scaling

Gioia and Chakraborty (2006)

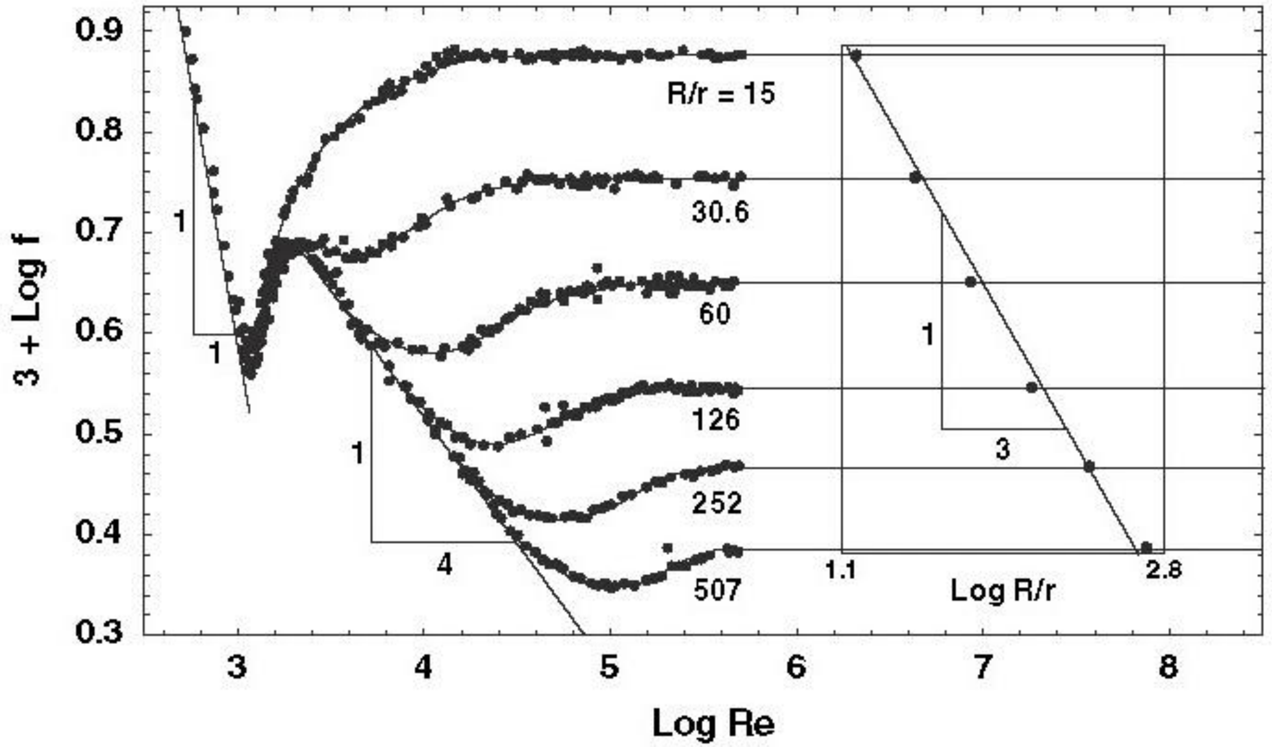


FIG. 1. Nikuradse's data. Up to a Re of about 3000 the flow is streamlined (free from turbulence) and $f \sim 1/\mathrm{Re}$. Note that for very rough pipes (small R/r) the curves do not form a belly at intermediate values of Re. Inset: verification of Strickler's empirical scaling for f at high Re, $f \sim (r/R)^{1/3}$.

Roughness-induced criticality is exhibited in Nikuradze data

Scaling of Nikuradse's data

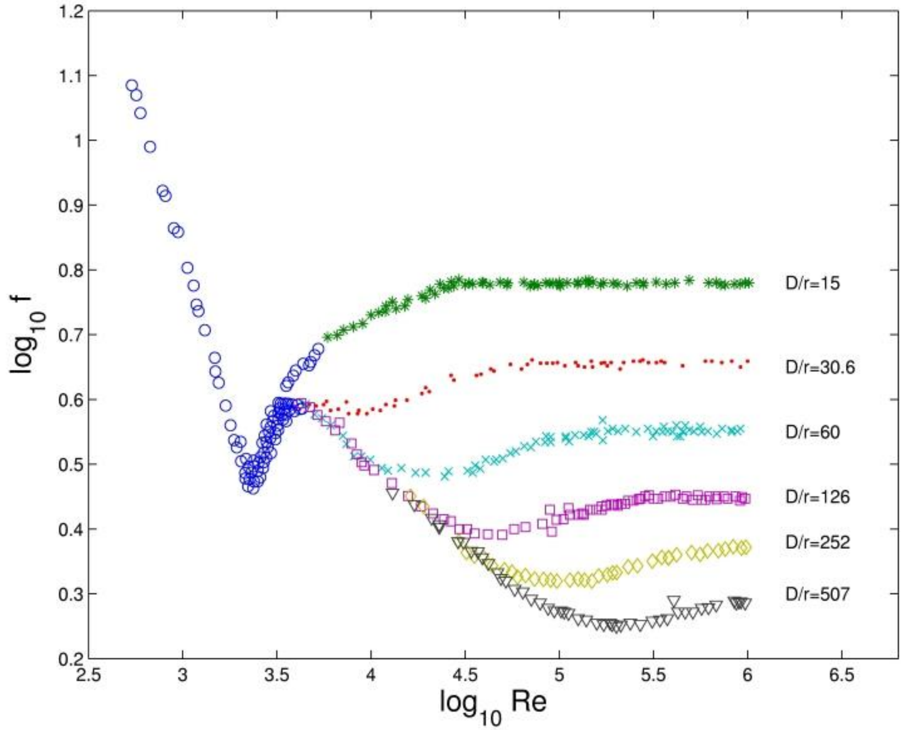
	Critical phenomena	Turbulence
Temperature control	$t \to 0$	$1/Re \to 0$
Field control	$h \to 0$	$r/D \to 0$

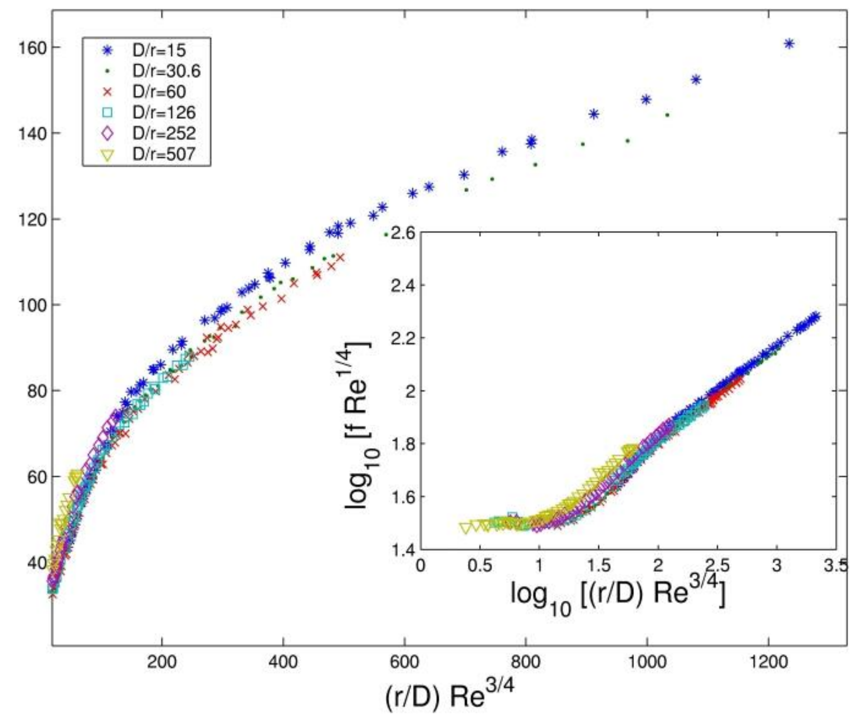
Scaling of Nikuradse's Data

- **In the turbulent regime, the extent of the Blasius regime is apparently roughness dependent.**
  - $f \sim Re^{-1/4}$ as $r/D \rightarrow 0$
- **At large Re, f is independent of roughness.**
  - $f \sim (r/D)^{1/3}$ for $Re \rightarrow \infty$
- **Combine into unified scaling form**
  - $f = Re^{-1/4}\, g([r/D]\, Re^{\alpha})$
    - Determine α by scaling argument: Re dependence must cancel out at large Re to give Strickler scaling
  - Exponent $\alpha = 3/4$ and the scaling function $g(z) \sim z^{1/3}$ for $z \rightarrow \infty$
- $f = Re^{-1/4}\, g([r/D]\, Re^{3/4})$

Scaling of Nikuradse's Data

- **Is it true that $f = Re^{-1/4}\, g([r/D]\, Re^{3/4})$?**
  - **Check by plotting $f\, Re^{1/4}$ vs. $[r/D]\, Re^{3/4}$**
    - **Do data as a function of two variables collapse onto a single universal curve?**





Intermittency corrections

Mehrafarin & Pourtolami (2008)

1. **Definition of intermittency exponent**

$$\langle \delta v_l^2 \rangle \sim (\langle \varepsilon_l \rangle l)^{2/3} \sim l^{2/3+\eta}$$

2. **Momentum transfer** $f \sim \frac{v_r}{V}$ $\quad f \sim r^{\alpha}$ $\quad \alpha = \frac{1}{3} + \frac{\eta}{2}$

3. **Flow transformation under rescaling**
   - **Boundary layer viscosity** $\sim r v_r$ (length $\times$ velocity), i.e., as $\sim r^{\alpha+1}$.

$$f\left(l\frac{r}{R}, l^{-\alpha-1}\,\mathrm{Re}\right) = l^{\alpha} f\left(\frac{r}{R}, \mathrm{Re}\right)$$

4. **Putting it all together**

$$f = \mathrm{Re}^{-(2+3\eta)/(8+3\eta)}\, g\left(\frac{r}{R}\,\mathrm{Re}^{6/(8+3\eta)}\right)$$

Data collapse in Nikuradse's data

PRL **96,** 044503 (2006) PHYSICAL REVIEW LETTERS week ending 3 FEBRUARY 2006

Roughness-Induced Critical Phenomena in a Turbulent Flow

Nigel Goldenfeld
Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois, 61801-3080, USA

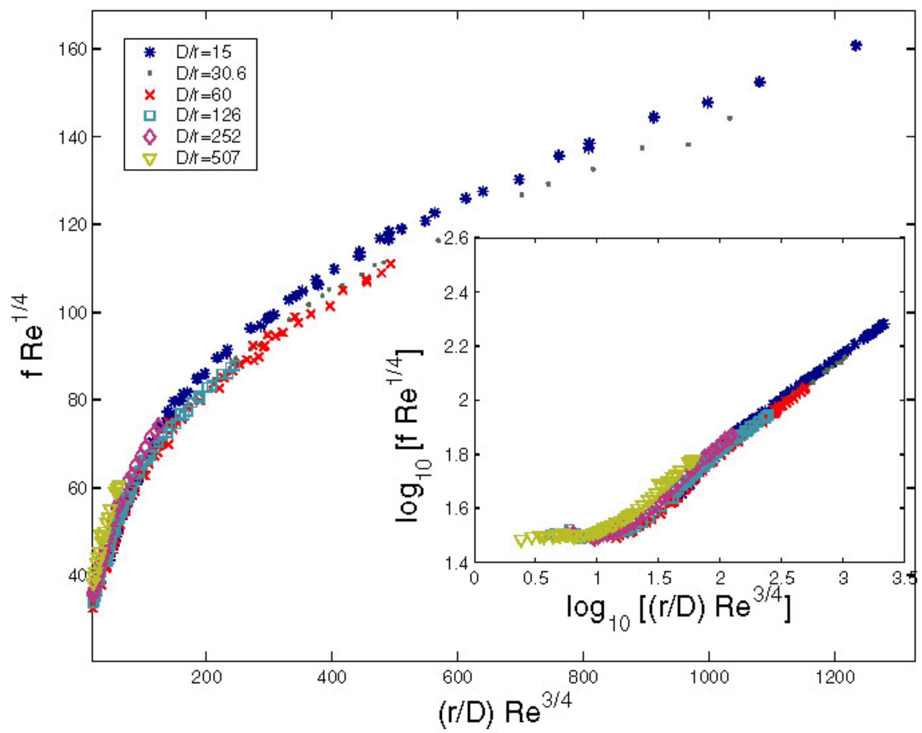
(Received 16 September 2005; published 30 January 2006)

PHYSICAL REVIEW E **77**, 055304(R) (2008)

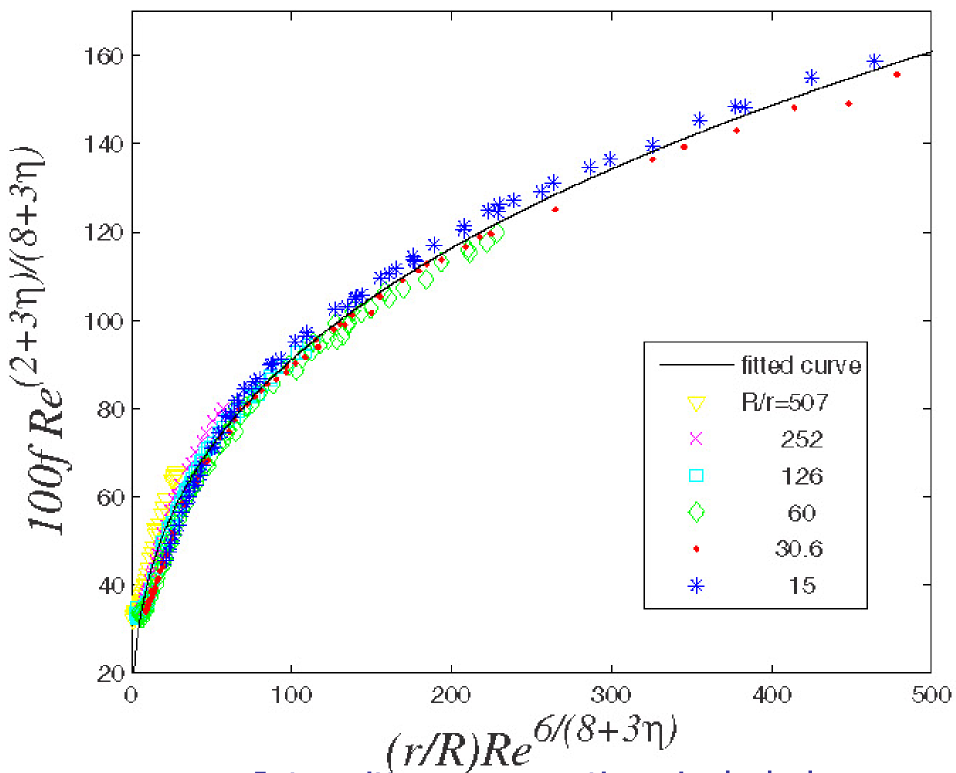
Intermittency and rough-pipe turbulence

Mohammad Mehrafarin* and Nima Pourtolami
Department of Physics, Amirkabir University of Technology, Tehran 15914, Iran

(Received 23 February 2008; published 15 May 2008)



Mean field (Kolmogorov 41) exponents



Intermittency corrections included
Value of η ~ 0.02 consistent with spectral estimates

By simply measuring the pressure drop across a pipe, Nikuradse in 1933 measured the anomalous spectral exponents (intermittency corrections) 8 years before Kolmogorov's spectral theory!

This is completely analogous to determining anomalous critical exponents in phase transitions from measurements of the M(H) scaling at T_c

Roughness-induced criticality

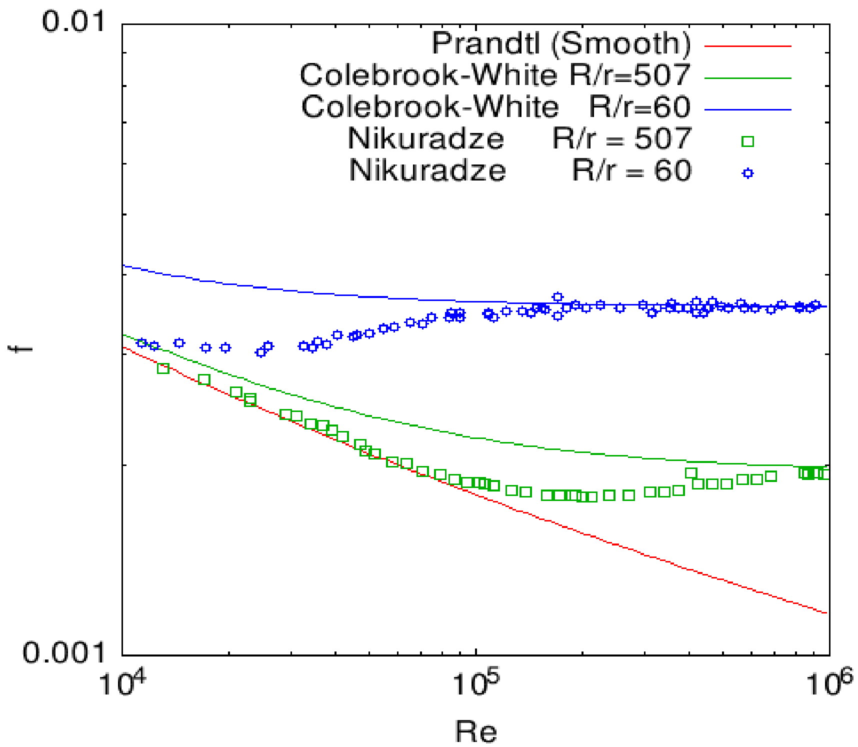
- **Multiscale structure of turbulence reminiscent of critical phenomena at phase transitions …**
- **What would be the signatures of turbulence as a dynamic critical phenomenon?**
- **Roughness-induced: laminar pipe flow is linearly stable, but boundary roughness is a relevant variable, coupling to turbulent state.**
  - **Symmetry is the enemy of instability**
- **Critical: theory predicts new scaling laws in Re, roughness (r)**
- **Spectral connection: macroscopic flow properties directly connected to correlations in fluctuations**
  - **Analogous to non-equilibrium fluctuation-dissipation theorems**

Prediction of friction factor from momentum transfer

Prandtl Theory

The friction factor can be expressed in terms of the shear at the wall:

$$f = \frac{8\tau}{\rho U^2}$$



To connect τ and U must know the velocity profile. Dimensional analysis and assumption of complete similarity suggests:

$$\partial_y u = \frac{\sqrt{\tau/\rho}}{y} g(\mathrm{Re}); g(\mathrm{Re}) \to const$$

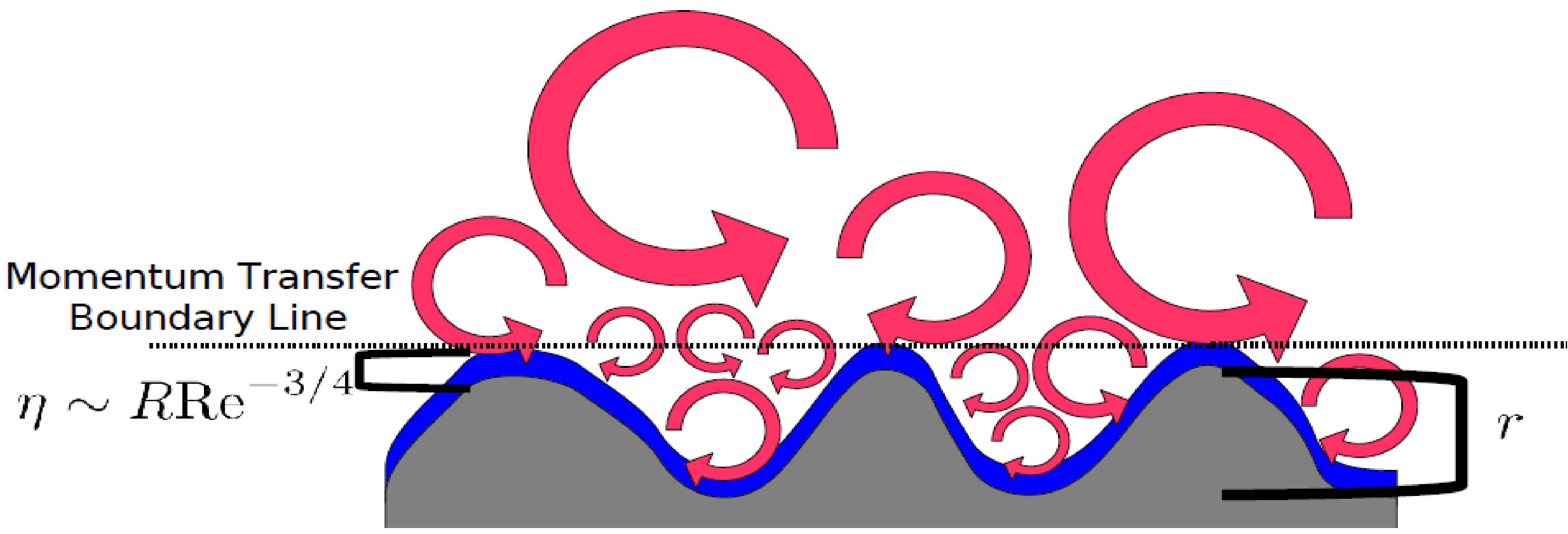
Solve and determine the value of τ such that the average velocity from the profile is equal to U:

$$\frac{1}{\sqrt{f}} = \frac{1}{\kappa} \ln\left(\frac{1}{\mathrm{Re}\sqrt{f}}\right) + B$$

The Colebrook-White equation generalizes this to rough pipes by introducing an offset to the viscous layer position.

Momentum transfer

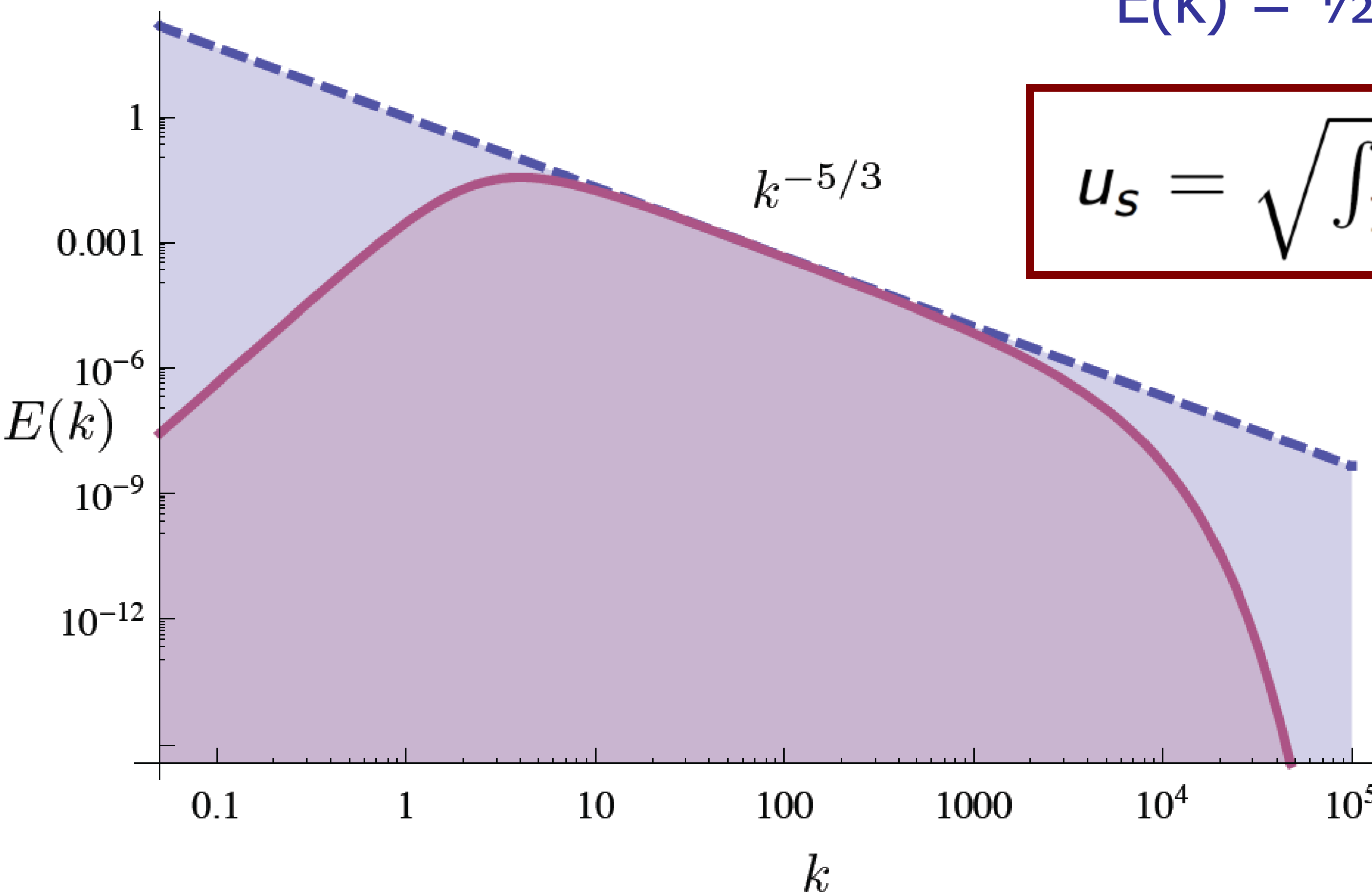
- **Theory of Gioia and Chakraborty (2006)**
  - **Roughness and dissipation scale filter turbulent structures near the wall**
  - **Momentum transfer between wall and flow**
    - **structures on scales smaller than the filter scale have little momentum contrast**
    - **structures on scales much larger than the filter scale have too small a vertical component to make significant contribution**



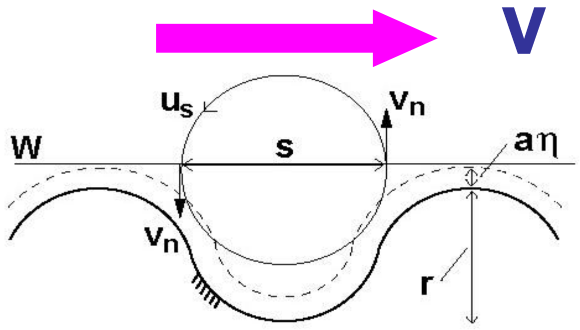
The spectral connection

$E(k) = ½\, d(u_k^2)/dk$

$$u_s = \sqrt{\int_{1/s}^{\infty} E(k)dk}$$



Scaling argument for Blasius and Strickler regimes



- $f \sim \rho\, V\, u_s / \rho V^2 \sim u_s / V$
  - Contribution to friction factor from dominant eddy on scale of roughness element, $s = r + a\eta$

$$f \propto u_s \propto \left(\int_{1/s}^{\infty} E(k)\,dk\right)^{1/2}$$

- K41: Use $E(k) \sim k^{-5/3}$

$$f \sim \left(\frac{r}{R} + ab\,\mathrm{Re}^{-3/4}\right)^{1/3}$$

  - Large Re: $s \sim r$ and $f \sim (r/D)^{1/3}$ Strickler law predicted!
  - Small Re: $s \sim \eta$ and $f \sim \mathrm{Re}^{-1/4}$ Blasius law predicted!

- Friction factor formula satisfies roughness-induced criticality scaling relation

(Gioia and Chakraborty 2006)

Evaluation of friction factor

- **Now include the dissipation range and integral scale**

$f = \kappa_\tau u_s / V$ or

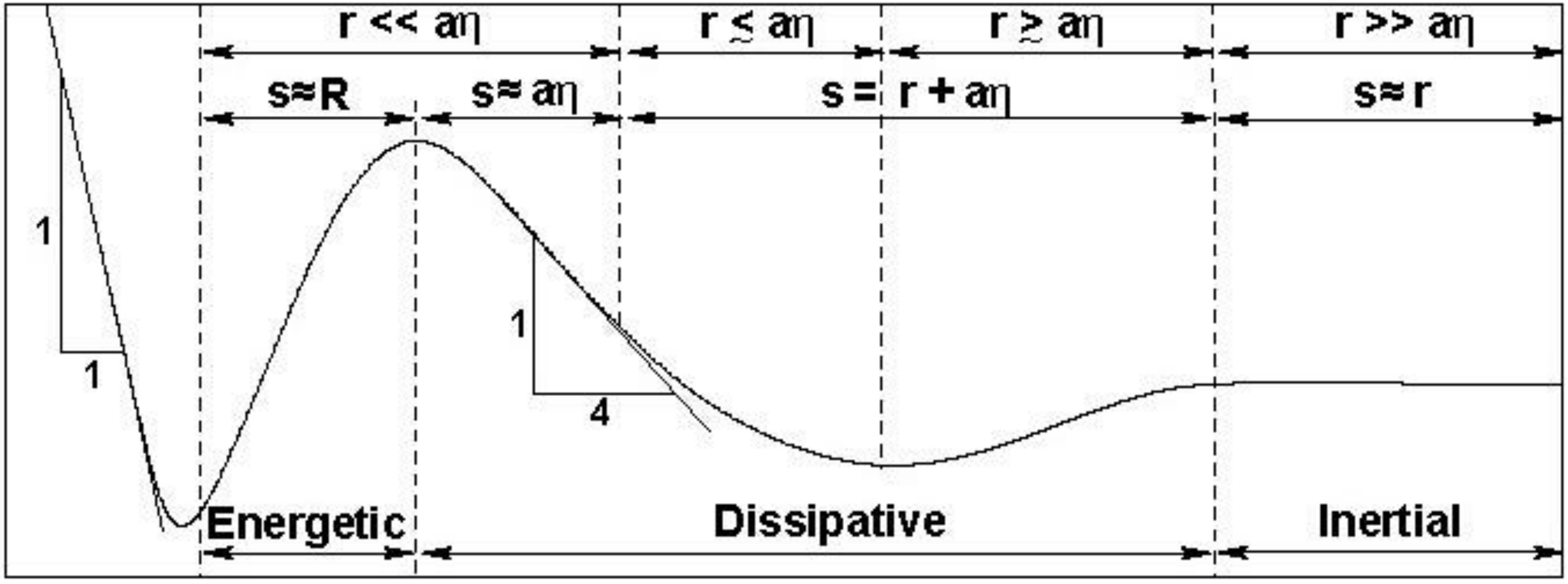
$$f = K\left(\int_0^{s/R} x^{-1/3} c_d(b\,\mathrm{Re}^{-3/4}/x)\, c_e(x)\, dx\right)^{1/2}, \quad (1)$$

where $K \equiv \kappa_\tau \kappa_u \sqrt{2/3}$, $s/R = r/R + ab\,\mathrm{Re}^{-3/4}$, and $b \equiv (\kappa_\varepsilon \kappa_u^3)^{-1/4}$. Equation (1) gives f as an explicit function of the Reynolds number Re and the roughness r/R.

Dissipation range

Integral scale

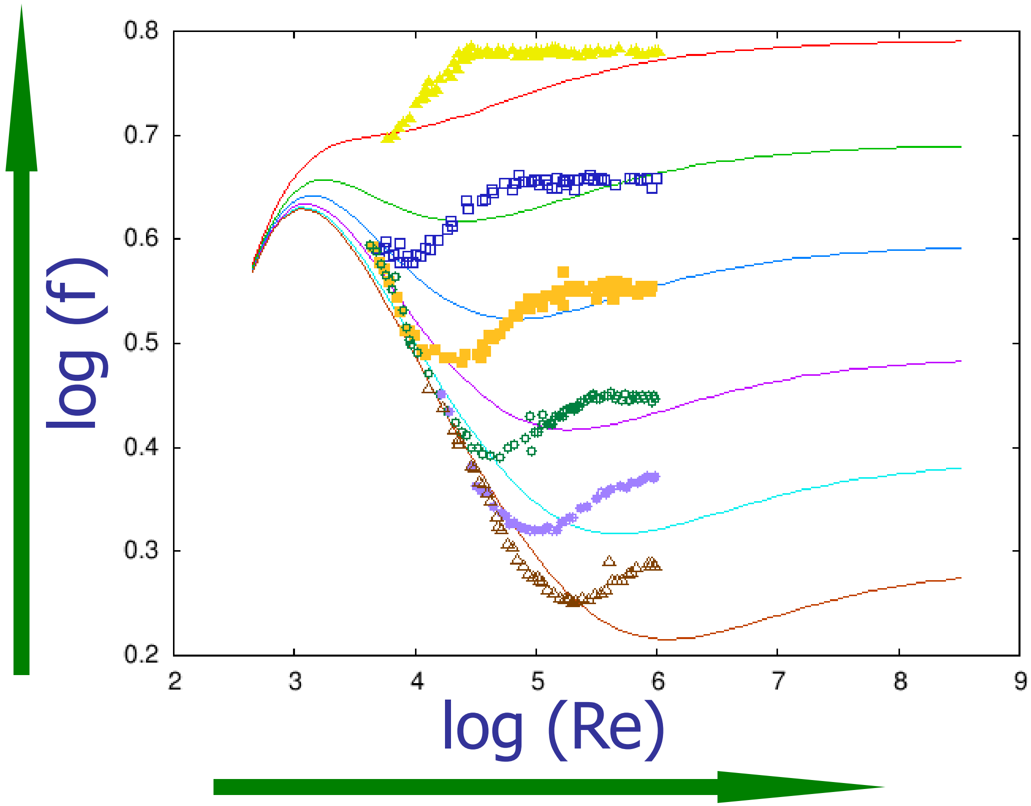
Friction factor contributions



Gioia and Chakraborty (2006)

Boundary layer structure

- **How many adjustable parameters in Gioia-Chakraborty model?**
  - a = 5, so that thickness of viscous layer ~ 5 η
  - b measured to be 11.4 (Antonia and Pearson (2000)
- **Model essentially completely determined.**
- **But: scale of curves do not match data!**
  - Need to have proper integration of theory with velocity profile



Testing the Spectral Connection

The central testable difference between momentum transfer theory and the Prandtl theory is the dependence of the friction factor on the energy spectrum.

How can we determine whether the friction factor depends on the turbulent energy spectrum?

Testing the Spectral Connection

The central testable difference between momentum transfer theory and the Prandtl theory is the dependence of the friction factor on the energy spectrum.

How can we determine whether the friction factor depends on the turbulent energy spectrum?

We must find a flow with a different energy spectrum!

Difference between Turbulence in a Two-Dimensional Fluid and in a Three-Dimensional Fluid

T. D. Lee
Department of Physics, University of California, Berkeley, California
(Received January 19, 1951)

THE difference between a two-dimensional and a three-dimensional fluid can easily be seen from the vorticity equation given as

$$\dot{\boldsymbol{\omega}}+(\mathbf{v},\nabla)\boldsymbol{\omega}=\nu\Delta\boldsymbol{\omega}+(\boldsymbol{\omega},\nabla)\mathbf{v}, \qquad (1)$$

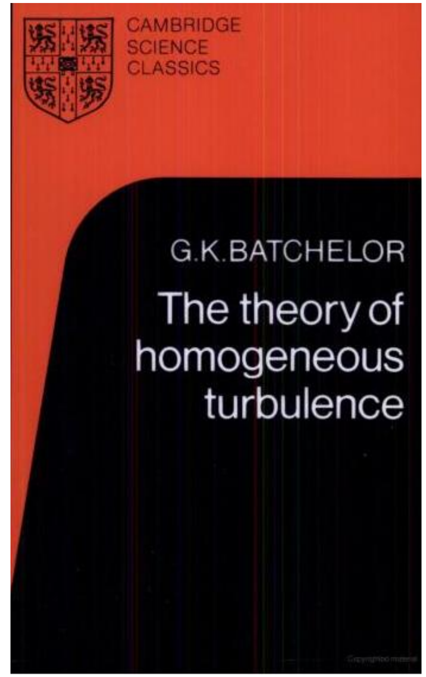
where $\boldsymbol{\omega}$, $\mathbf{v}$, and ν are the vorticity, velocity, and kinematic viscosity of the fluid. In the two-dimensional case, $(\boldsymbol{\omega},\nabla)\mathbf{v}$ is identically zero. Hence, if one neglects viscosity in a system moving with the fluid, the vorticity never changes, and the scattering of energy between eddies does not lead to any change in vorticity. This conservation law forbids the fulfillment of an ergodic hypothesis for a two-dimensional fluid. Indeed, if one

Enstrophy = mean squared vorticity

T.D. Lee, Journal of Applied Physics, Vol. 22, p.524 (1951) dimensional motion we should beware of assuming too close a relation between two- and three-dimensional turbulence). Hence for a motion with zero viscosity, the integrals

$$\int_0^\infty E(\kappa)\,d\kappa, \quad \int_0^\infty \kappa^2 E(\kappa)\,d\kappa$$

are constant; even when ν is finite but small the integrals will be constant until such time as energy has been transferred to high wave-numbers at which viscous forces are significant. The effect of the non-linear term of the equation will be to transfer energy over an increasingly wide range of wave-numbers, and if we imagine the initial state to be such that all the energy lies in the range $0 < \kappa < \kappa'$, one of the effects of the non-linear term will be to transfer energy to wave-numbers $\kappa > \kappa'$. But if there is a transfer of energy across $\kappa = \kappa'$, the constancy of $\int_0^\infty \kappa^2 E(\kappa)\,d\kappa$ demands that there should be an even greater flow of energy in the opposite direction



1953

THE PHYSICS OF FLUIDS VOLUME 10, NUMBER 7 JULY 1967

Inertial Ranges in Two-Dimensional Turbulence

Robert H. Kraichnan
Peterborough, New Hampshire
(Received 1 February 1967)

Two-dimensional turbulence has both kinetic energy and mean-square vorticity as inviscid constants of motion. Consequently it admits two formal inertial ranges, $E(k) \sim \epsilon^{2/3}k^{-5/3}$ and $E(k) \sim \eta^{2/3}k^{-3}$, where ϵ is the rate of cascade of kinetic energy per unit mass, η is the rate of cascade of mean-square vorticity, and the kinetic energy per unit mass is $\int_0^\infty E(k)\ dk$. The $-\frac{5}{3}$ range is found to entail

Cascades in 2D turbulence

- **Energy cascade**
  - **Direction of energy flow is from small to large scales**

$$E(k) \propto \epsilon^{2/3} k^{-5/3}$$

- **Enstrophy cascade**
  - **Direction of enstrophy flow is from large to small scales**

$$E(k) \propto \lambda^{2/3} k^{-3}$$

Momentum-transfer/roughness-induced criticality theory vs. Prandtl theory

- **Prandtl**
  - Assumes complete similarity - no characteristic scale
- **Momentum-transfer**
  - Characteristic scale set by larger of Kolmogorov scale or wall-roughness

Momentum-transfer/roughness-induced criticality theory vs. Prandtl theory

- **Prandtl**
  - Assumes complete similarity - no characteristic scale
    - Law of the wall
    - Zero roughness is not recognized to be singular
- **Momentum-transfer**
  - Characteristic scale set by larger of Kolmogorov scale or wall-roughness
    - Power-law velocity profile in intermediate asymptotic regime
    - Zero roughness is a singular limit (roughness-induced criticality)

Momentum-transfer/roughness-induced criticality theory vs. Prandtl theory

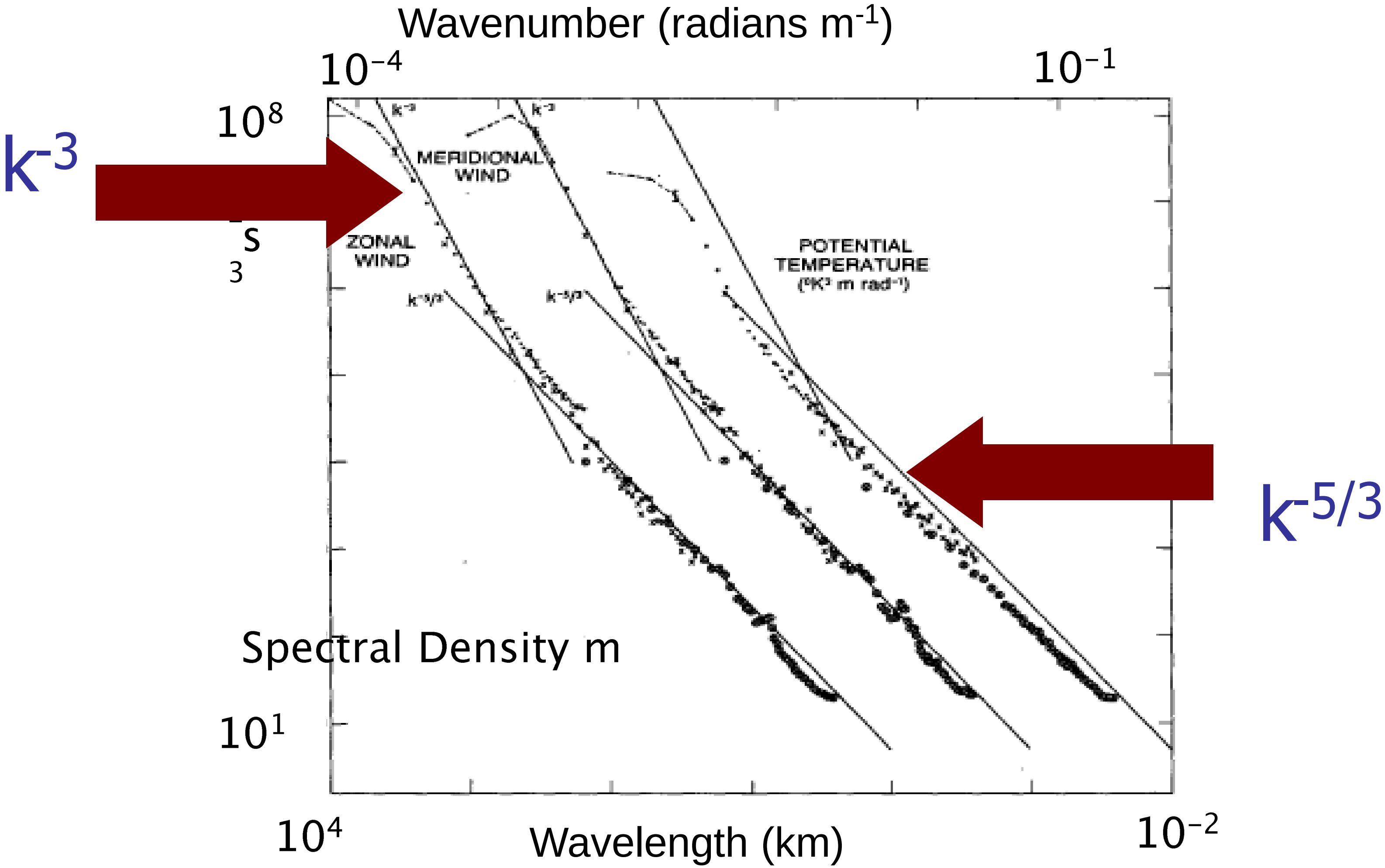
- **Prandtl**
  - Assumes complete similarity - no characteristic scale
    - ➔ Law of the wall
    - ➔ Zero roughness is not recognized to be singular
  - No representation of underlying nature of turbulent flow
- **Momentum-transfer**
  - Characteristic scale set by larger of Kolmogorov scale or wall-roughness
    - ➔ Power-law velocity profile in intermediate asymptotic regime
    - ➔ Zero roughness is a singular limit (roughness-induced criticality)
  - Nature of underlying flow is represented by the form of the energy spectrum:

Momentum-transfer/roughness-induced criticality theory vs. Prandtl theory

- **Prandtl**
  - Assumes complete similarity - no characteristic scale
    - ➔ Law of the wall
    - ➔ Zero roughness is not recognized to be singular
  - No representation of underlying nature of turbulent flow
    - Unable to make predictions for friction factor in 2D
    - No connection with spectral structure of turbulence

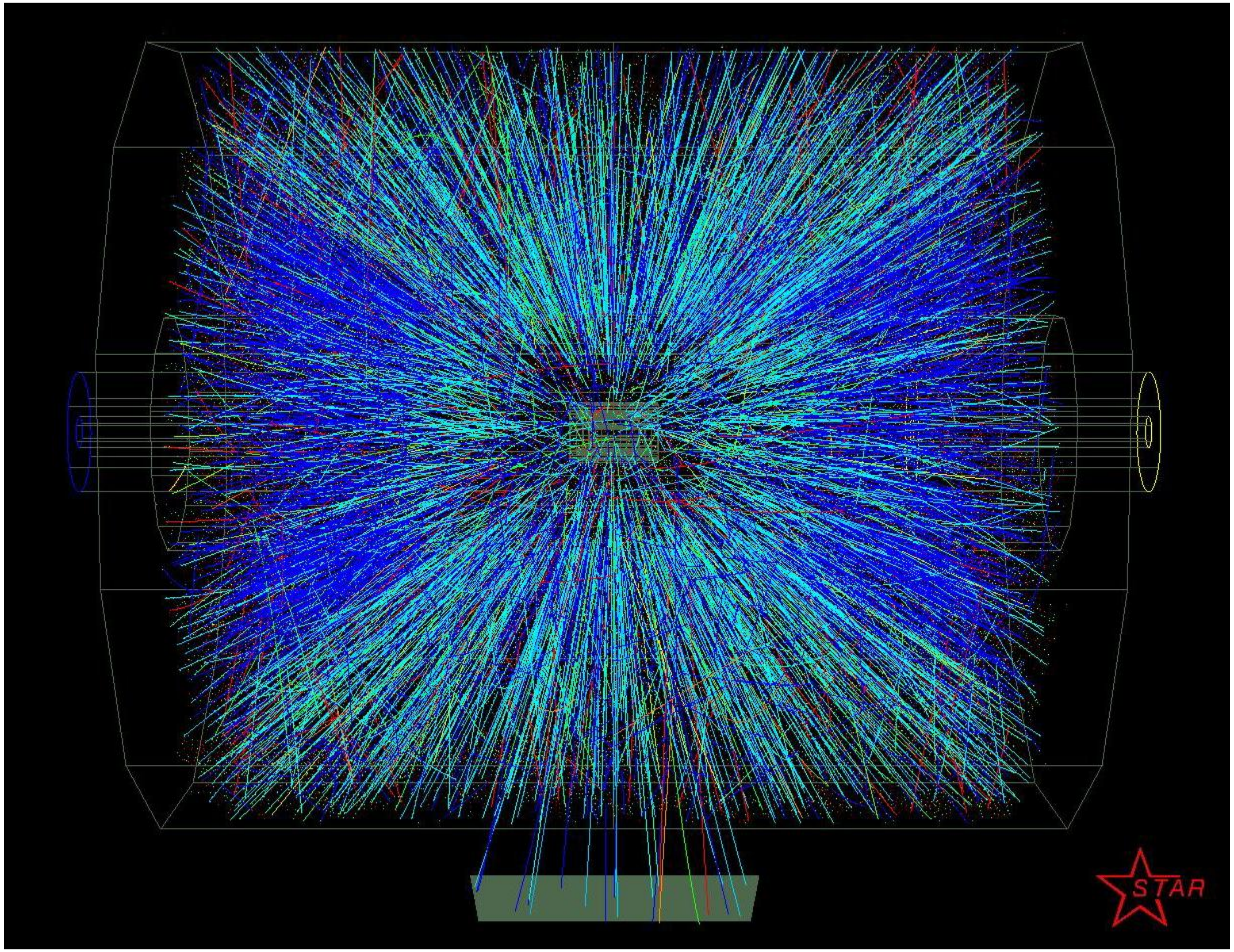
- **Momentum-transfer**
  - Characteristic scale set by larger of Kolmogorov scale or wall-roughness
    - ➔ Power-law velocity profile in intermediate asymptotic regime
    - ➔ Zero roughness is a singular limit (roughness-induced criticality)
  - Nature of underlying flow is represented by the form of the energy spectrum:
    - E.g. Vortex stretching present or absent?
    - 3D – forward energy cascade
    - 2D – forward enstrophy and/or inverse energy cascade

Atmospheric turbulence

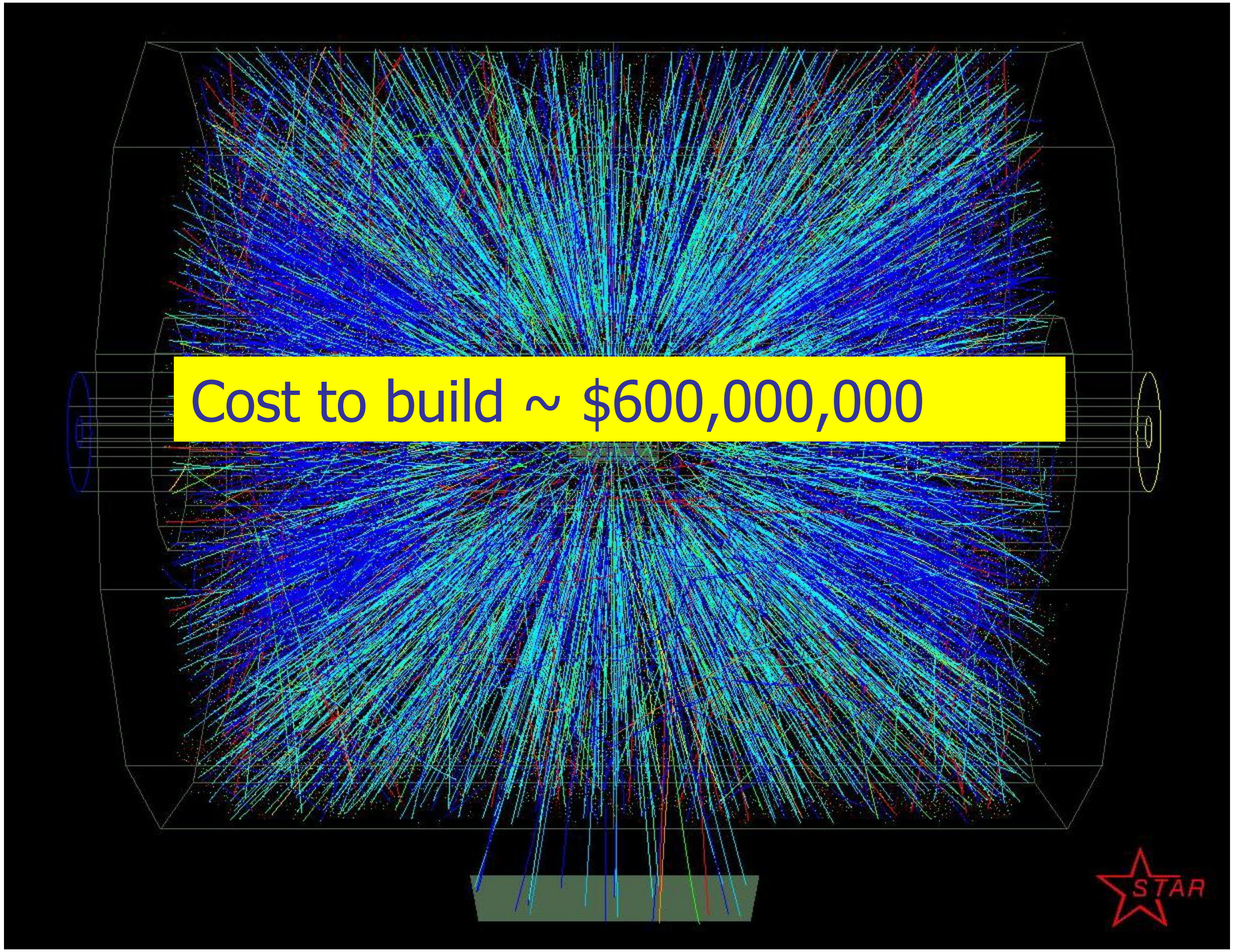


G. D. Nastrom and K. S. Gage, "A Climatology of Atmospheric Wavenumber Spectra of Wind and Temperature Observed by Commercial Aircraft", Jour. Atmos. Sci. vol 42, 1985 p953

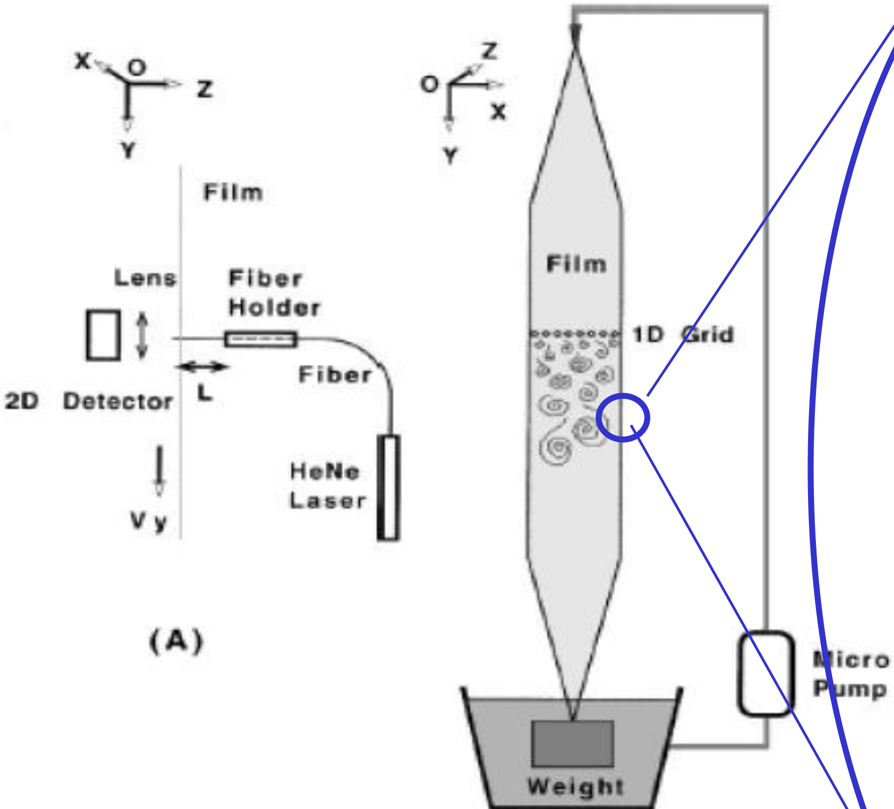
Quark-gluon liquid at RHIC



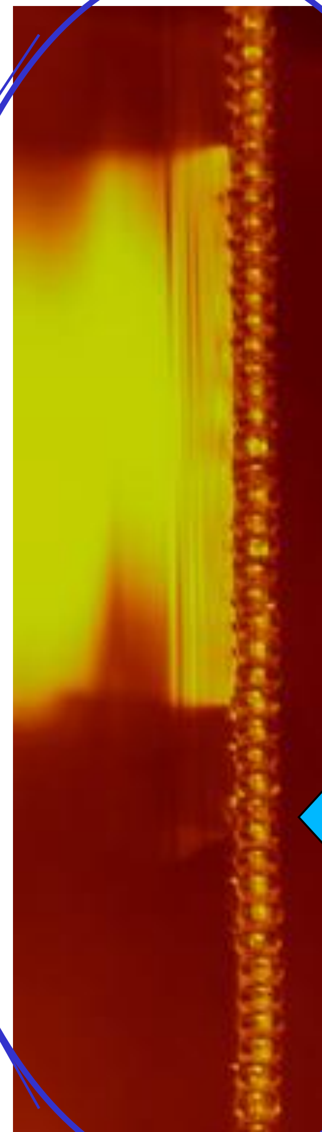
Quark-gluon liquid at RHIC



How to make a 2D turbulent rough-pipe



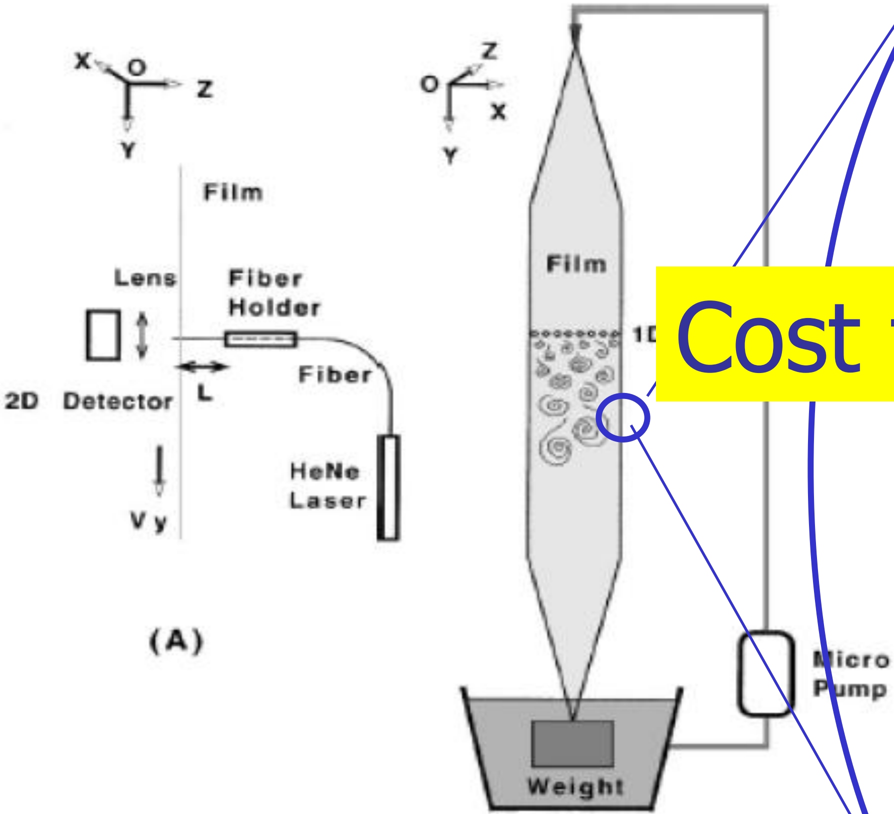
Kellay and Goldburg (2002)



H. Kellay 2008

Beads or wire-wrap make roughness elements

How to make a 2D turbulent rough-pipe



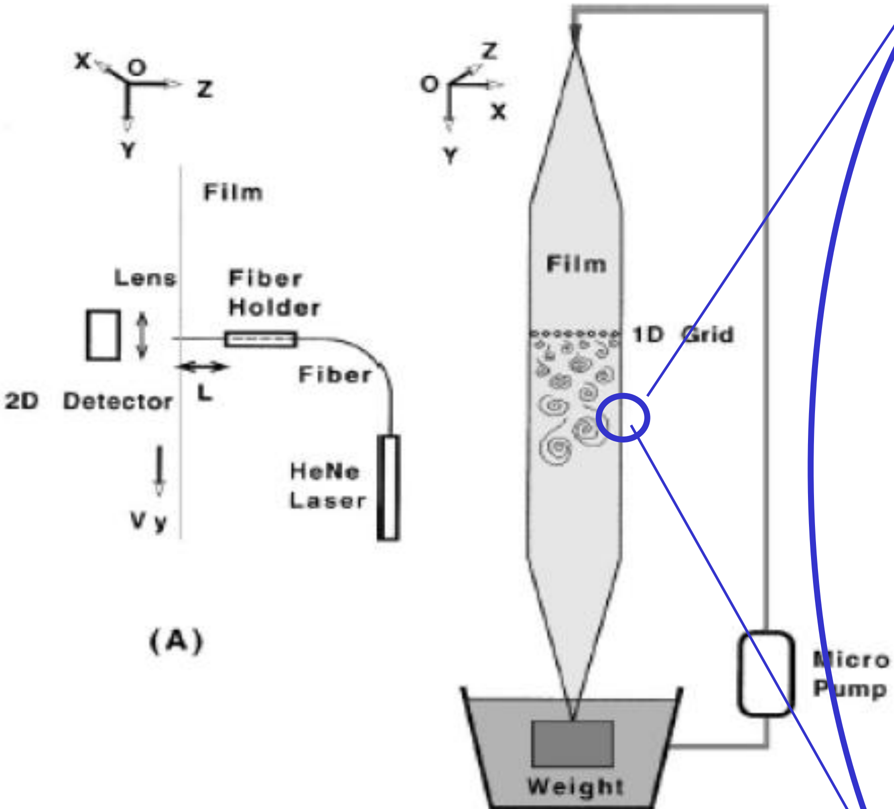
Kellay and Goldburg (2002)

Cost to build $7.23

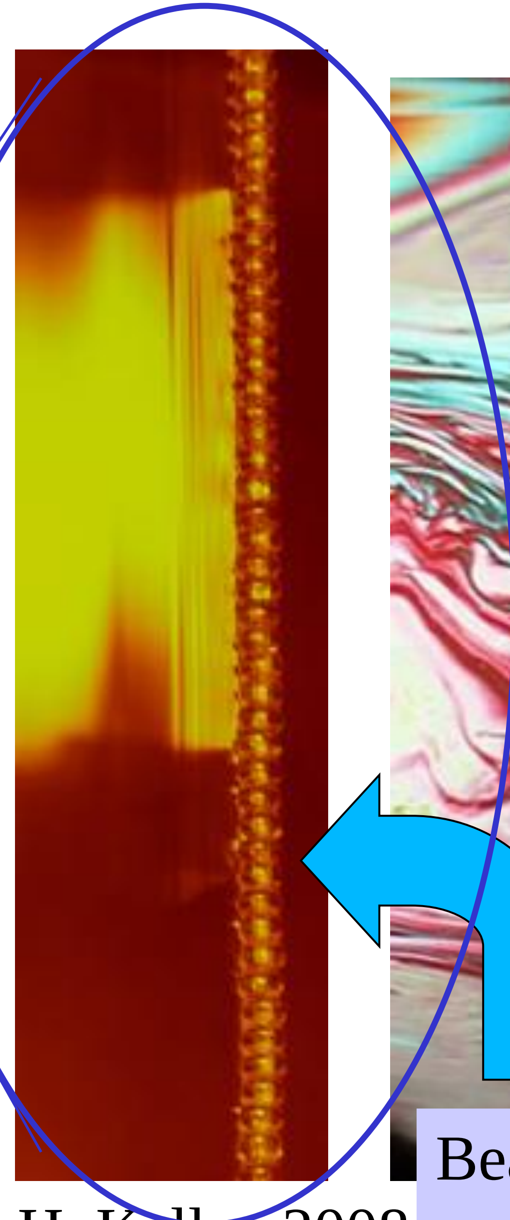
H. Kellay 2008

Beads or wire-wrap make roughness elements

How to make a 2D turbulent rough-pipe



Kellay and Goldburg (2002)



H. Kellay 2008

Beads or wire-wrap make roughness elements

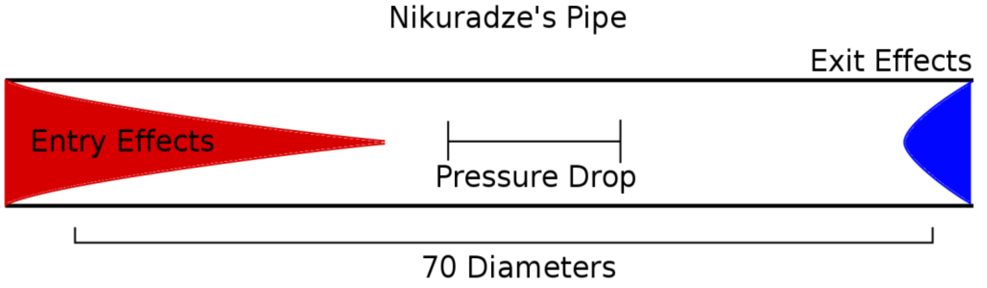
2D friction factor scaling in r→ 0 limit

- **In 3D the inertial range, energy is conserved.**
- **Rate of energy transfer between scales: ε**

$$E(k) \propto \epsilon^{2/3} k^{-5/3}$$

- **In 2D, enstrophy is conserved. Constructing a spectrum from the rate of enstrophy transfer λ**

$$E(k) \propto \lambda^{2/3} k^{-3}$$



$$E(k) \propto U^2 L^{(1-\alpha)} k^{-\alpha}$$

Momentum transfer theory predicts:

$$f \propto \mathrm{Re}^{(1-\alpha)/(1+\alpha)}$$

In 2D, the friction factor in the Blasius regime will have an exponent that depends on the cascade

Generalized momentum transfer theory

$$E(k) \propto U^2 L^{(1-\alpha)} k^{-\alpha}$$

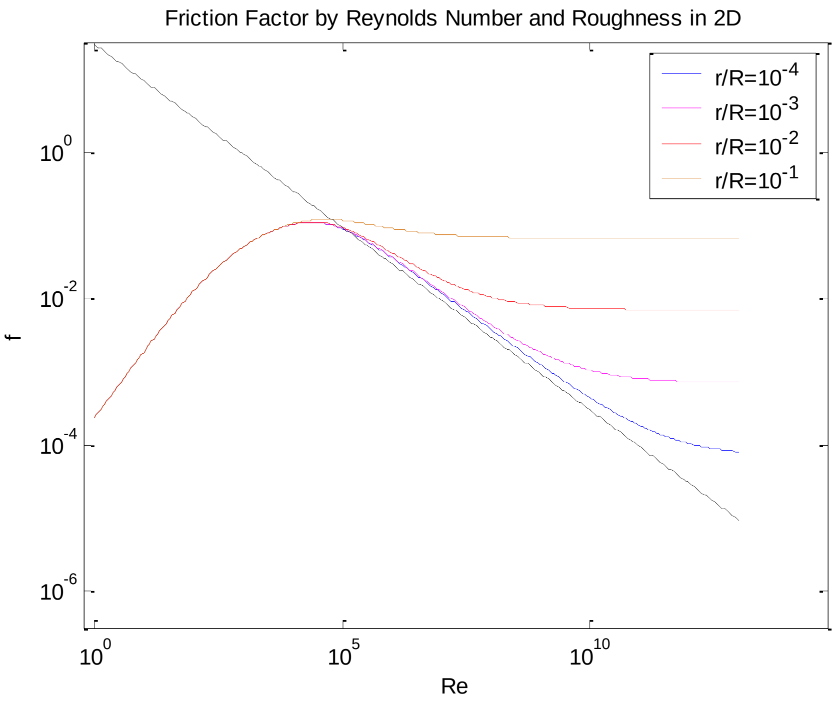
$$u_s \propto U(s/L)^{(\alpha-1)/2}$$

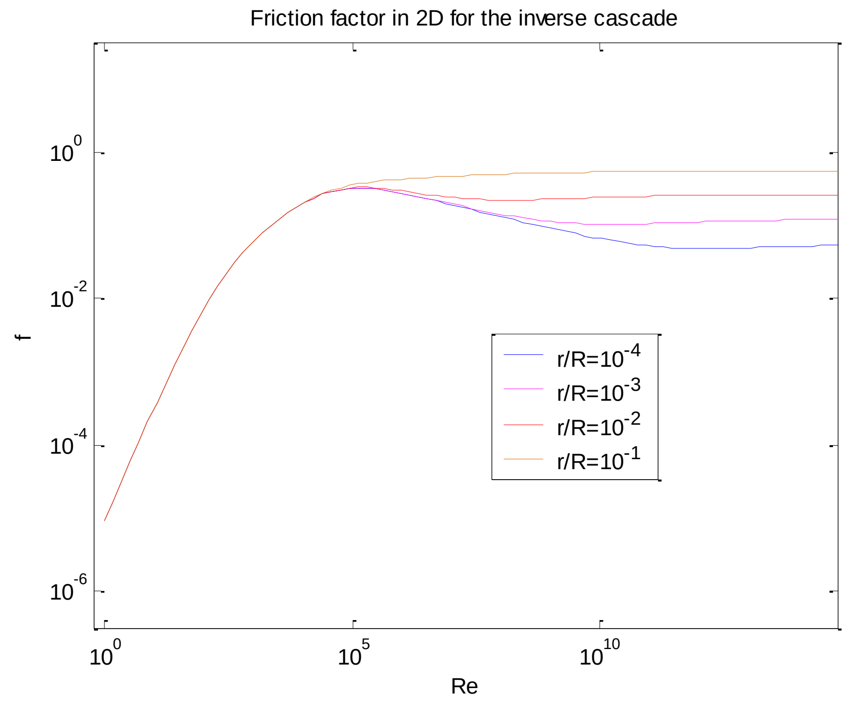
$$\mathrm{Re}_\eta = u_\eta \eta/\nu \approx 1, \text{ or } u_\eta \propto \nu/\eta.$$

$$u_\eta \propto U\,\mathrm{Re}^{(1-\alpha)/(1+\alpha)}$$

$$f \propto \mathrm{Re}^{(1-\alpha)/(1+\alpha)}$$

2D friction factor scaling in both Re and r





Enstrophy cascade
$f \sim Re^{-1/2}$ (Blasius)
$f \sim (r/D)$ (Strickler)

Inverse cascade
$f \sim Re^{-1/4}$ (Blasius)
$f \sim (r/D)^{1/3}$ (Strickler)

Simulations

- **2D turbulence via direct numerical simulation**
- **Pressure at cell center, velocity at cell walls.**
- **Spectral method to solve pressure equation for incompressibility:**

$$\nabla^2 P = (\nabla \cdot \mathbf{V})/\Delta t$$

- **SMART (Sharp and Monotonic Algorithm for Realistic Transport) algorithm for advection. 3rd order nonlinear: preserves maxima and minima (Gaskell & Lau 1988)**

- **Rough walls – conformal mapping.**

Conformal Mapping

- **Navier-Stokes equation before mapping:**

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = \nu \nabla^2 \mathbf{V} - \nabla \mathbf{P}$$

- **Map coordinates:** $x + iy = f(u + iv)$

$$\bar{\mathbf{V}} = (y_v V_x - x_v V_y)\hat{\mathbf{u}} + (x_u V_y - y_u V_x)\hat{\mathbf{v}}$$

$$|g'|^2 \equiv x_u^2 + x_v^2 \qquad \mathbf{A} \equiv \begin{bmatrix} x_u y_{uv} + x_v x_{uv} \\ x_u x_{uv} - x_v y_{uv} \end{bmatrix}$$

- **Navier-Stokes equation after mapping:**

$$|g'|^2 \frac{\partial \bar{\mathbf{V}}}{\partial t} + (\bar{\mathbf{V}} \cdot \nabla)\bar{\mathbf{V}} = \nu \nabla^2 \bar{\mathbf{V}} + \frac{|\bar{\mathbf{V}}|^2}{|g'|^2}\mathbf{A} + \frac{2\nu}{|g'|^2}\mathbf{A}^{\perp}(\nabla \times \bar{\mathbf{V}})_z - \nabla \mathbf{P}$$

Meaning of New Terms

$$|g'|^2 \frac{\partial \bar{\mathbf{V}}}{\partial t} + (\bar{\mathbf{V}} \cdot \nabla)\bar{\mathbf{V}} = \nu \nabla^2 \bar{\mathbf{V}} + \frac{|\bar{\mathbf{V}}|^2}{|g'|^2}\mathbf{A} + \frac{2\nu}{|g'|^2}\mathbf{A}^{\perp}(\nabla \times \bar{\mathbf{V}})_z - \nabla \mathbf{P}$$

- **Two new body forces as a result of the mapping:**

$$\frac{|\bar{\mathbf{V}}|^2}{|g'|^2}\mathbf{A}$$

Body force due to acceleration around contours of the boundary

$$\frac{2\nu}{|g'|^2}\mathbf{A}^{\perp}(\nabla \times \bar{\mathbf{V}})_z$$

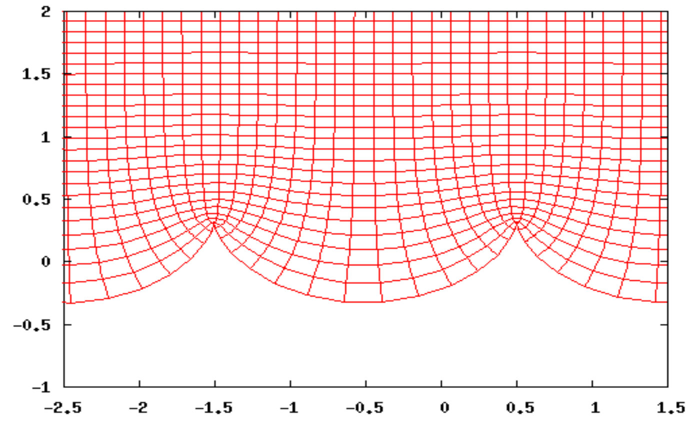
Body force due to curvature in the map corresponding to vorticity of the real-space velocity field.

Representation of Roughness

- **Want to generate walls with a particular lengthscale or set of lengthscales using the conformal map.**
- **At lower boundary (v=0), try:**

$$z = w + re^{ikw}$$

- **Coordinate singularities when**



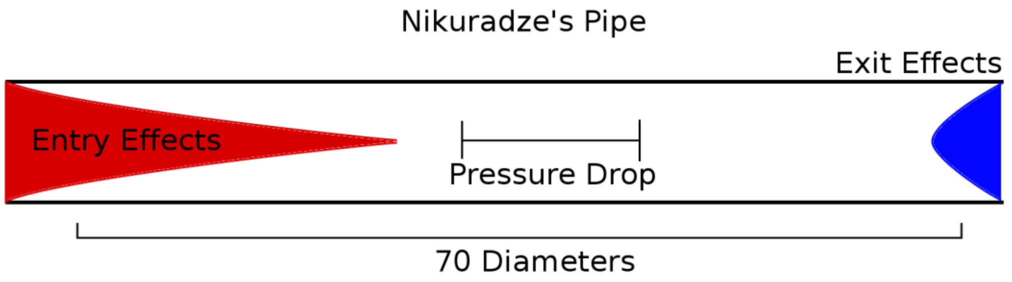
$$|g'|^2 = (1 - rk\sin(ku)e^{-kv})^2 + (rk\cos(ku)e^{-kv})^2 = 0$$
$$ku = \frac{\pi}{2} + n\pi, e^{-kv} = rk$$

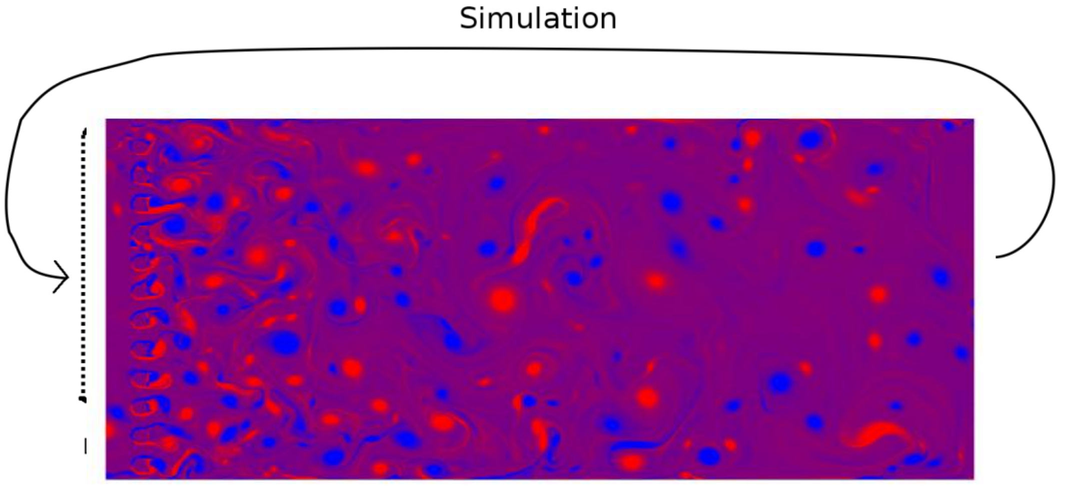
- $rk < 1$ in order to prevent singularities inside the computational domain.
- Parameters r and k.

Simulations

Entry Effects

We use periodic boundaries, with a pressure drop applied to the pipe to keep the average velocity constant – this is the friction factor pressure drop.





Generating Turbulence

Two ways:

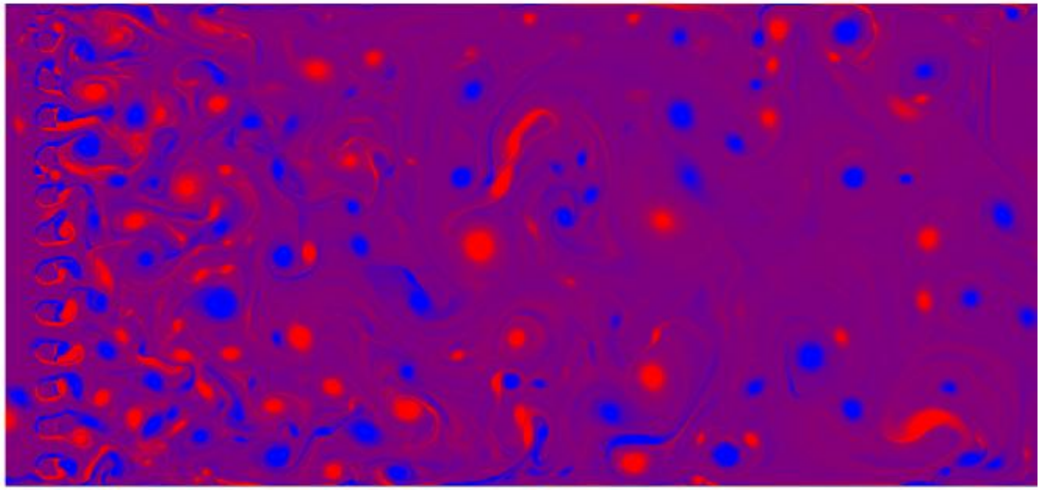
- Roughness generated
    - Inverse cascade
- Grid generated
    - Enstrophy cascade

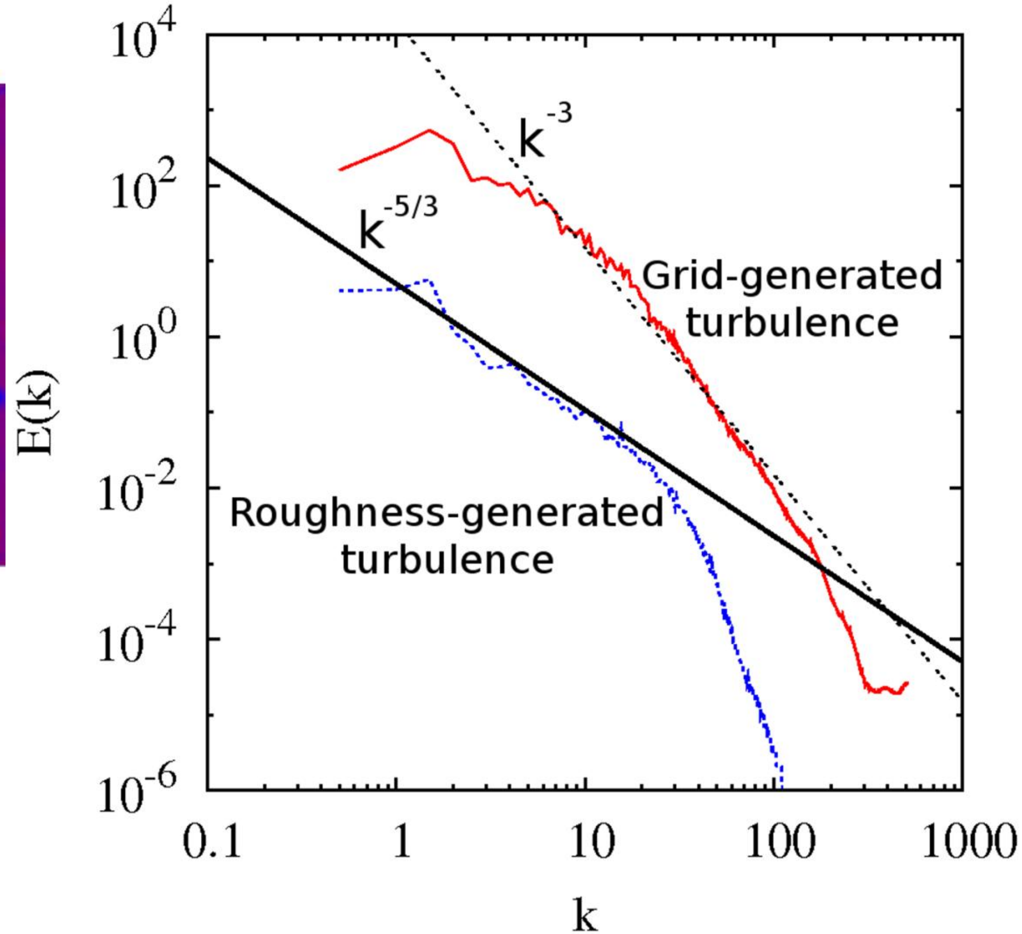
Grid generated turbulence:
Simulate with a grid for several pipe transits, then remove the grid and start measuring the friction factor.

Direct numerical simulations

- Rough walls domain mapped to a rectangular one by conformal maps
- Rough walls generate turbulence

Spectra





- **Good verification of inverse and enstrophy cascade in our simulations.**

Blasius friction factor

Blasius scalings compare well with analytic predictions from momentum transfer theory

Grid-generated

Measured: -0.42±0.05

Expected: -1/2

Roughness-generated

Measured: -0.22±0.03

Expected: -1/4

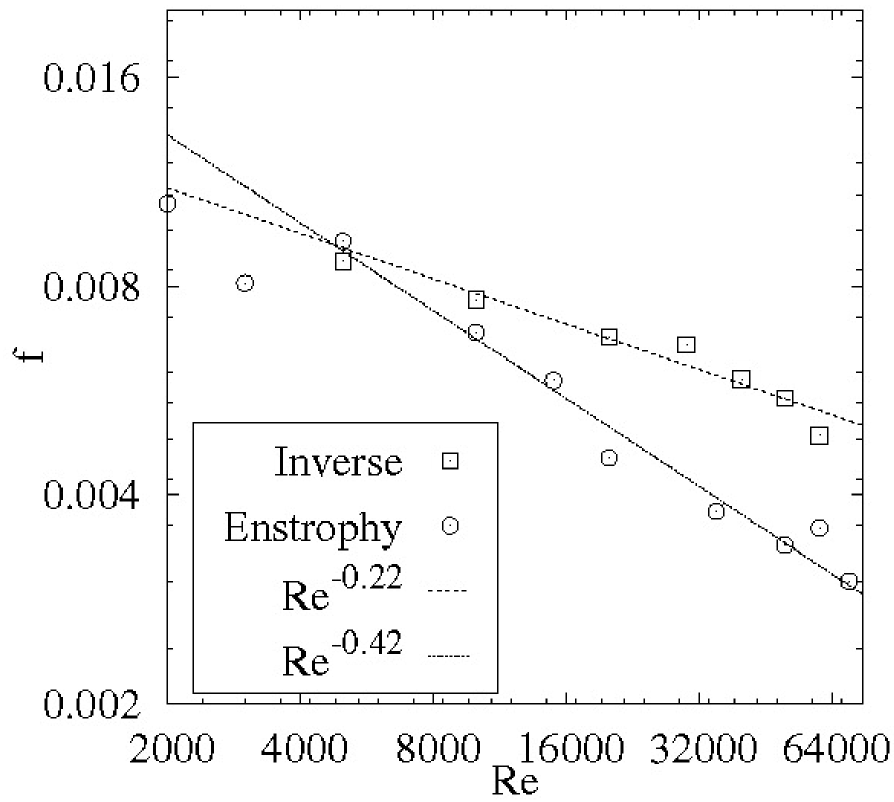


FIG. 2: Scaling of the friction factor with respect to Re for inverse cascade and enstrophy cascade dominated flows in 2D. The roughness is $r/R = 0.067$, and the data have been averaged over a time of 5 pipe transits.

Data collapse in 2D

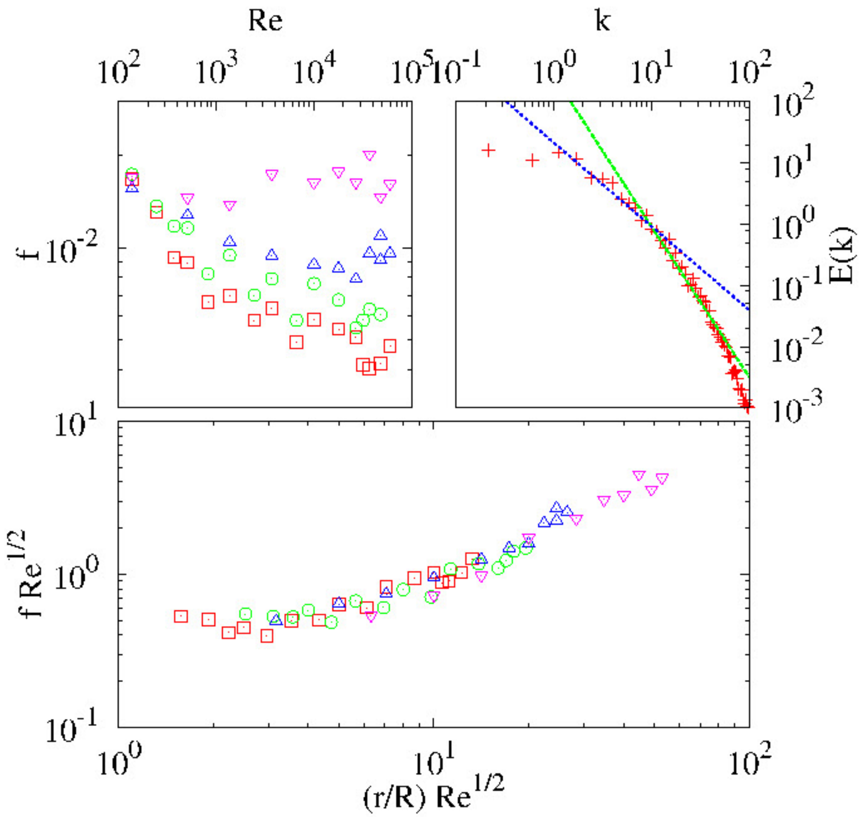


FIG. 3: (Color online). The bottom inset shows the enstrophy cascade data collapse of the friction factor curves for nondimensional roughness 0.05 (○), 0.08 (□), 0.1 (△), and 0.2 (▽) over a range of Reynolds numbers from 1000 to 80000. The top left insent shows the unscaled friction factor data. The top right inset shows the energy spectrum at $r/R = 0.08$ and $\mathrm{Re} = 80000$. The straight lines correspond to $k^{-5/3}$ and k^{-3}.

- **Rough pipe simulations with small amount of random noise → enstrophy-dominated cascade**
  - **Data collapse using enstrophy predictions works well**

$$f = \mathrm{Re}^{-1/2} g\left(\frac{r}{R}\mathrm{Re}^{1/2}\right)$$

Data for non-dimensional roughness from 0.08 to 0.2

Reynolds numbers up to 80000.

Implication for Blasius regime

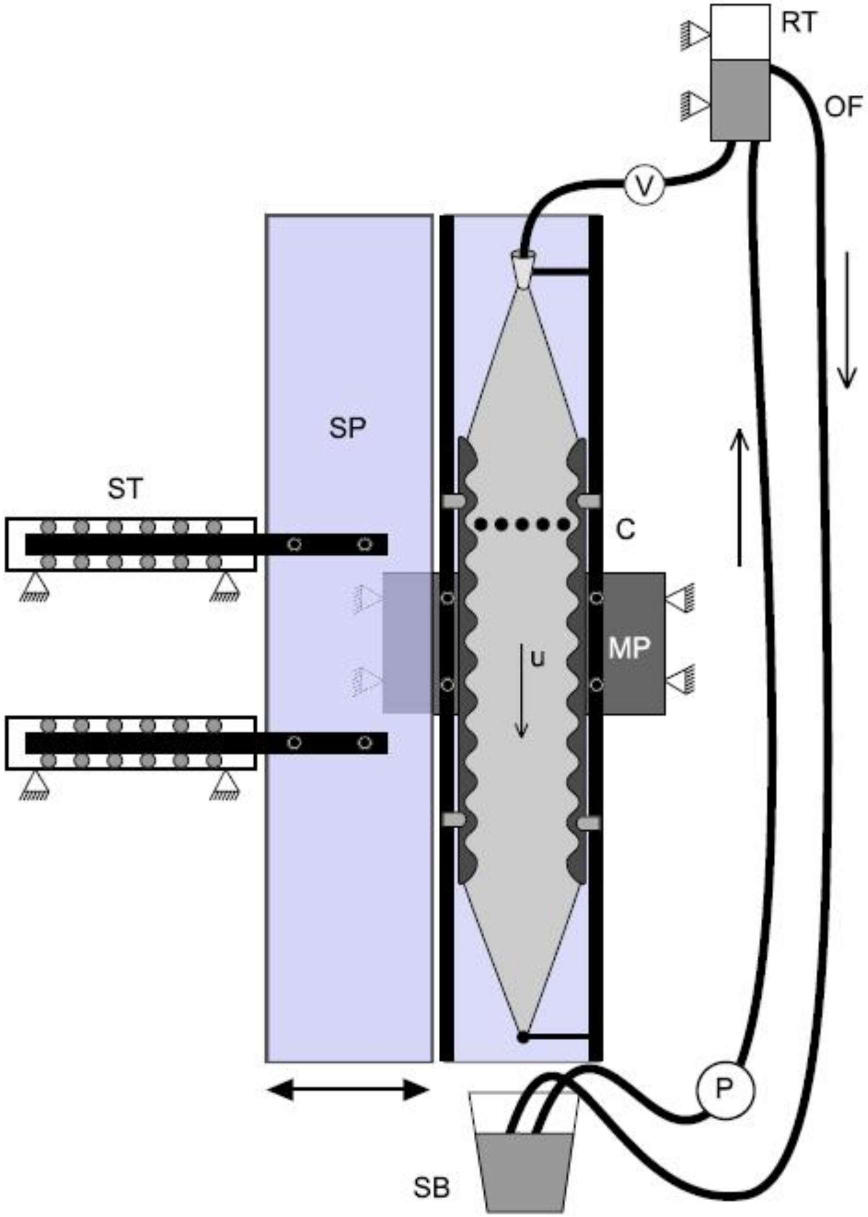
- **There is a Blasius regime in 2D pipe flow**
- **It is different from that in 3D**
- **The scaling with Re depends on the energy spectrum of turbulence**
  - **The scaling result is correctly predicted by momentum-transfer theory for both inverse energy and forward enstrophy regimes**
- **Prandtl theory is silent about making a prediction in 2D**
  - **Prandtl theory makes no prediction about the dependence of the friction factor in the Blasius regime on energy spectrum**

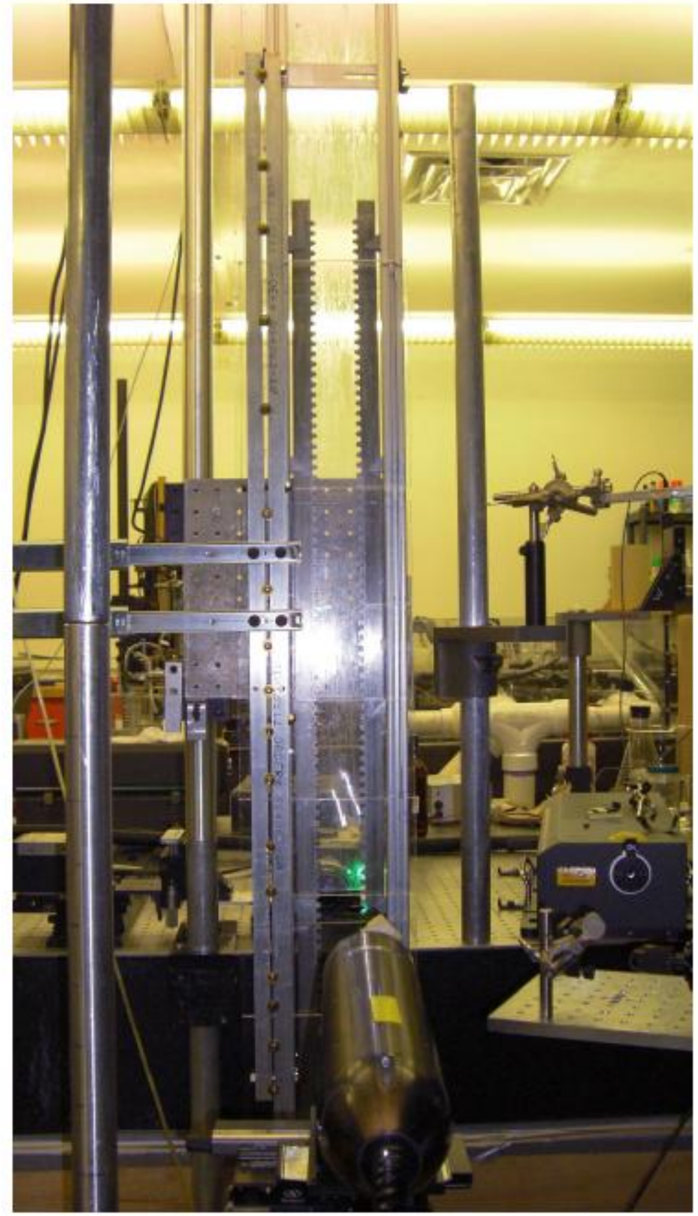
Implication of data collapse

- Data collapse occurs in 2D friction factor as well as 3D friction factor
- The data collapse is predicted by roughness-induced criticality
- 2D and 3D rough-pipe turbulence behave as if governed by a non-equilibrium critical point
- Boundary roughness is a relevant variable for understanding pipe flow turbulence
  - The zero roughness limit is a singular one

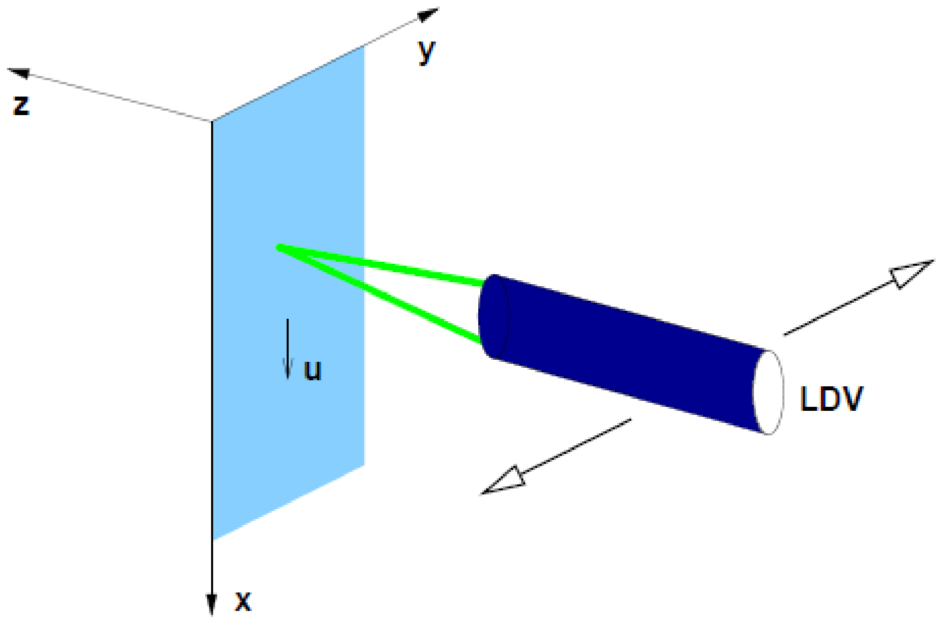
Experimental results in 2D

Experiments at Pittsburgh and Bordeaux using turbulent soap-films

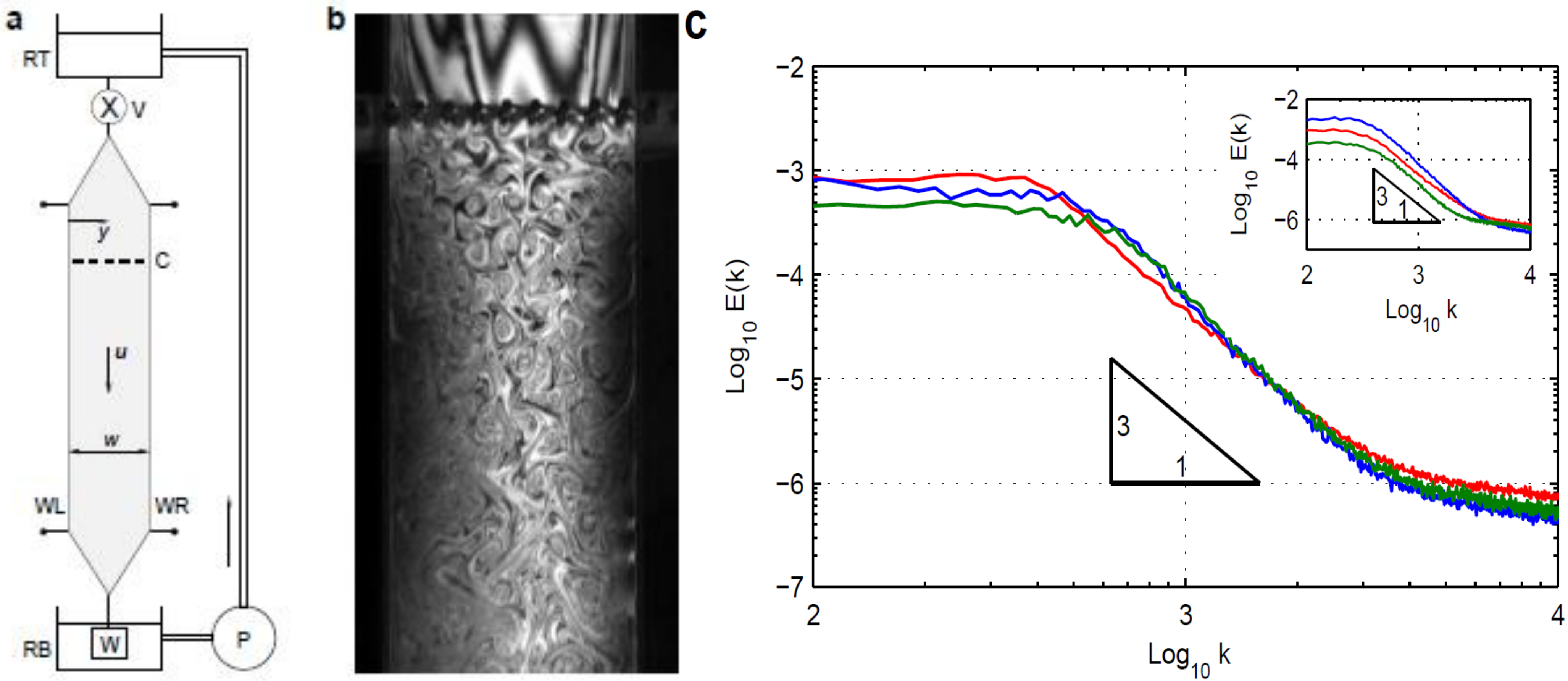

RT
OF
V
SP
ST
C
MP
u
P
SB



Laser Doppler Velocimetry



- Solution is seeded with particles of size 0.4μm
- For a fixed position, LDV measures vertical velocity $u_i = u(t_i)$ and horizontal velocity $v_i = v(t_i)$ of particles passing through the measuring volume
- From velocity times series, it is possible to calculate:
  - Time-averaged velocity $\langle u \rangle$, $\langle v \rangle$
  - Reynolds stress $\tau_{\text{Re}} = \rho \langle (u - \langle u \rangle)(v - \langle v \rangle) \rangle$
  - Power spectrum E_{11} and E_{22}
- LDV can be moved stepwise in horizontal (y) direction (stepsize can be as small as 10μm)


a
RT
V
y
C
u
w
WL
WR
RB
W
P
b
c
Log10 E(k)
Log10 k
3
1
-2
-3
-4
-5
-6
-7
2
3
4

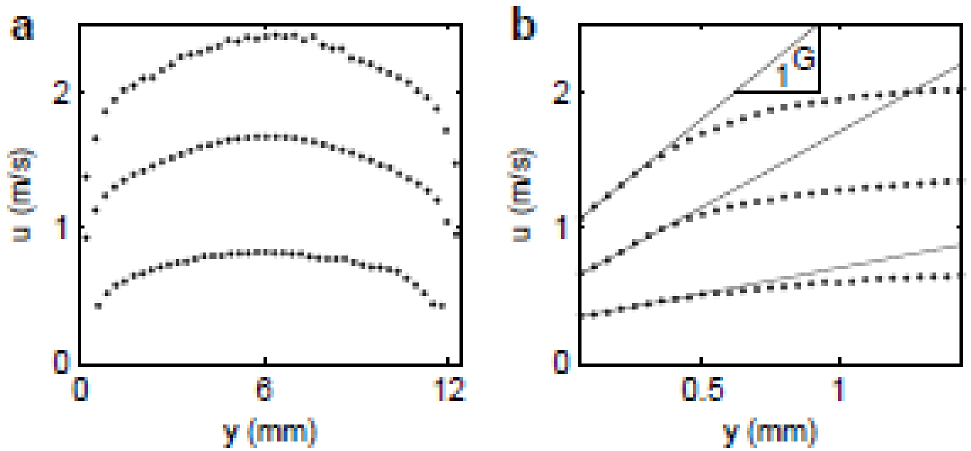


Figure 2: The mean velocity profile $u(y)$ in turbulent soap-film flows, from LDV measurements. **(a)** Typical plots of $u(y)$ in a film of width $w = 12$ mm. **(b)** Typical plots of $u(y)$ close to one of the wires, in the viscous layer where $du(y)/dy = G$ and the thickness of the film is uniform and $\approx 10\,\mu$m (Supplementary Information). The velocity profiles correspond to Re $= 7893, 17648, 25912$. Points on the film closer than $\approx 20\mu$m (the diameter of the beam of the LDV) from the edge of the wire cannot be probed with the LDV; thus the first data point, which we position at $y = 0$, is at a distance of $\approx 20\mu$m from the edge of the wire. The apparent slip velocity is likely to represent 3D and surface-tension effects associated with the complex flow at the contact between the film and the wire.

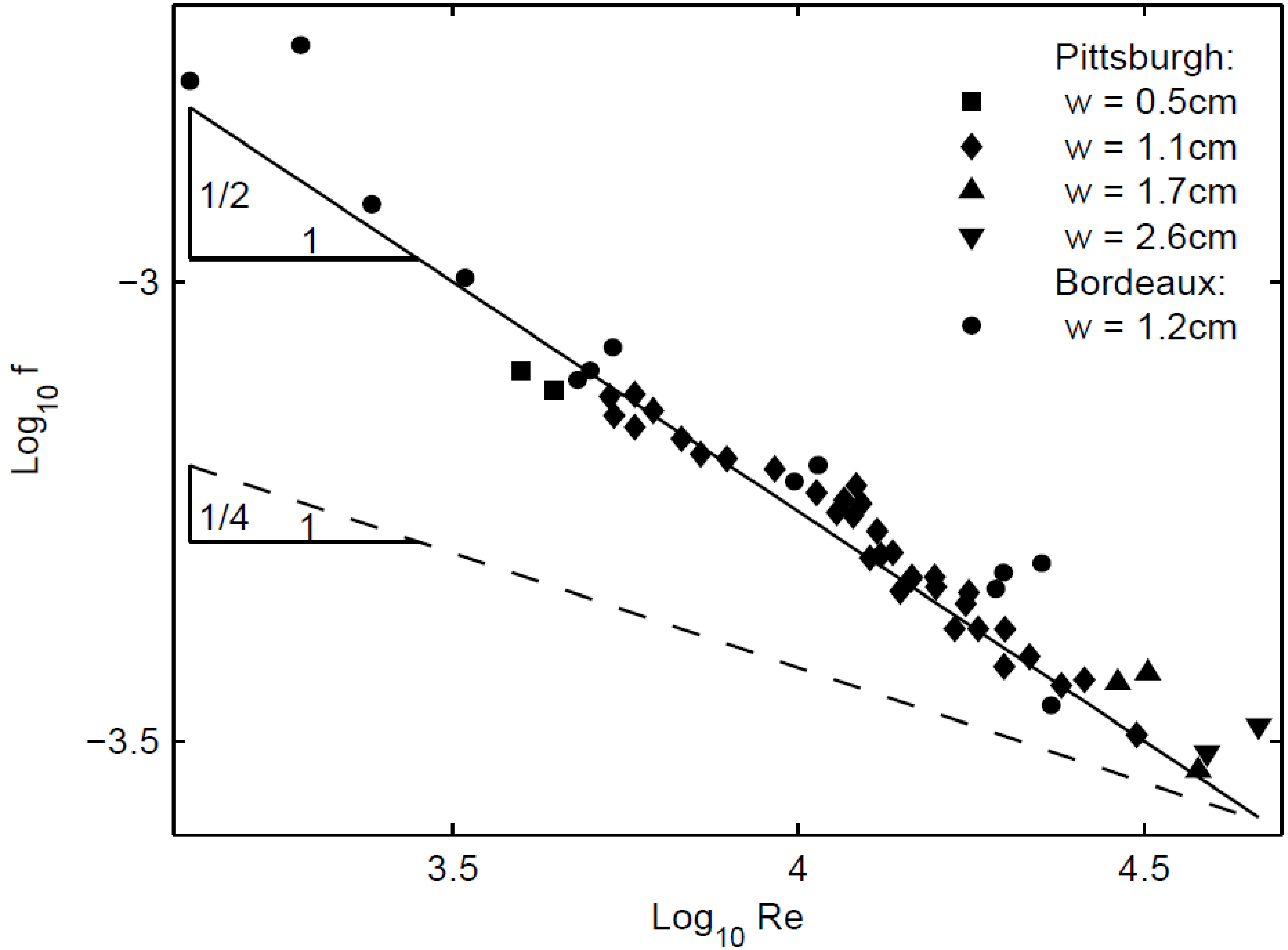
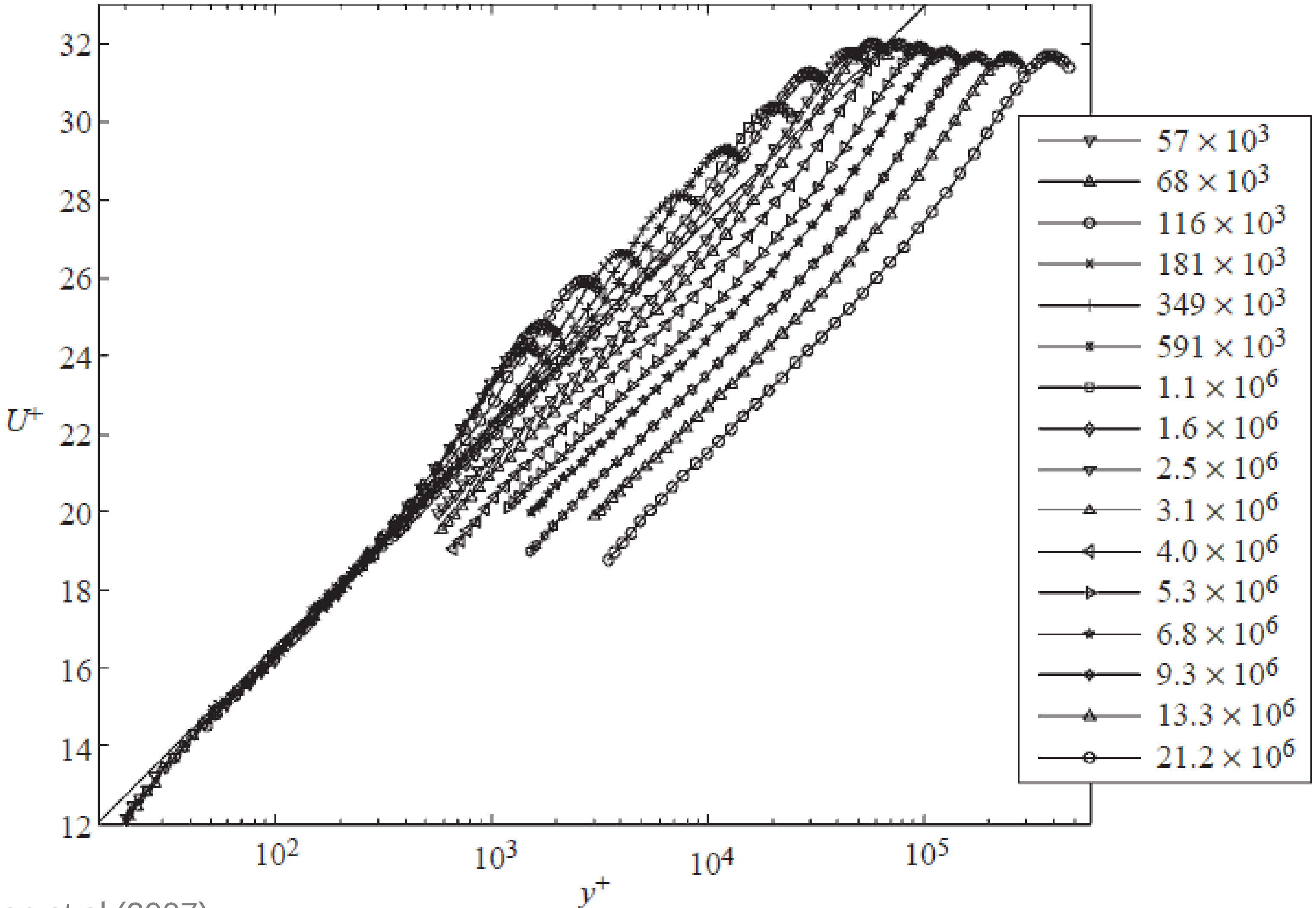


Figure 3: Log-log plot of the frictional drag vs. the Reynolds number in 2D turbulent soap-film flows of Reynolds number $1300 \leq \mathrm{Re} \leq 25000$, from independent experiments performed in Pittsburgh and Bordeaux. The cloud of data points may be represented as a straight line of slope $1/2$, consistent with the scaling $f \propto \mathrm{Re}^{-1/2}$. The straight dashed line of slope $1/4$ corresponds to the Blasius empirical scaling, $f \propto \mathrm{Re}^{-1/4}$.

Conclusion of experiment

- **Friction factor exponent in Blasius regime in 2D enstrophy-dominated flow is ½, not ¼**
  - Clearly distinct from what happens in 3D
- **Friction factor exponent in Blasius regime in 2D inverse cascade is ¼**
  - Clearly distinct from enstrophy-dominated flow
- **Results in agreement with theoretical prediction**
- **Macroscopic flow property (friction factor) directly related to microscopic spectral property**
  - Predicted by Illinois theories of roughness-induced criticality and momentum transfer

The mean velocity profile



U ~ ln y

Allen et al (2007)

Figure 5. Velocity profiles over the full Reynolds number range. Figure from Shockling *et al.* (2006).

526 G. I. Barenblatt and V. M. Prostokishin

The mean velocity profile

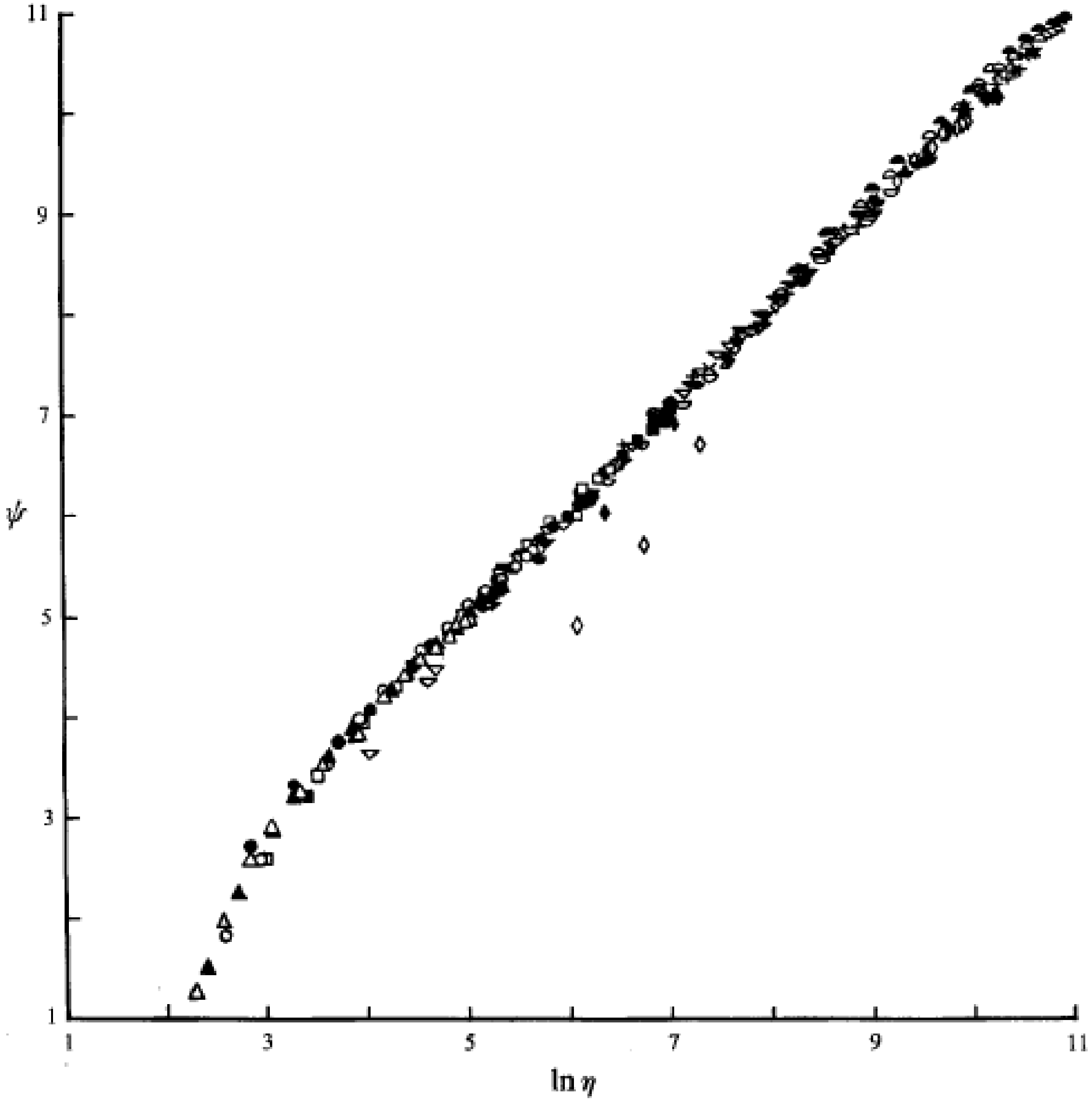


FIGURE 3. The experimental points in reduced coordinates $(\psi, \ln\eta)$ settle down, for large η, close to the bisectrix of the first quadrant, confirming the quasi-universal form of the scaling law. △, $Re = 4 \times 10^3$; ▲, $Re = 6.1 \times 10^3$; ○, $Re = 9.2 \times 10^3$; ●, $Re = 1.67 \times 10^4$; □, $Re = 2.33 \times 10^4$; ■, $Re = 4.34 \times 10^4$; ▽, $Re = 1.05 \times 10^5$; ▼, $Re = 2.05 \times 10^5$; ⊖, $Re = 3.96 \times 10^5$; ◒, $Re = 7.25 \times 10^5$; ◇, $Re = 1.11 \times 10^6$; ◆, $Re = 1.536 \times 10^6$; +, $Re = 1.959 \times 10^6$; ×, $Re = 2.35 \times 10^6$; ⌓, $Re = 2.79 \times 10^6$; ◓, $Re = 3.24 \times 10^6$.

$U \sim y^{\alpha}$

where $\alpha \sim 1/\ln Re$

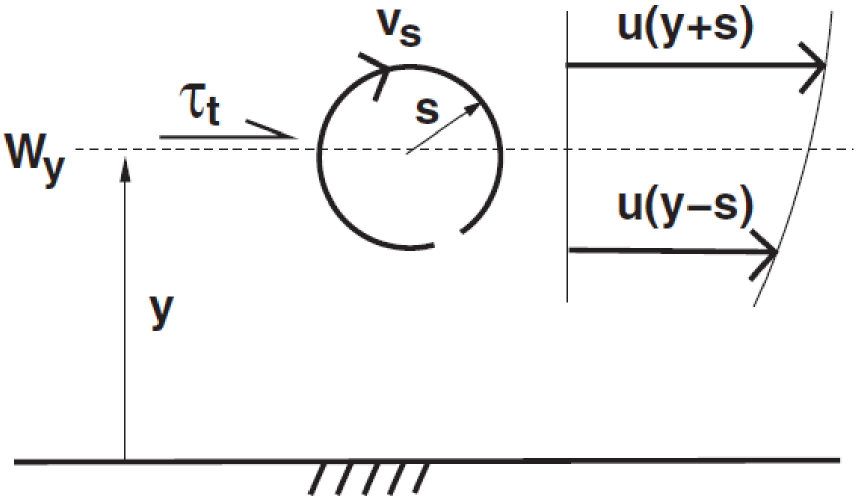
Spectral theory of mean velocity profile

- Prandtl theory and other approaches do not have a way to represent the nature of the turbulent state

- We derive a differential equation for the mean velocity profile in terms of the energy spectrum

- Outcome is that every intermediate asymptotic scaling regime in the mean velocity profile has a counterpart in the spectral structure

Spectral theory of mean velocity profile



$$u_s = \sqrt{\int_{1/s}^{\infty} E(k)dk}$$

- **Turbulent shear stress acting on a layer at y is**

$$\tau_t = \kappa_\tau \rho y v_y u'(y)$$

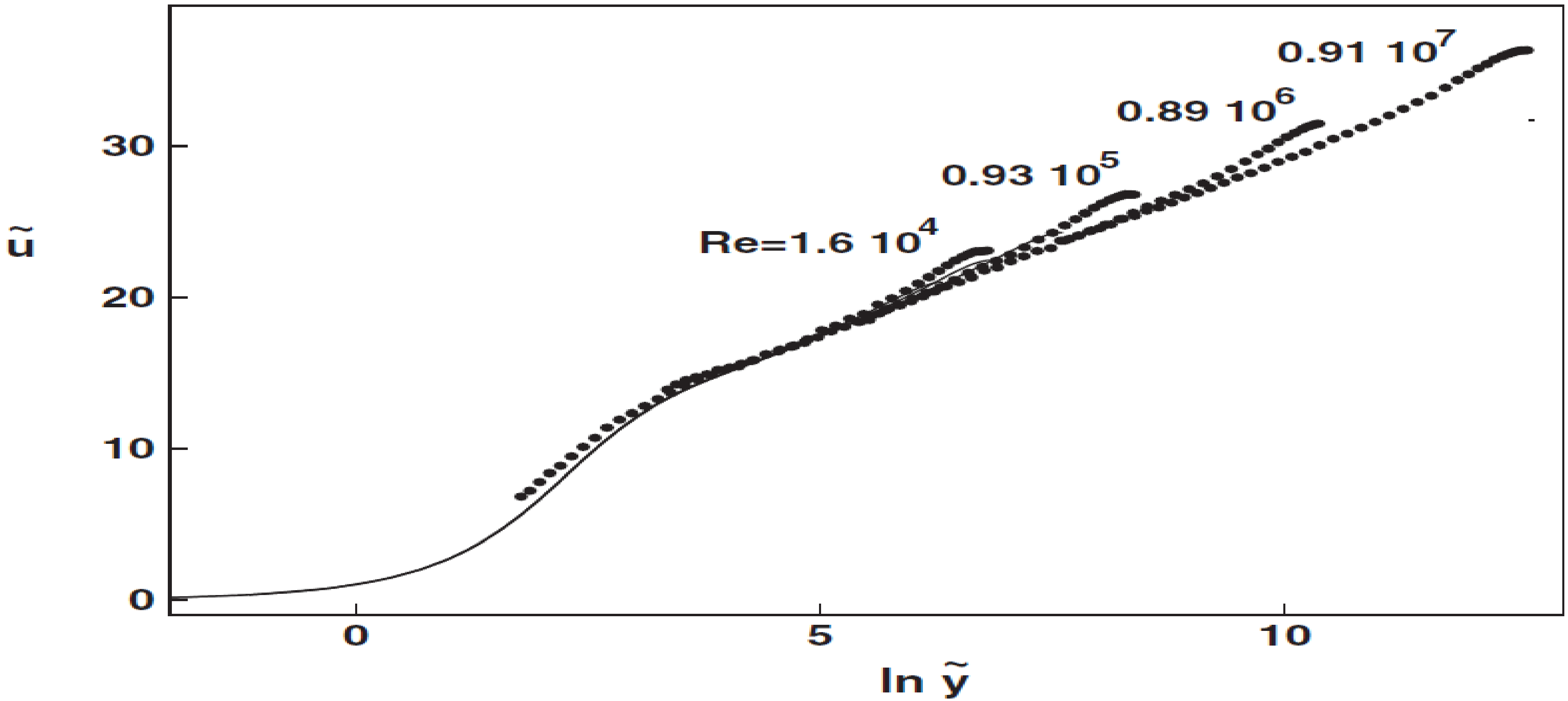
- **The shear stress in terms of the spectrum is**

$$\tau_t = \kappa^2 \rho I^{3/4} y^2 u'(y)^2$$

- **The total shear stress (turbulent + viscous) is**

$$\tau_t + \rho \nu u' = \tau_0 (1 - y/R)$$

Anatomy of the mean velocity profile



Viscous	Buffer	Log	Wake
Dissipation	Dissipation	Inertial	Integral

Spectral connection

Not present in Prandtl theory

- **Turbulence is a critical point**
- **Fluctuations related to large-scale flow properties**
  - **analogous to fluctuations related to thermodynamics in phase transition theory**
- **Predictions about data collapse in (Re, r) observed in Nikuradze's data and tested in 2D DNS**

- **Momentum transfer calculations explicitly involve the energy spectrum in the formulae for the friction factor**
- **Friction factor scaling exponents predicted in 3D and 2D**
- **Predictions about Blasius regime in 2D enstrophy and inverse-cascade dominated flows tested in DNS and turbulent soap film experiments**
- **Spectral connection verified in preliminary data in 3D**

What about the million dollars?

- Virtually everything we know about turbulence did NOT come from the Navier-Stokes equations!!!
- They seem to be a bad place to start a theory
- We can "understand" turbulence without proving all the theorems that the Clay Institute requires
- Proving all the theorems that the Clay Institute requires may not allow us to "understand" turbulence
  - in the sense of relating microscopic spectral properties with macroscopic flow properties

References

- G. Gioia and P. Chakraborty, Turbulent Friction in Rough Pipes and the Energy Spectrum of the Phenomenological Theory. *Phys. Rev. Lett*. **96**, 044502 (2006).

- N. Goldenfeld, Roughness-induced criticality in turbulence. *Phys. Rev. Lett*. **96**, 044503 (2006).

- N. Guttenberg and N. Goldenfeld, The friction factor of two-dimensional rough boundary turbulent soap film flows. *Phys. Rev. E Rapid Communications* **79**, 065306R (2009)

- T. Tuan et al. Macroscopic effects of the spectral structure in turbulent flows. *Nature Physics*, **6**, 438-441 (2010).

- G. Gioia, N. Guttenberg, N. Goldenfeld, P. Chakraborty. Spectral theory of the turbulent mean-velocity profile. *Phys. Rev. Lett*. 108, 184501 (2010)