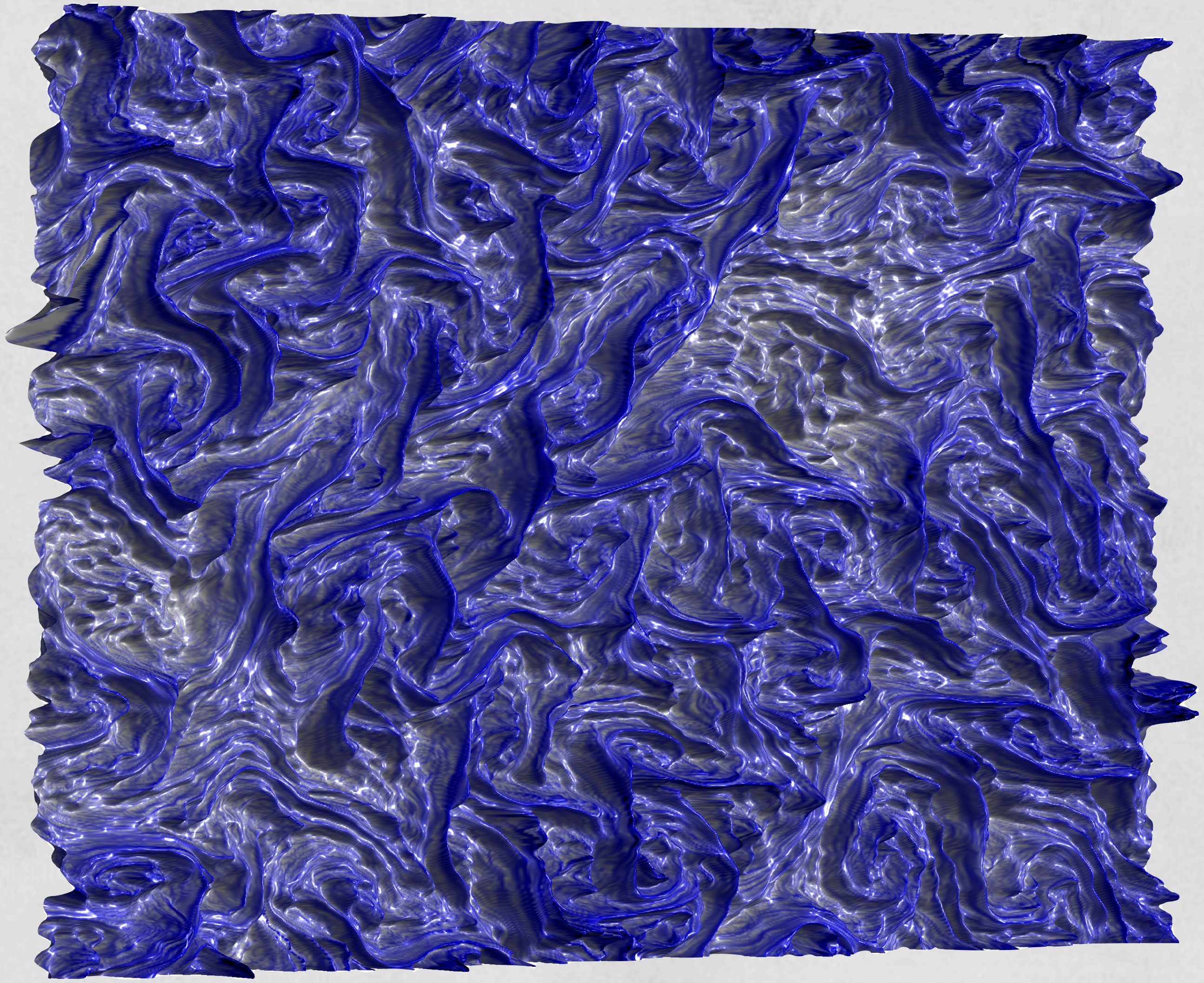
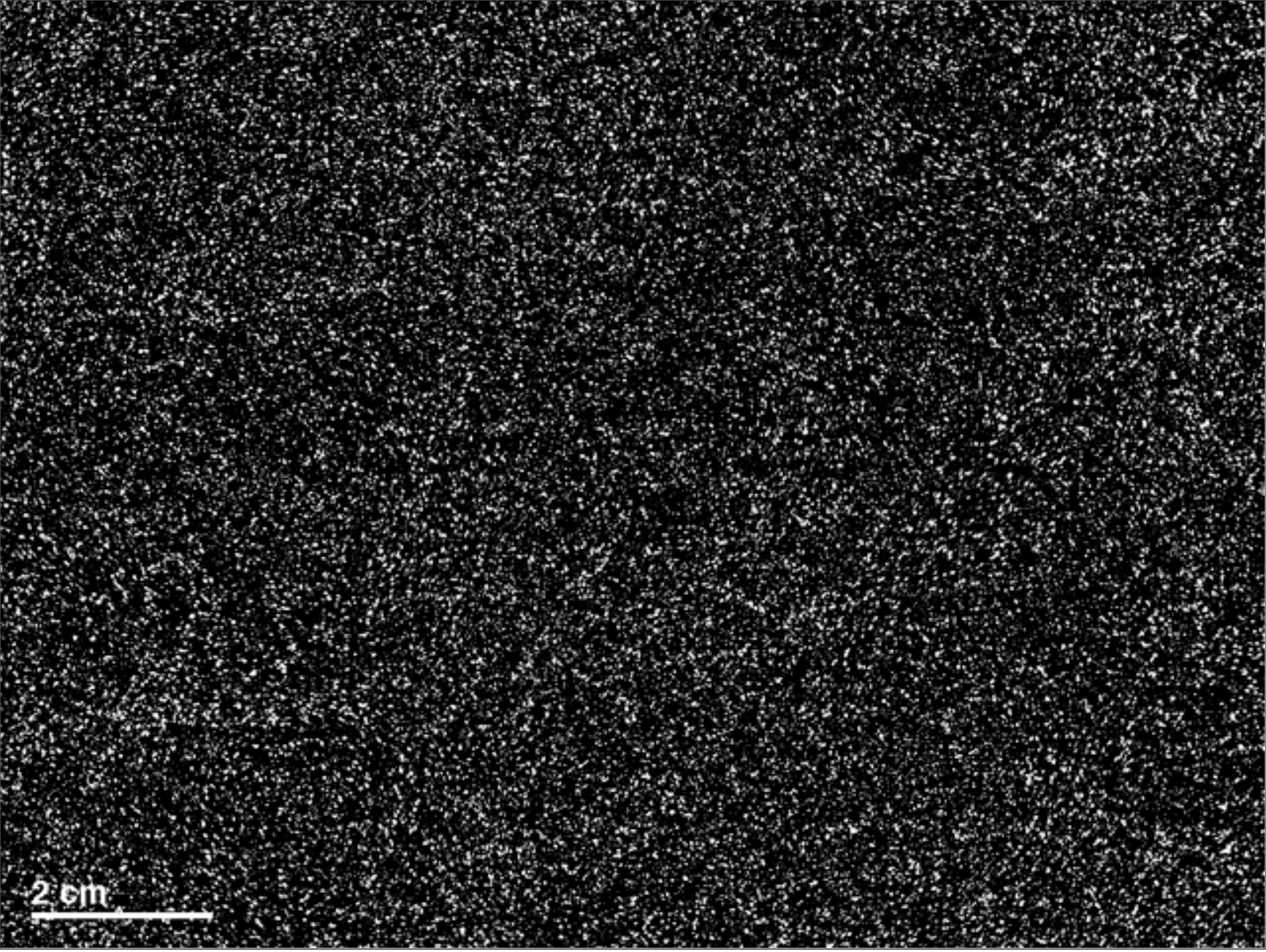


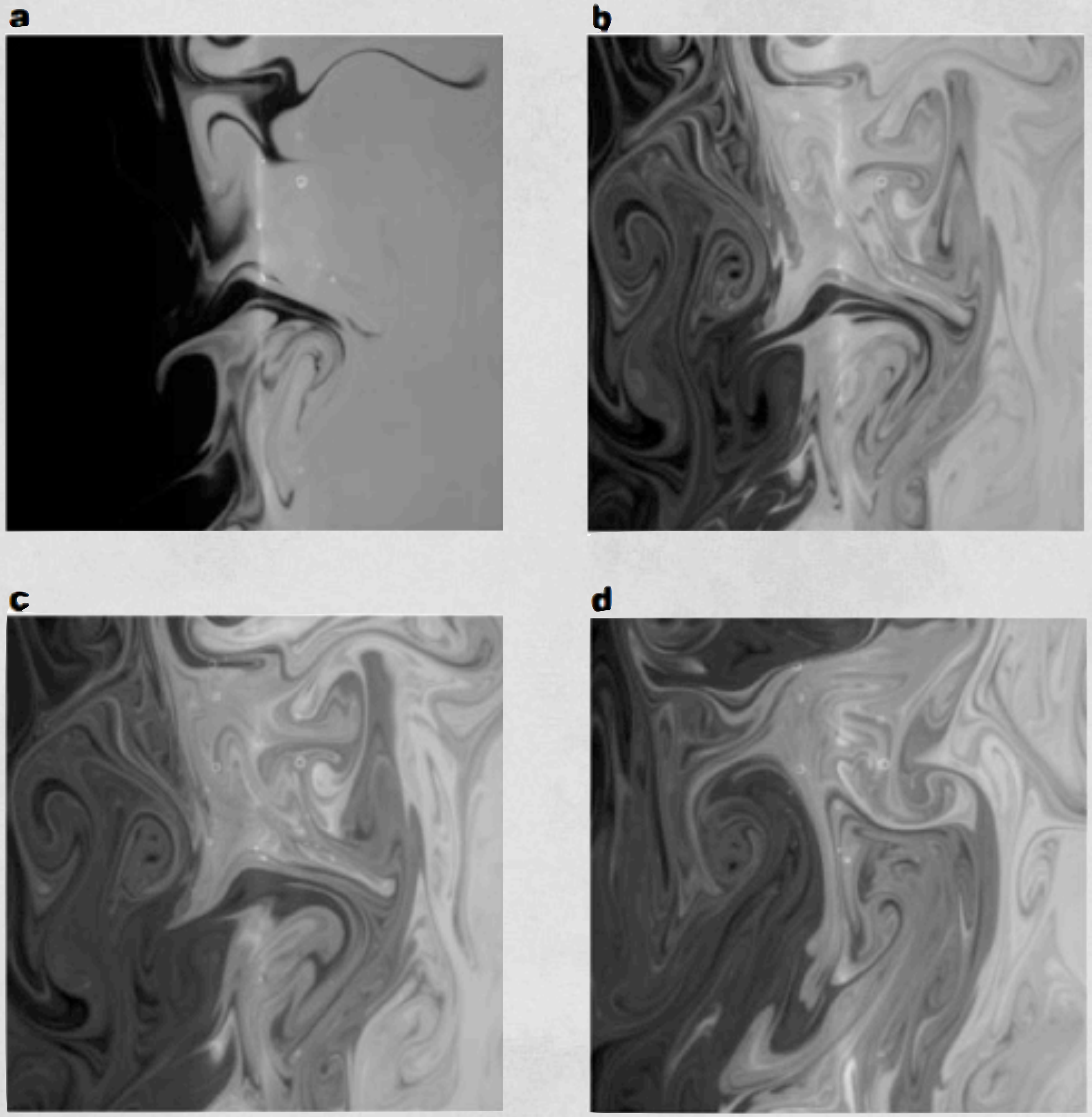
Lagrangian and Eulerian Nonlinearities in 2D Weak Turbulence

N. T. Ouellette

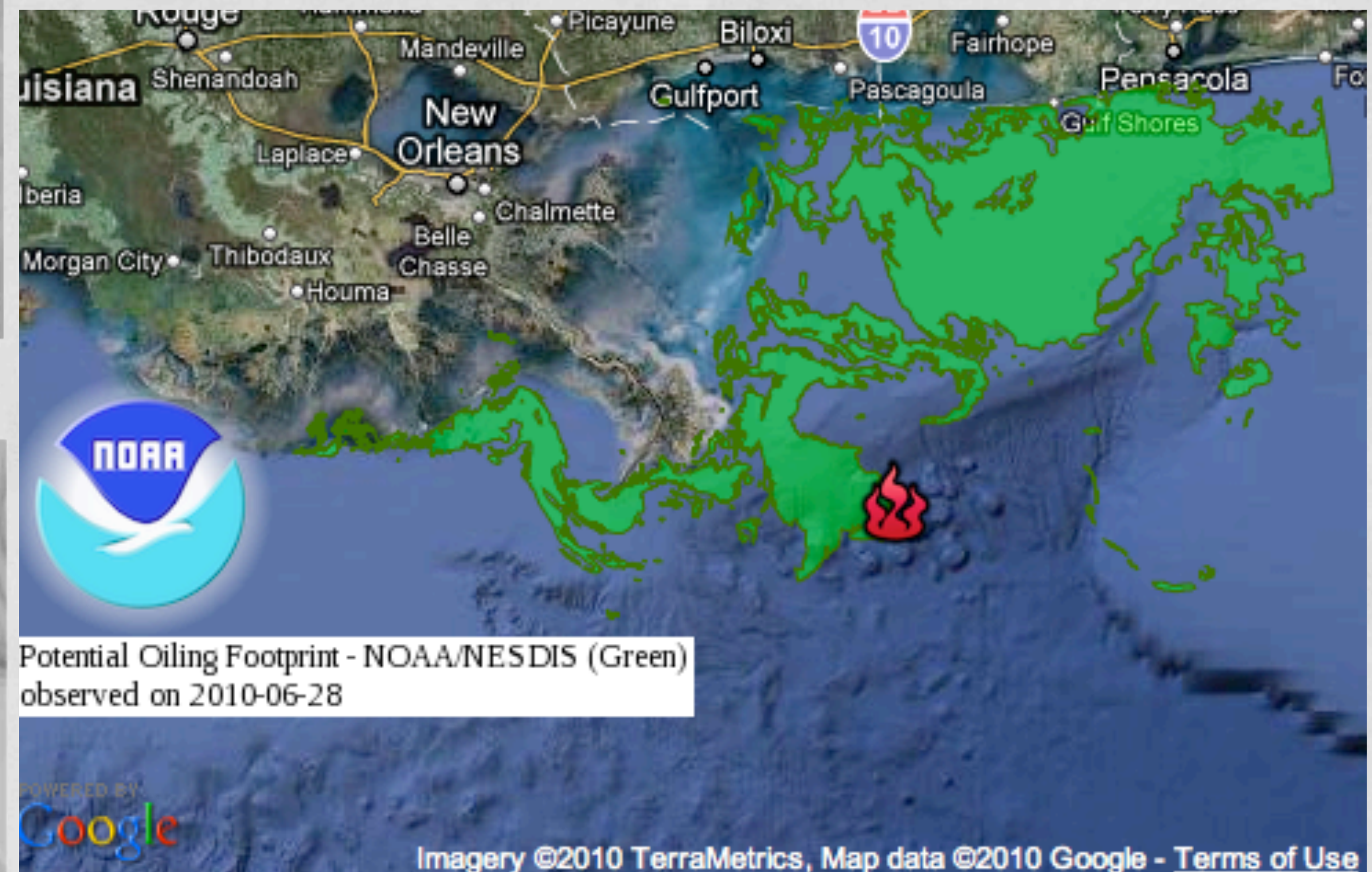
D. H. Kelley, Y. Liao


2 cm

Nonlinearity in Fluid Flow

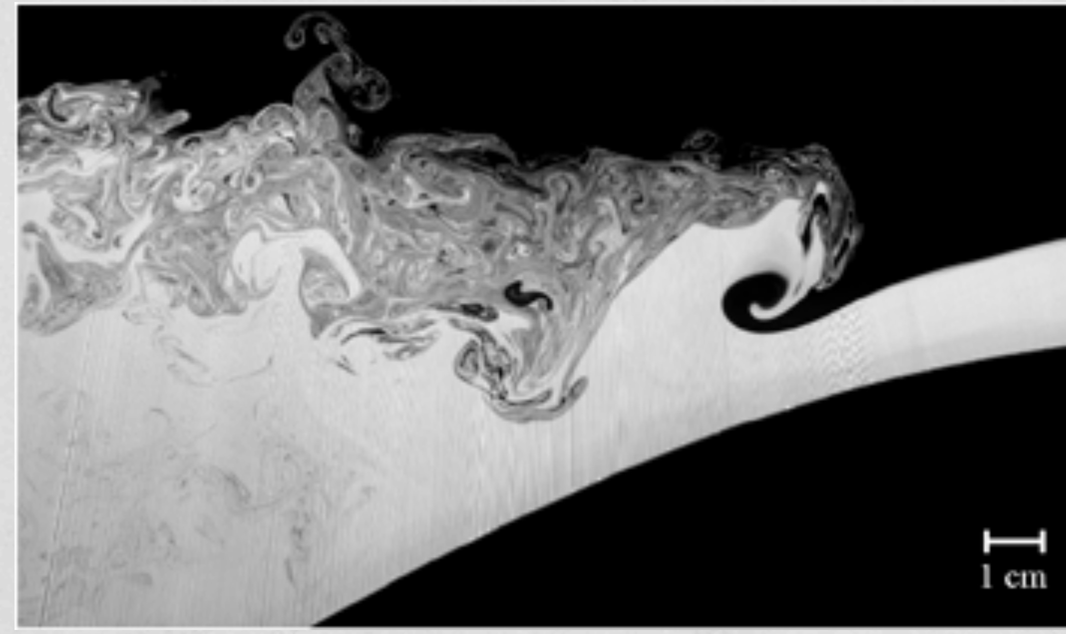


D. Rothstein, E. Henry, & J.P. Gollub, Nature (1999)



Drives efficient mixing

Nonlinearity in Fluid Flow

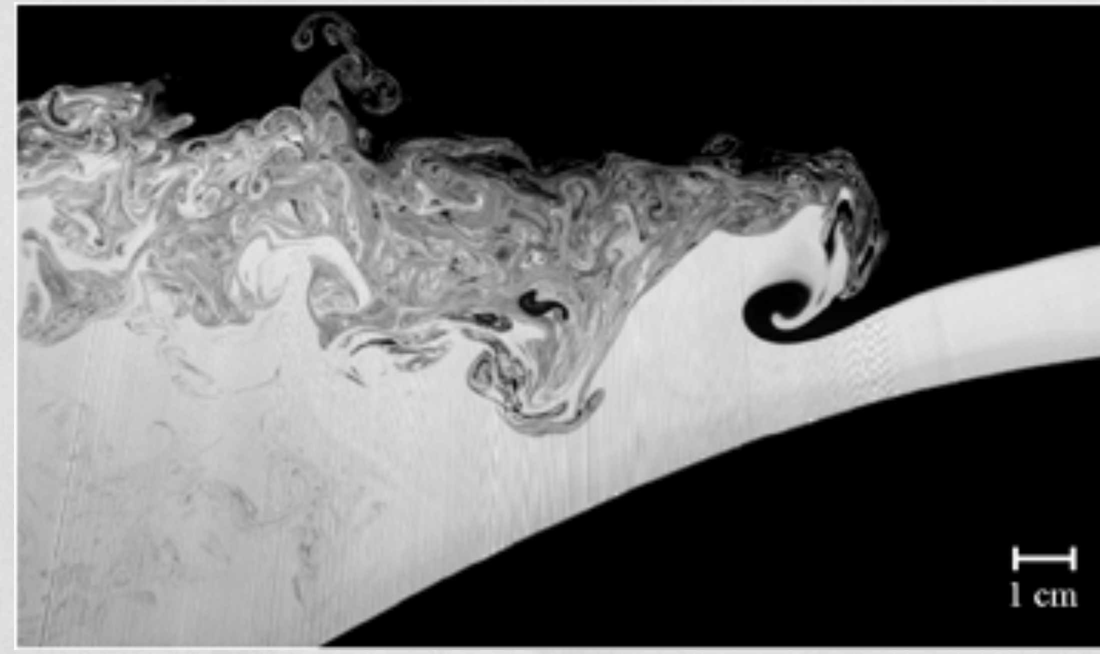


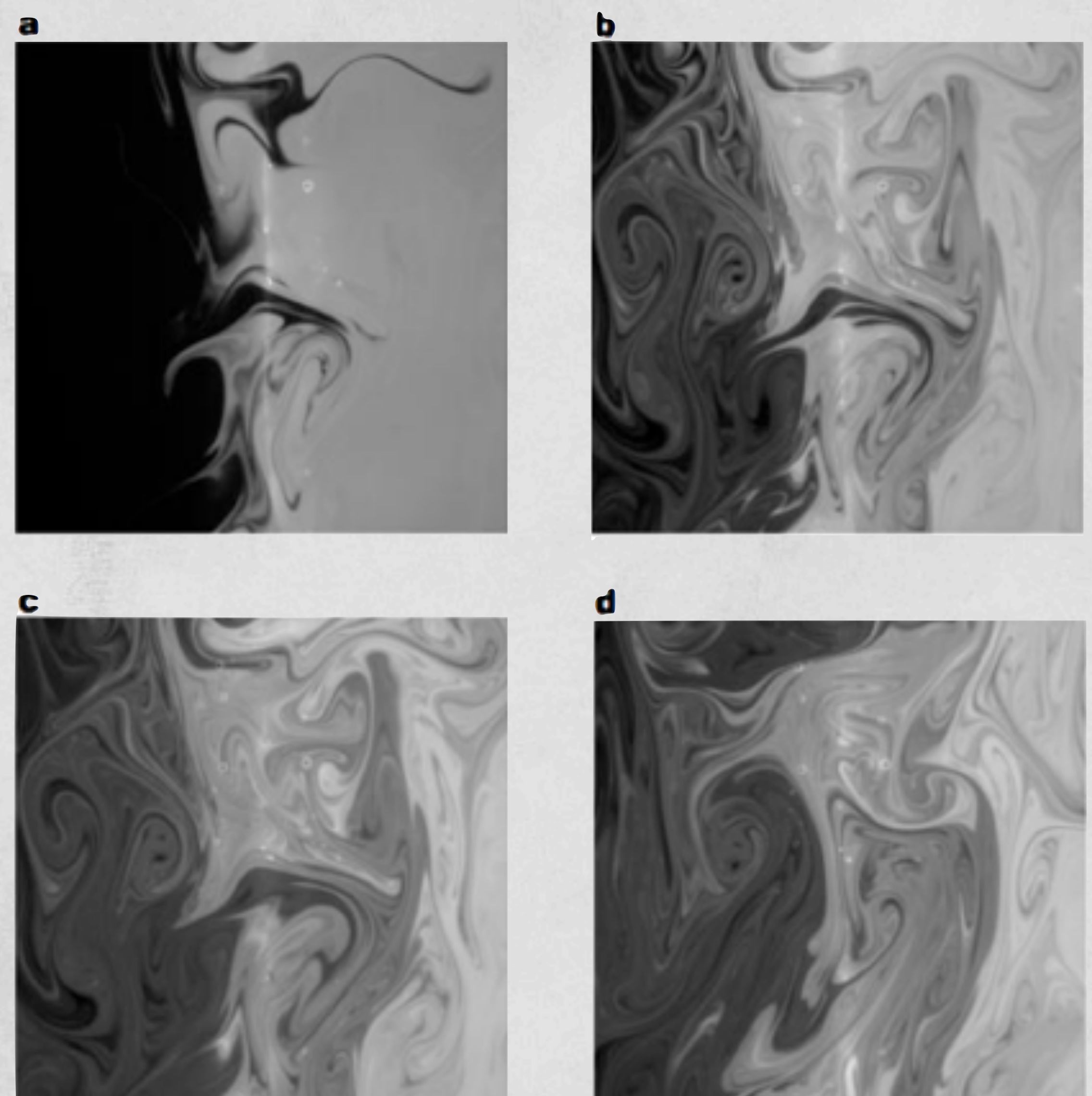
Drives production of small scales

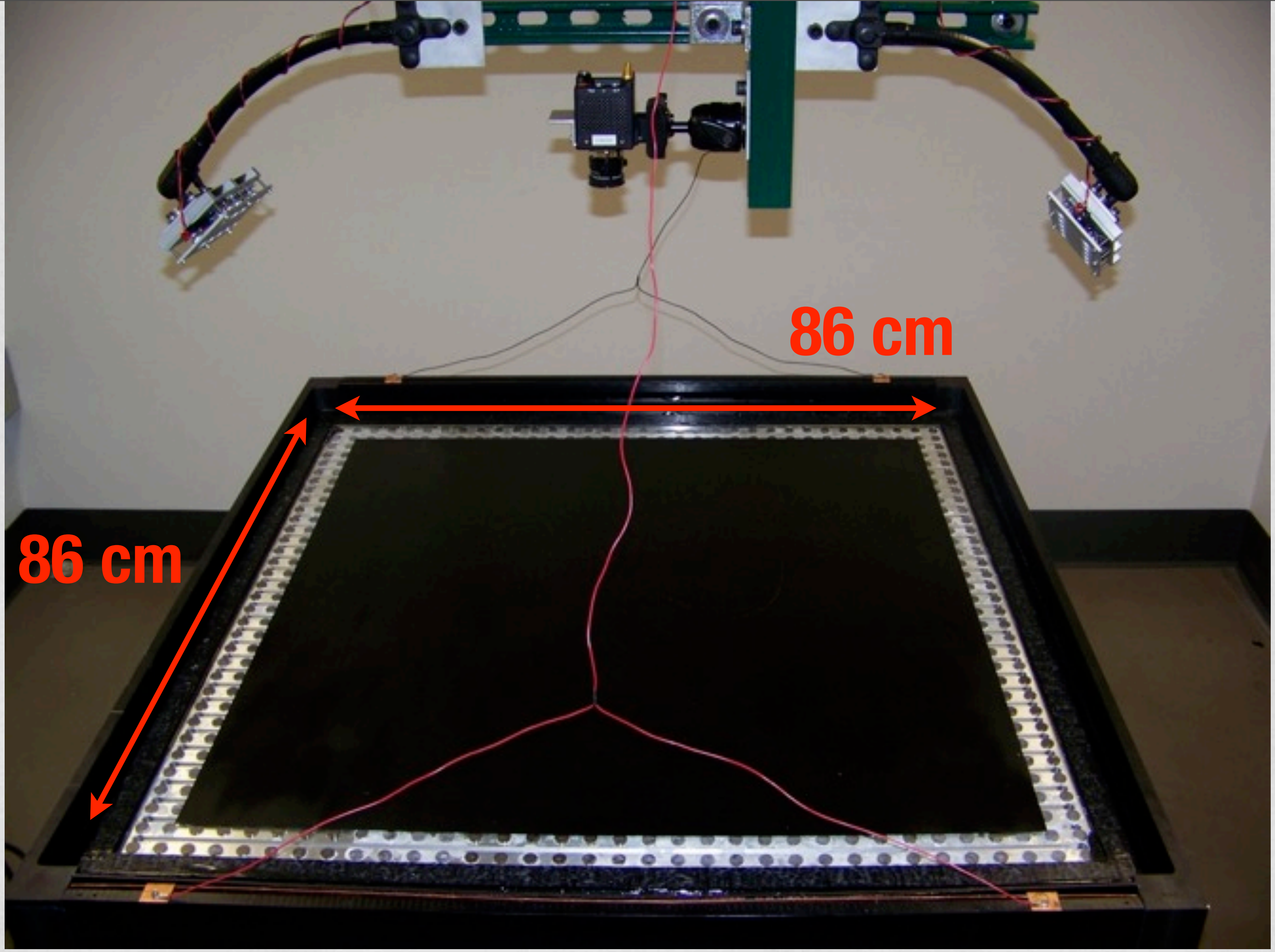
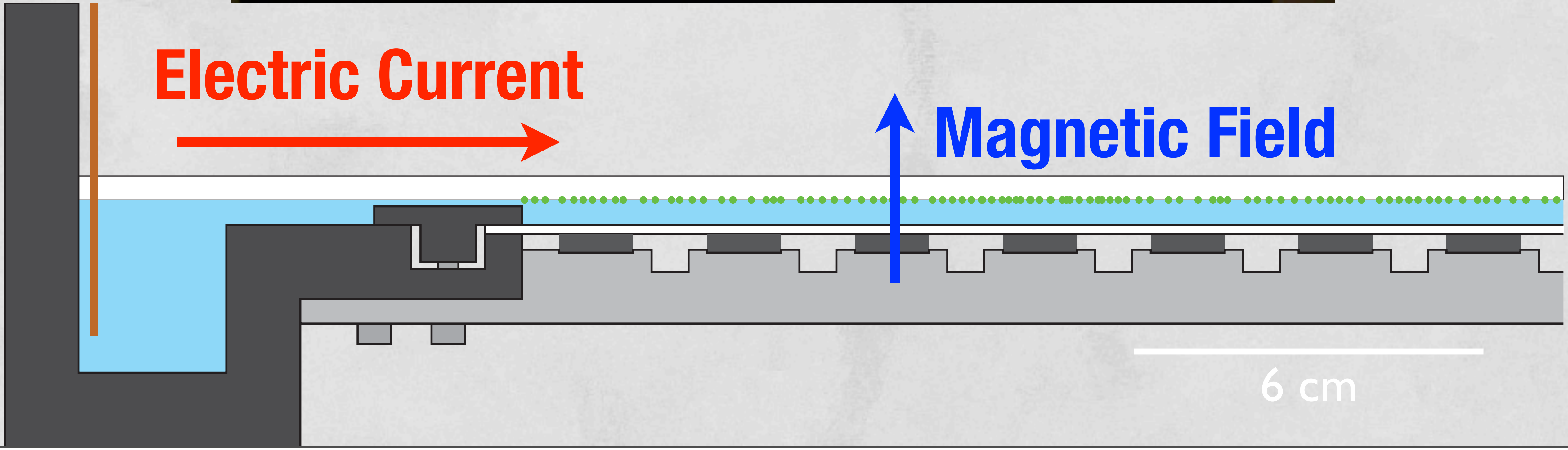
How can we characterize these manifestations of nonlinearity?

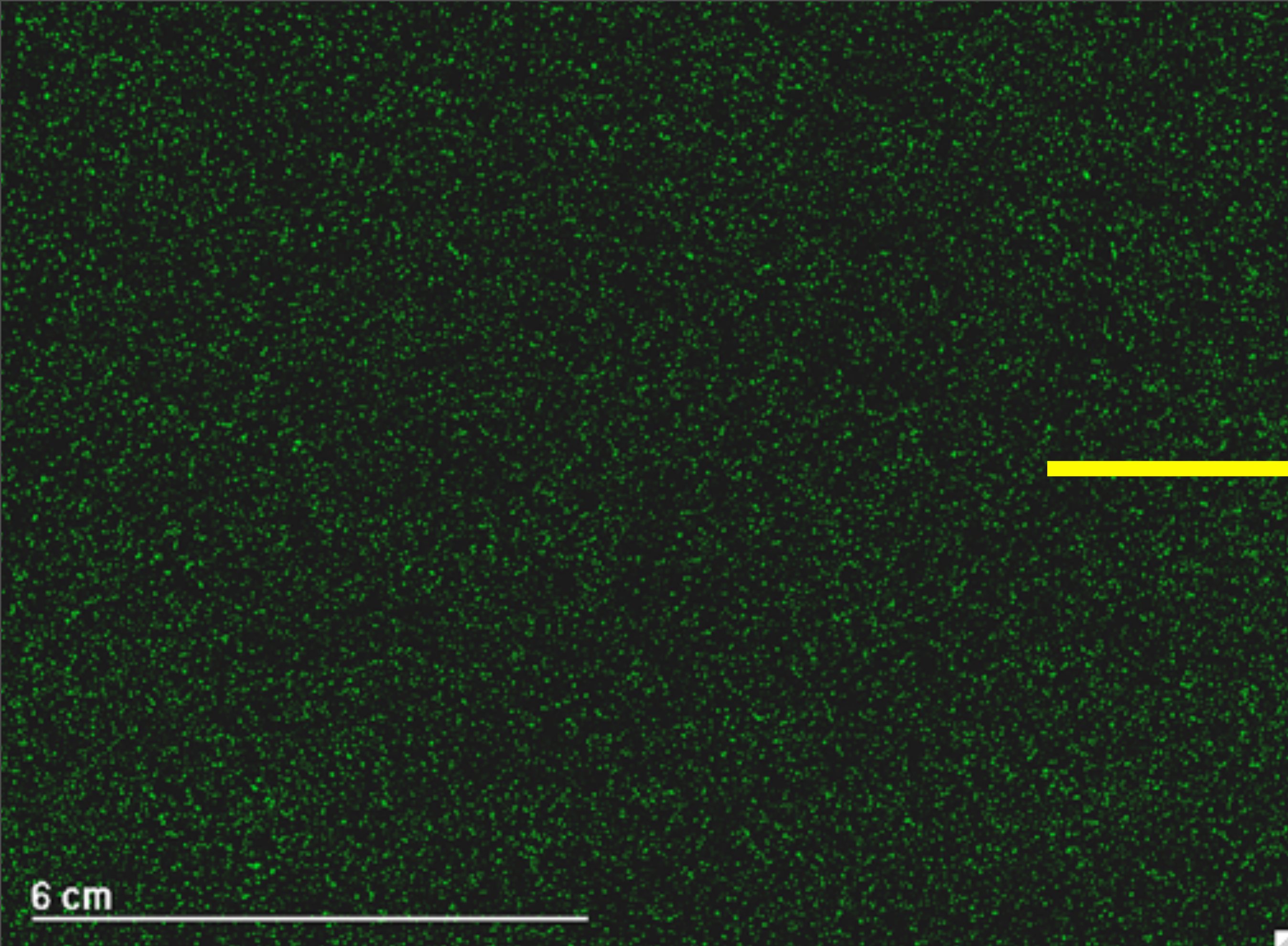
Can we link transport in real space and Fourier space?

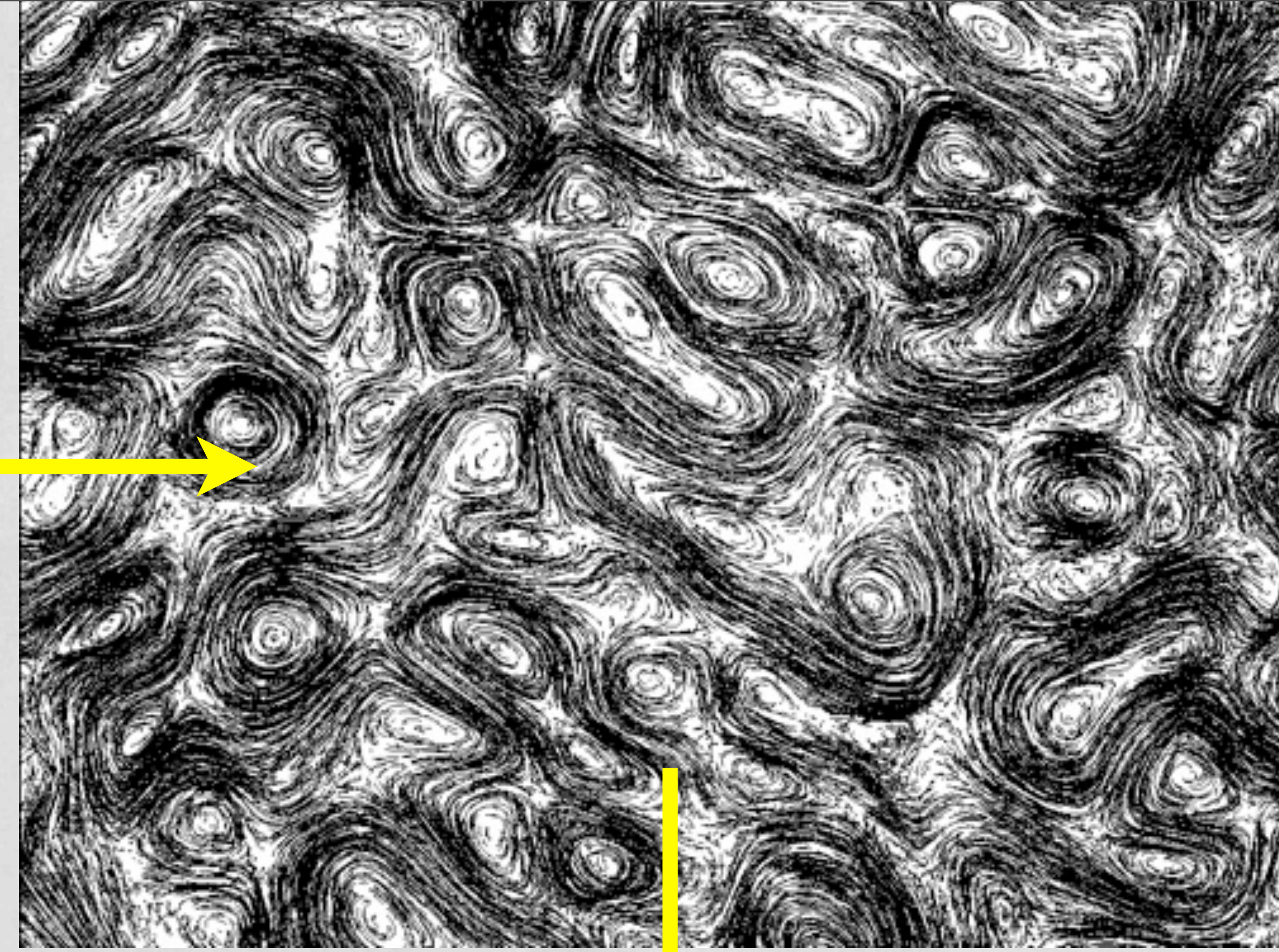
What are the spatial signatures of nonlinearity?

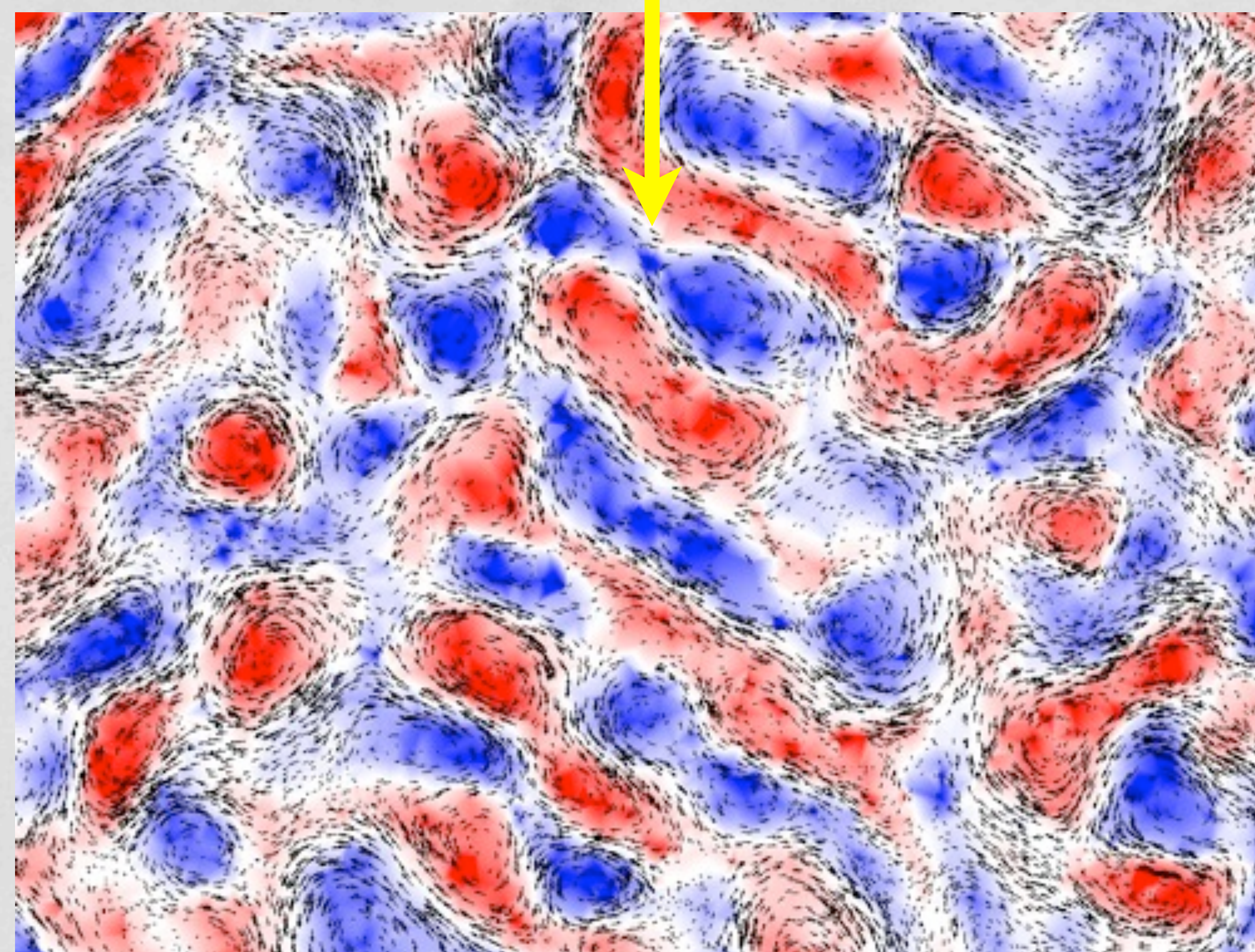





86 cm
86 cm

Electric Current
Magnetic Field
6 cm



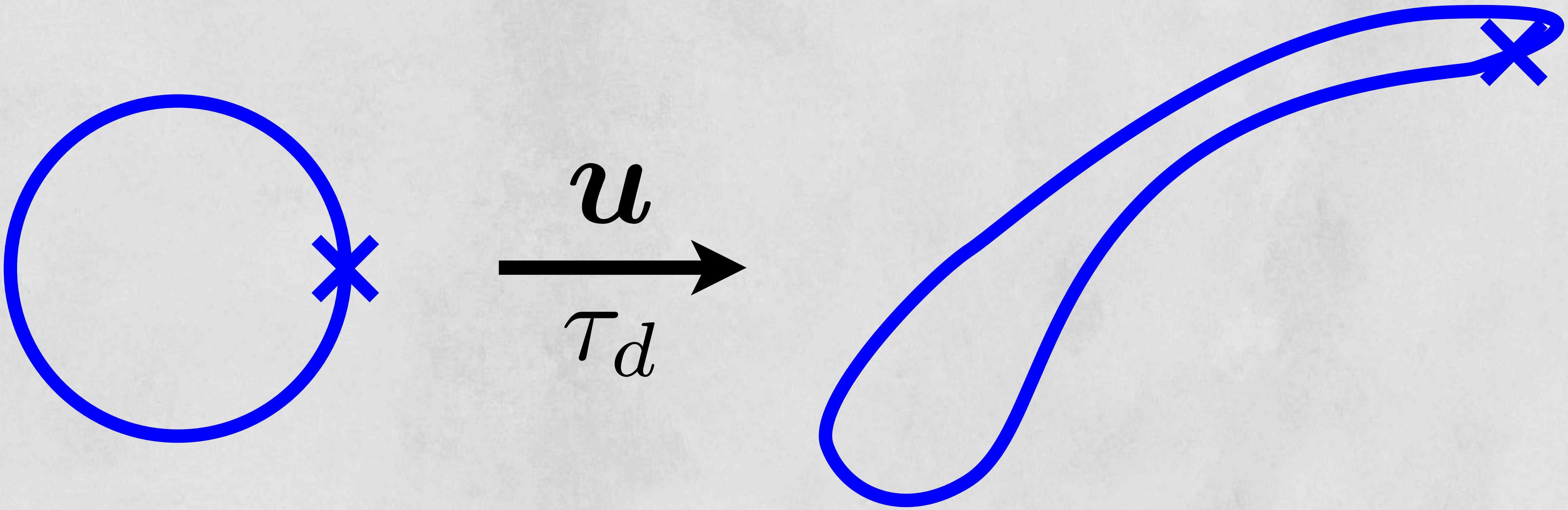




- Obtain velocity field with PTV
- 50 µm particles, ~35k per frame
- Advect virtual particles through field

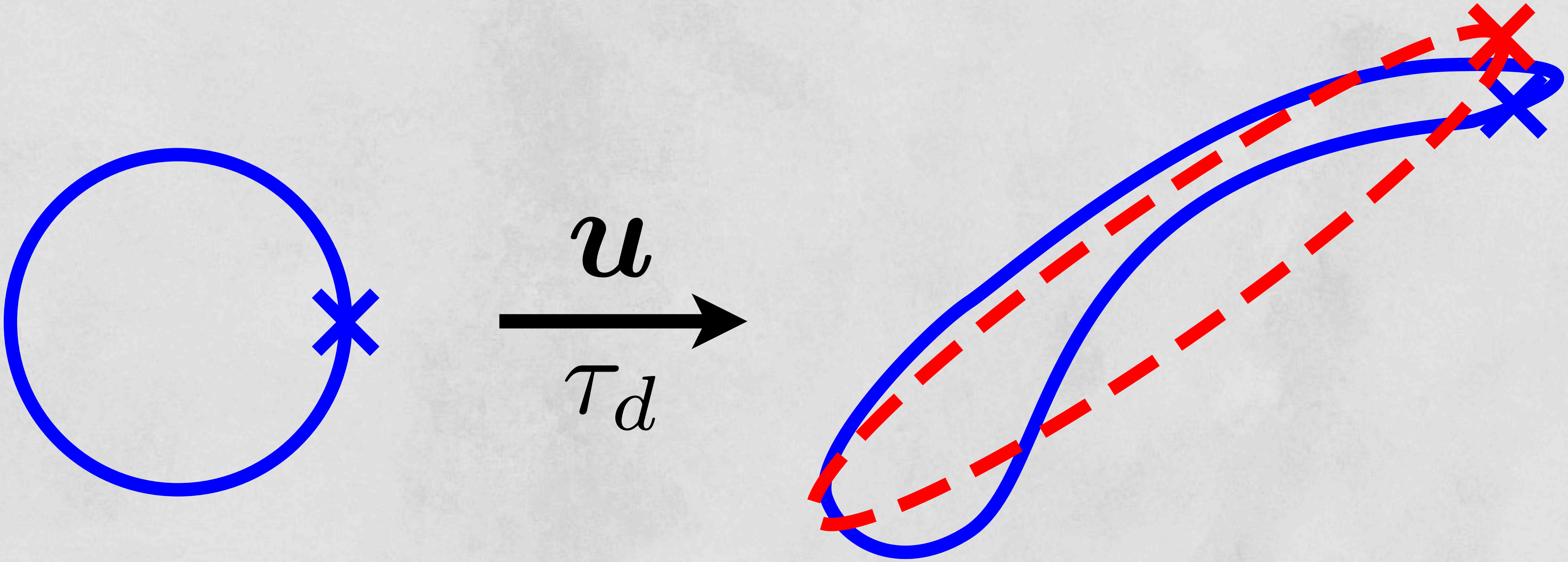
NTO, H. Xu, & E. Bodenschatz, Exp. Fluids (2006)
NTO, P.J.J. O'Malley, & J.P. Gollub, Phys. Rev. Lett. (2008)
S.T. Merrifield, D.H. Kelley, & NTO, Phys. Rev. Lett. (2010)

Lagrangian Deformation



M.L. Falk & J.S. Langer, Phys. Rev. E. (1998)

Lagrangian Deformation

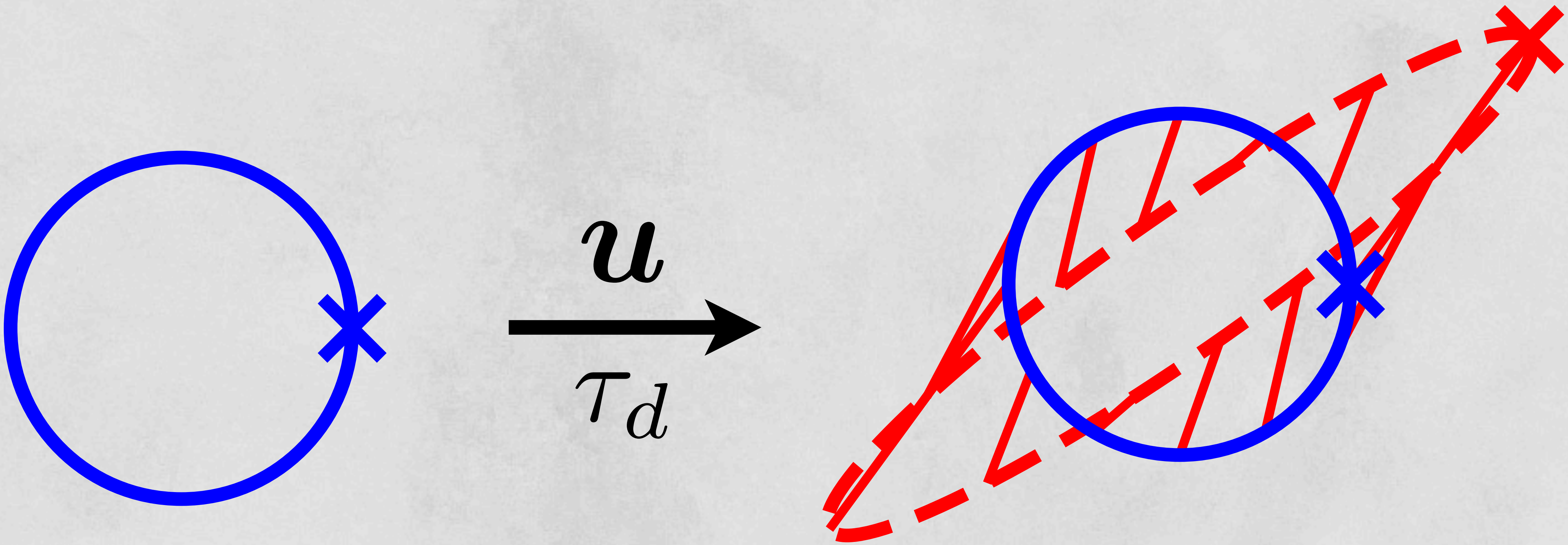


affine transformation:
translation, rotation,
shear, dilation,
compression

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}_0$$

M.L. Falk & J.S. Langer, Phys. Rev. E. (1998)

Lagrangian Deformation



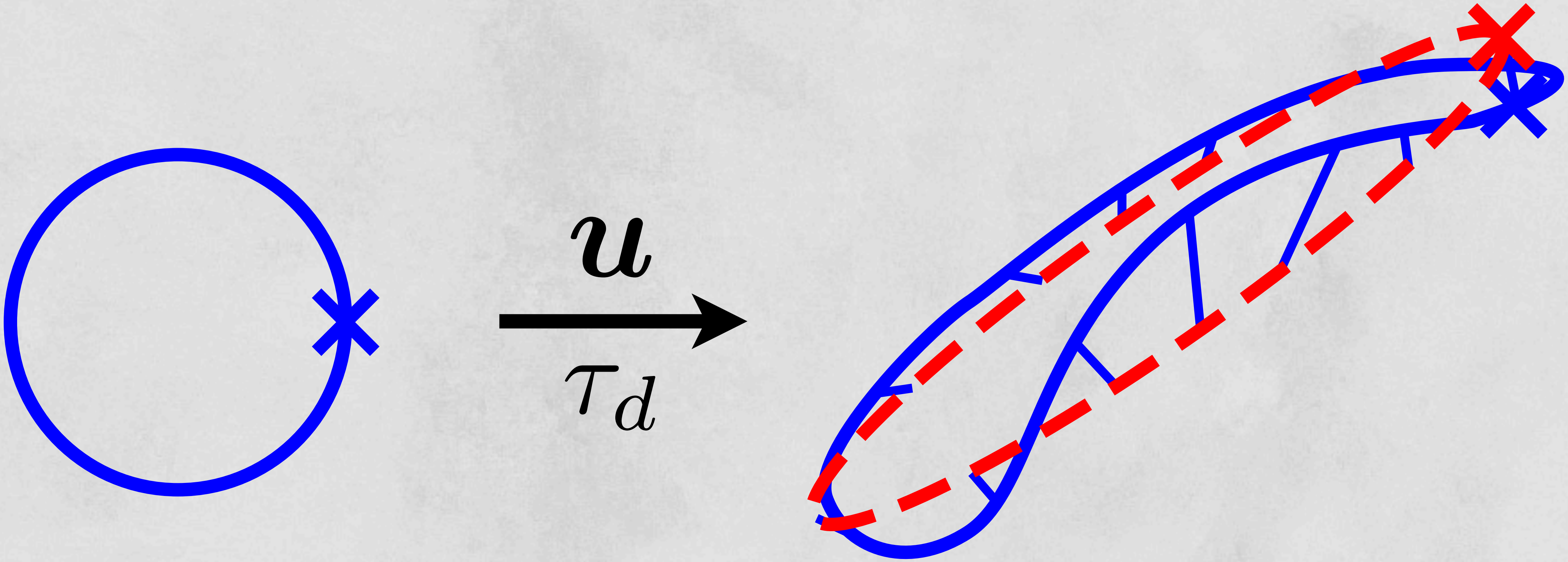
affine transformation:
translation, rotation,
shear, dilation,
compression

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}_0$$

linear deformation A^2

M.L. Falk & J.S. Langer, Phys. Rev. E. (1998)

Lagrangian Deformation



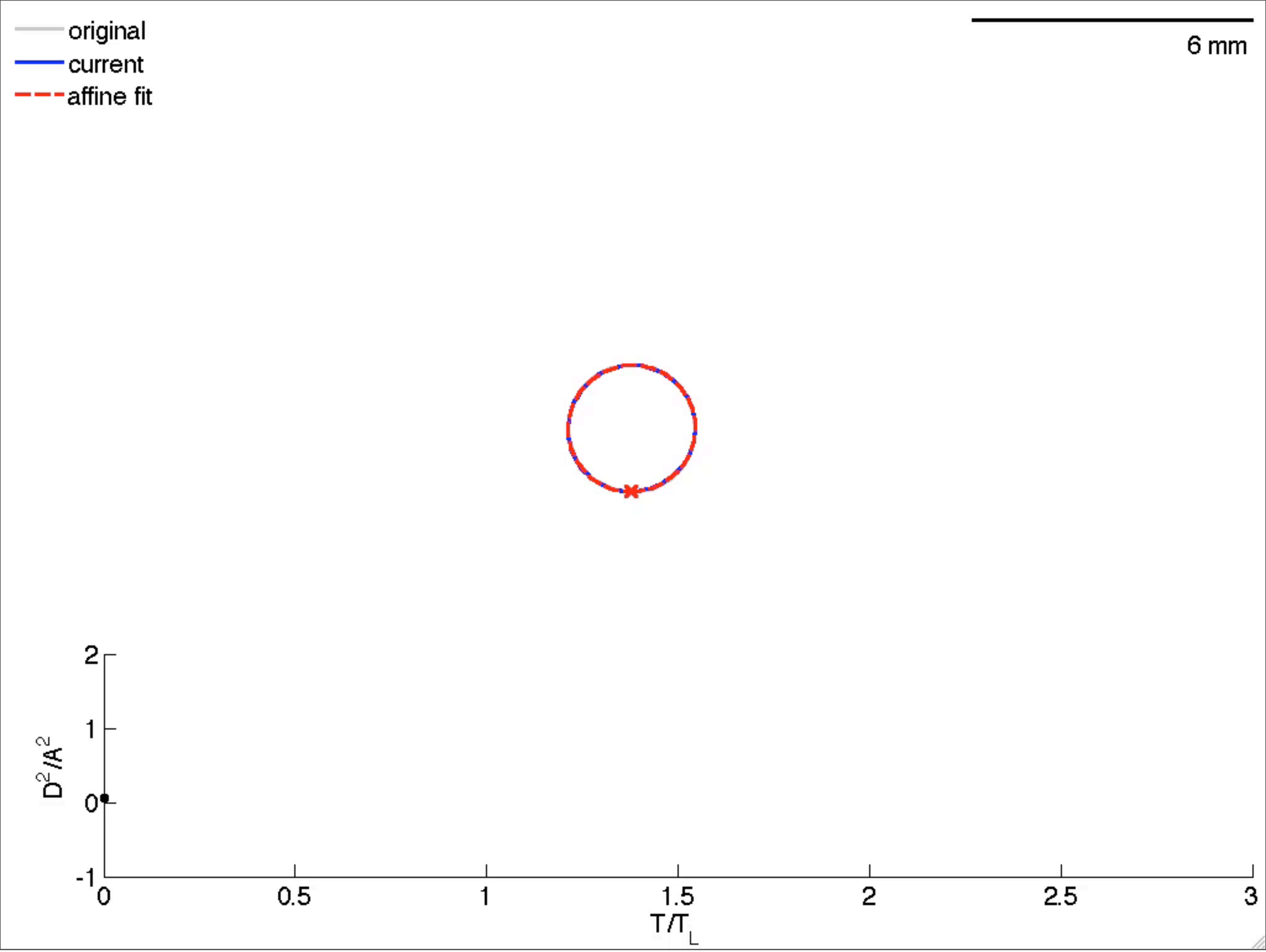
affine transformation:
translation, rotation,
shear, dilation,
compression

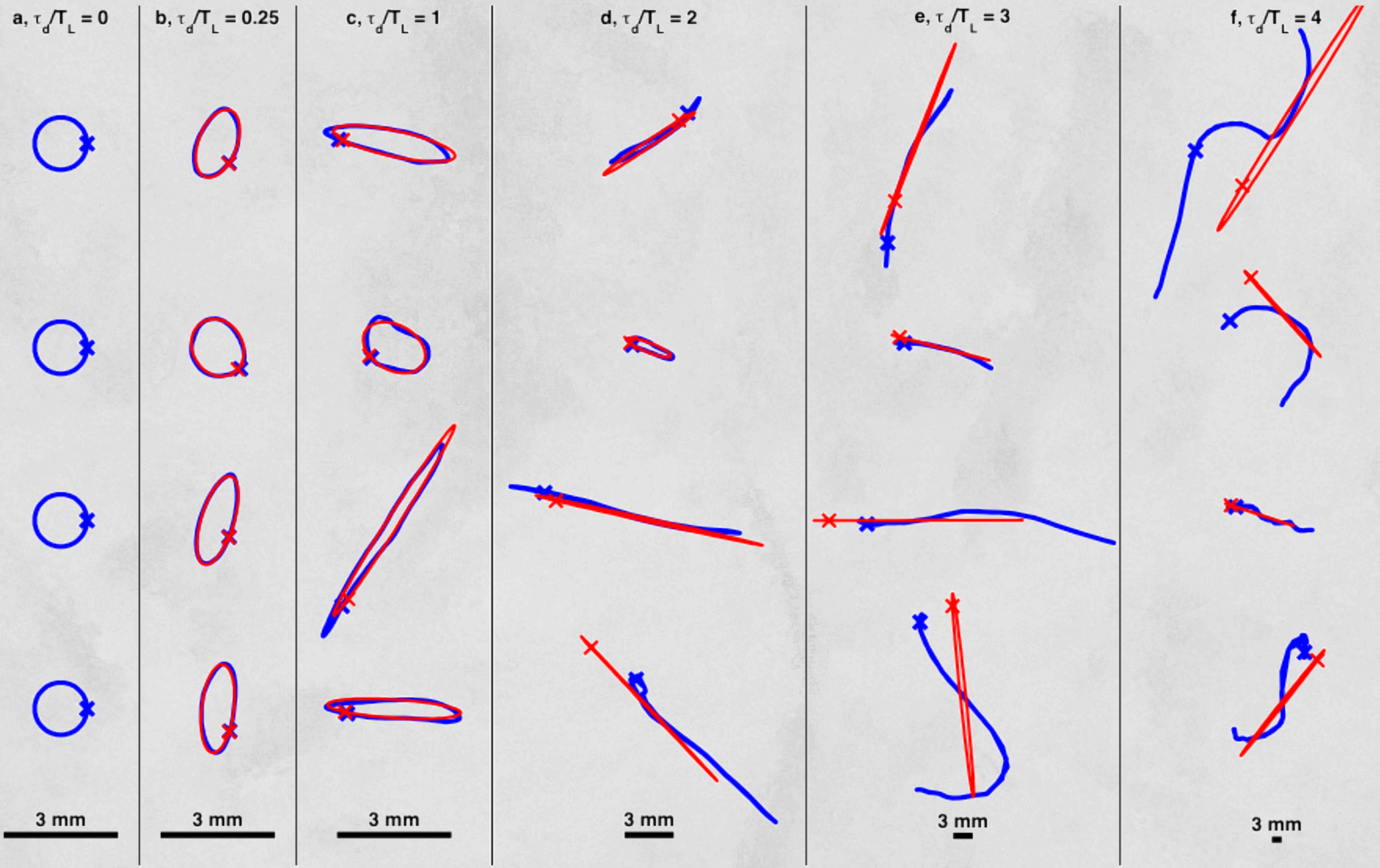
$$x = \boldsymbol{A} x_0$$

linear deformation A^2

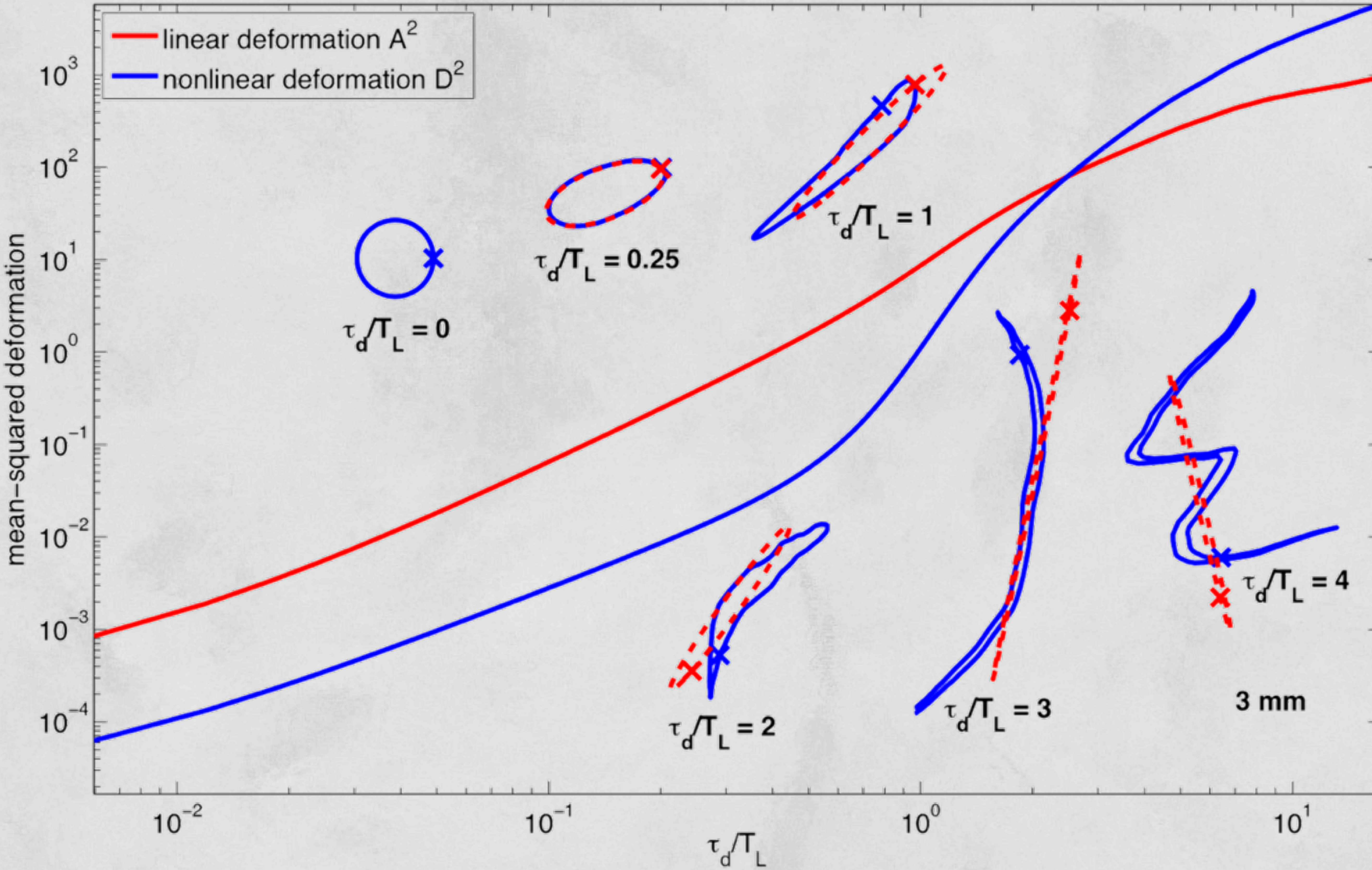
nonlinear deformation D^2

M.L. Falk & J.S. Langer, Phys. Rev. E. (1998)

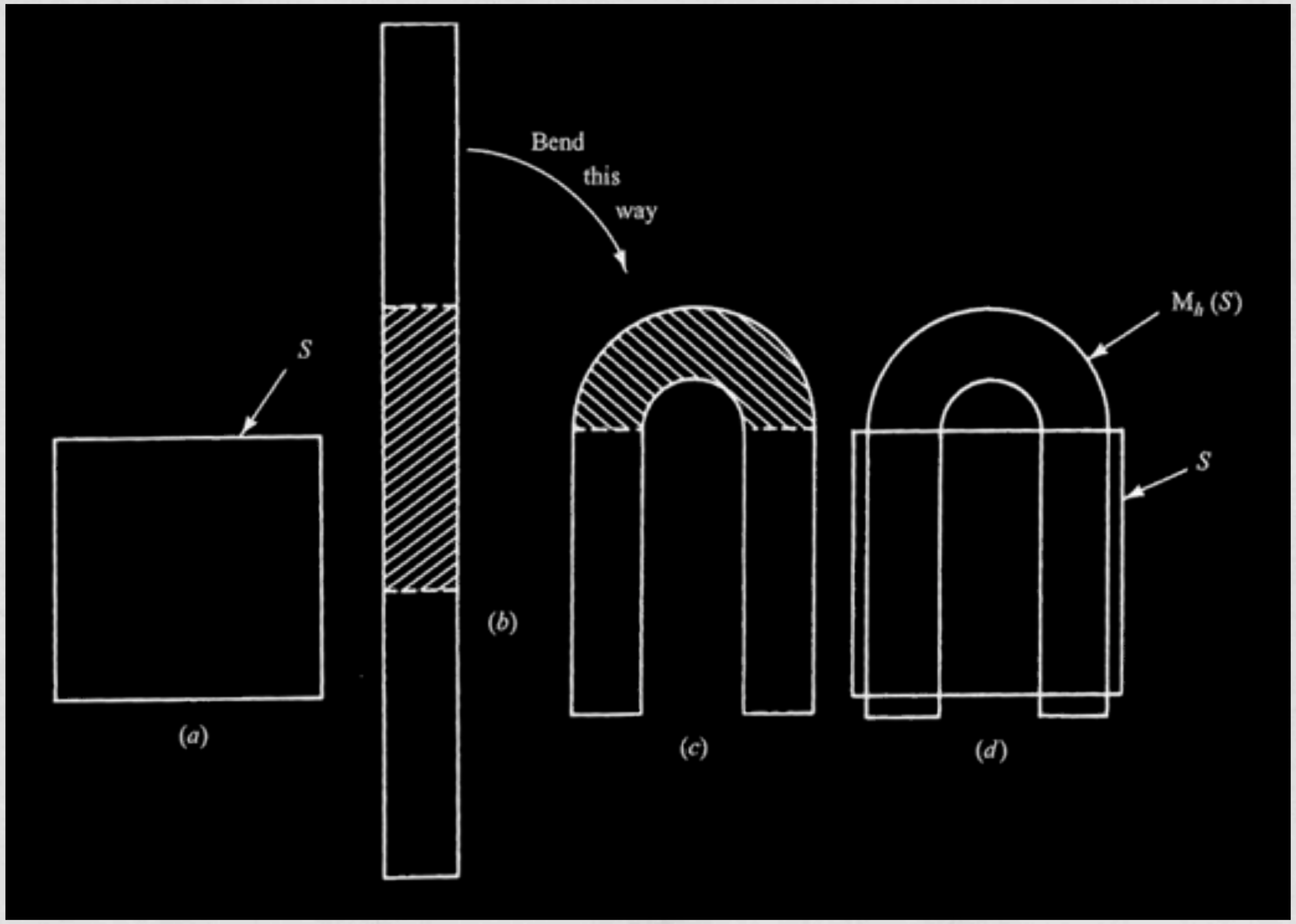

original
current
affine fit
6 mm
2
1
0
-1
D^2/A^2
0
0.5
1
1.5
2
2.5
3
T/T_L

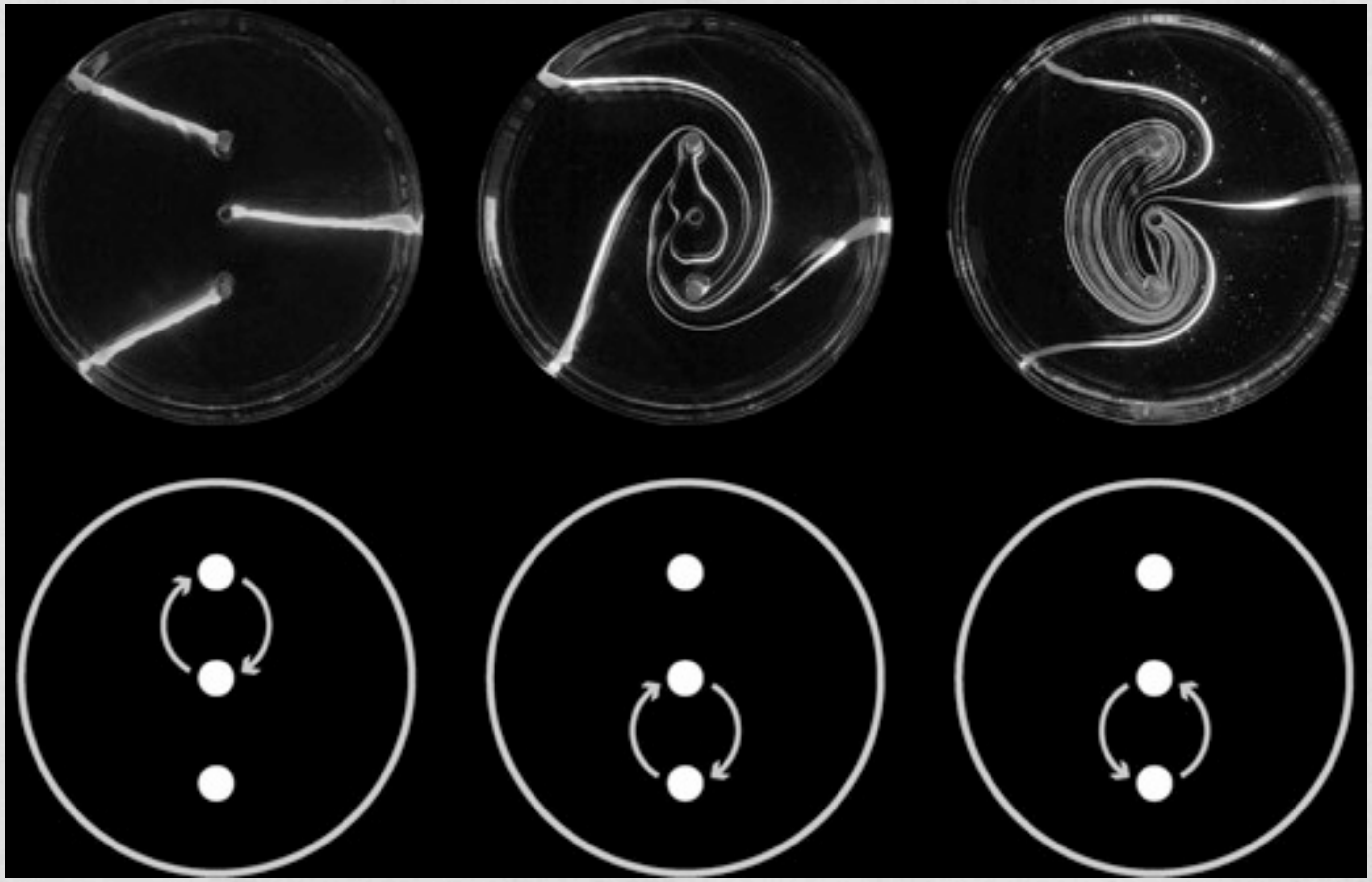


a, τd/TL = 0
b, τd/TL = 0.25
c, τd/TL = 1
d, τd/TL = 2
e, τd/TL = 3
f, τd/TL = 4
3 mm
3 mm
3 mm
3 mm
3 mm
3 mm

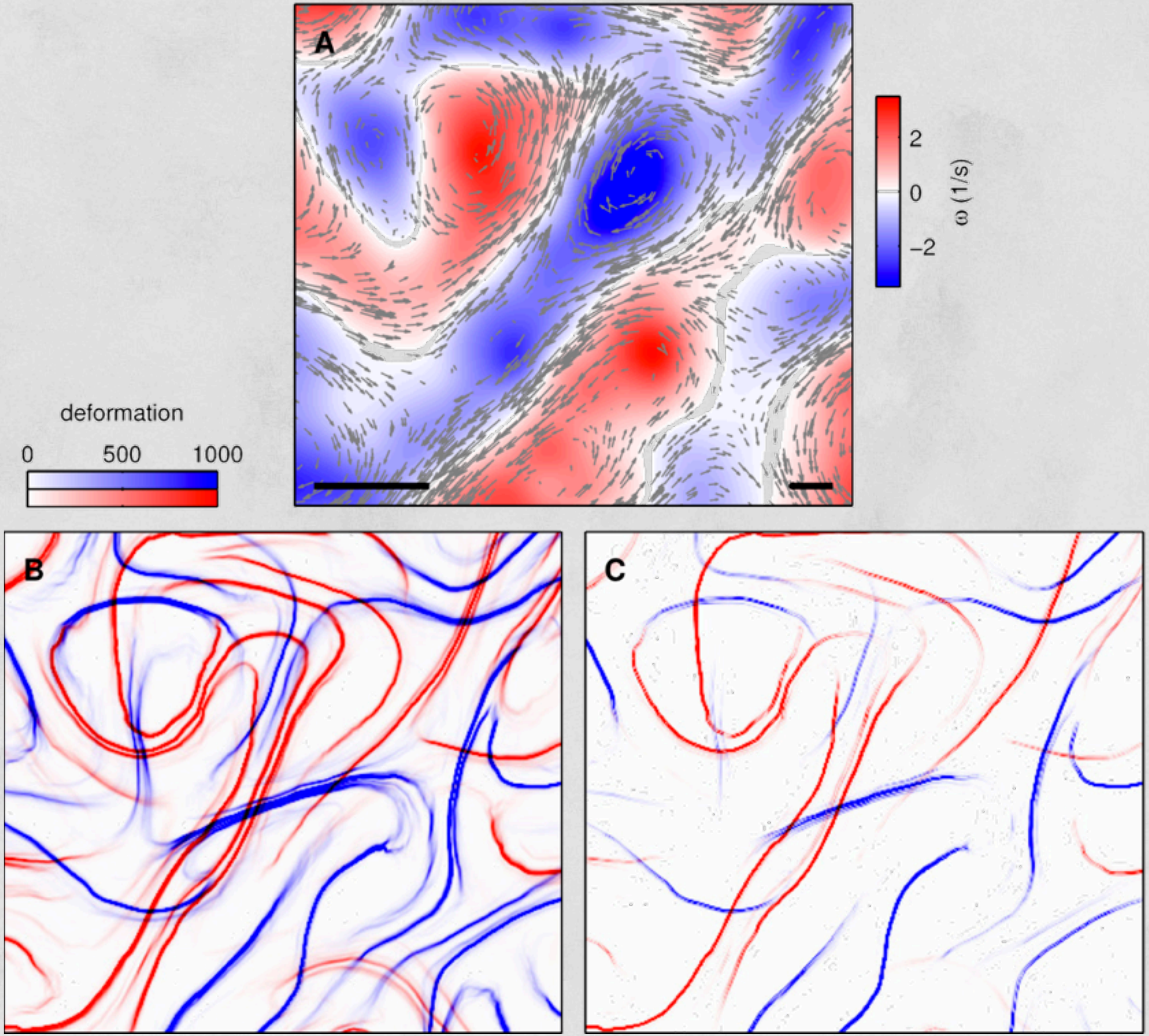


linear deformation A^2
nonlinear deformation D^2
$\tau_d/T_L = 0$
$\tau_d/T_L = 0.25$
$\tau_d/T_L = 1$
$\tau_d/T_L = 2$
$\tau_d/T_L = 3$
$\tau_d/T_L = 4$
3 mm
mean-squared deformation
τ_d/T_L
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{-2}
10^{-1}
10^{0}
10^{1}



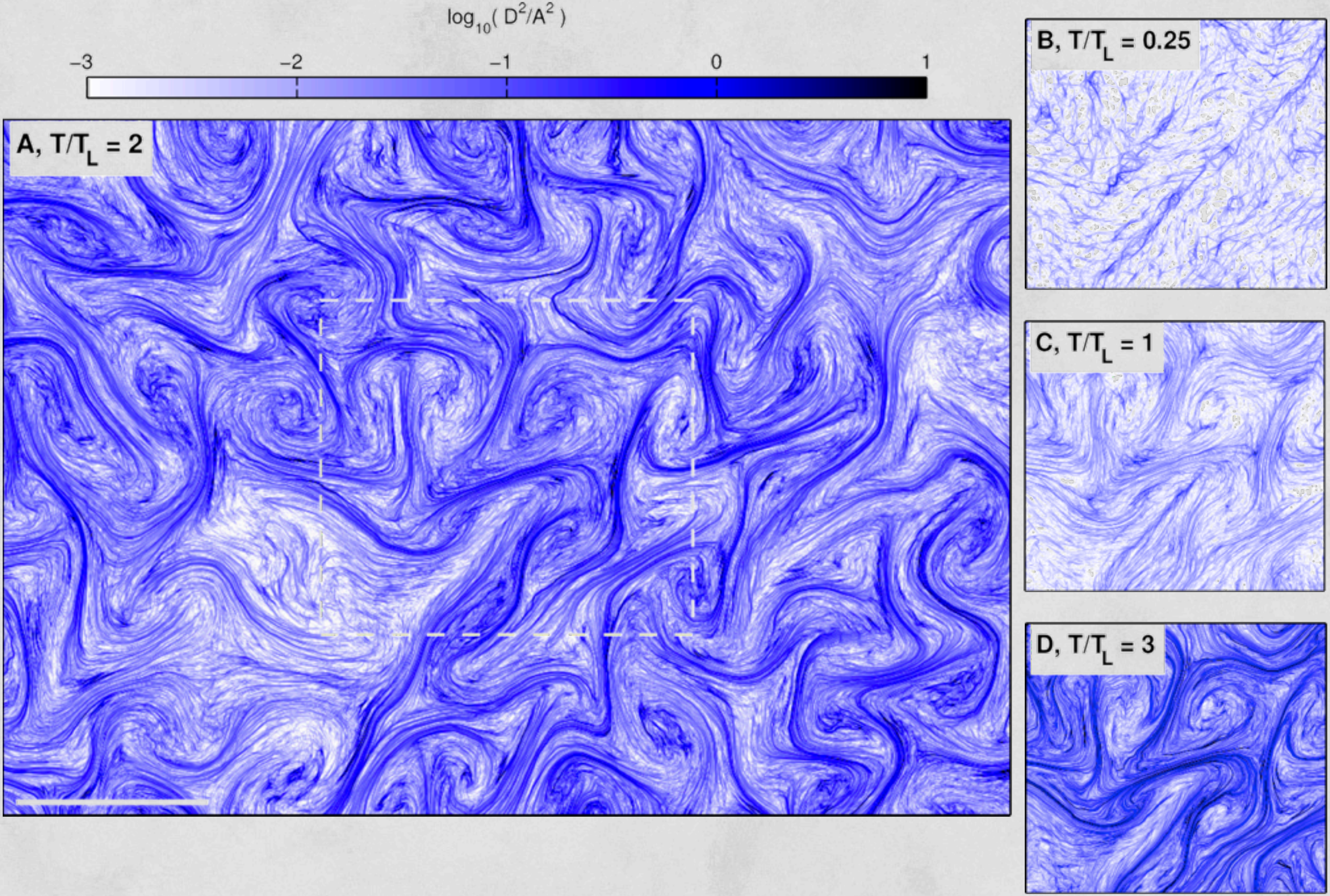


O. Reynolds, Nature (1894)
S. Smale, Bull. Am. Math. Soc. (1967)
E. Ott, Chaos in Dynamical Systems (2002)
P.L. Boyland, H. Aref, & M.A. Stremler, J. Fluid Mech. (2000)
M.A. Stremler & J. Chen, Phys. Fluids. (2007)

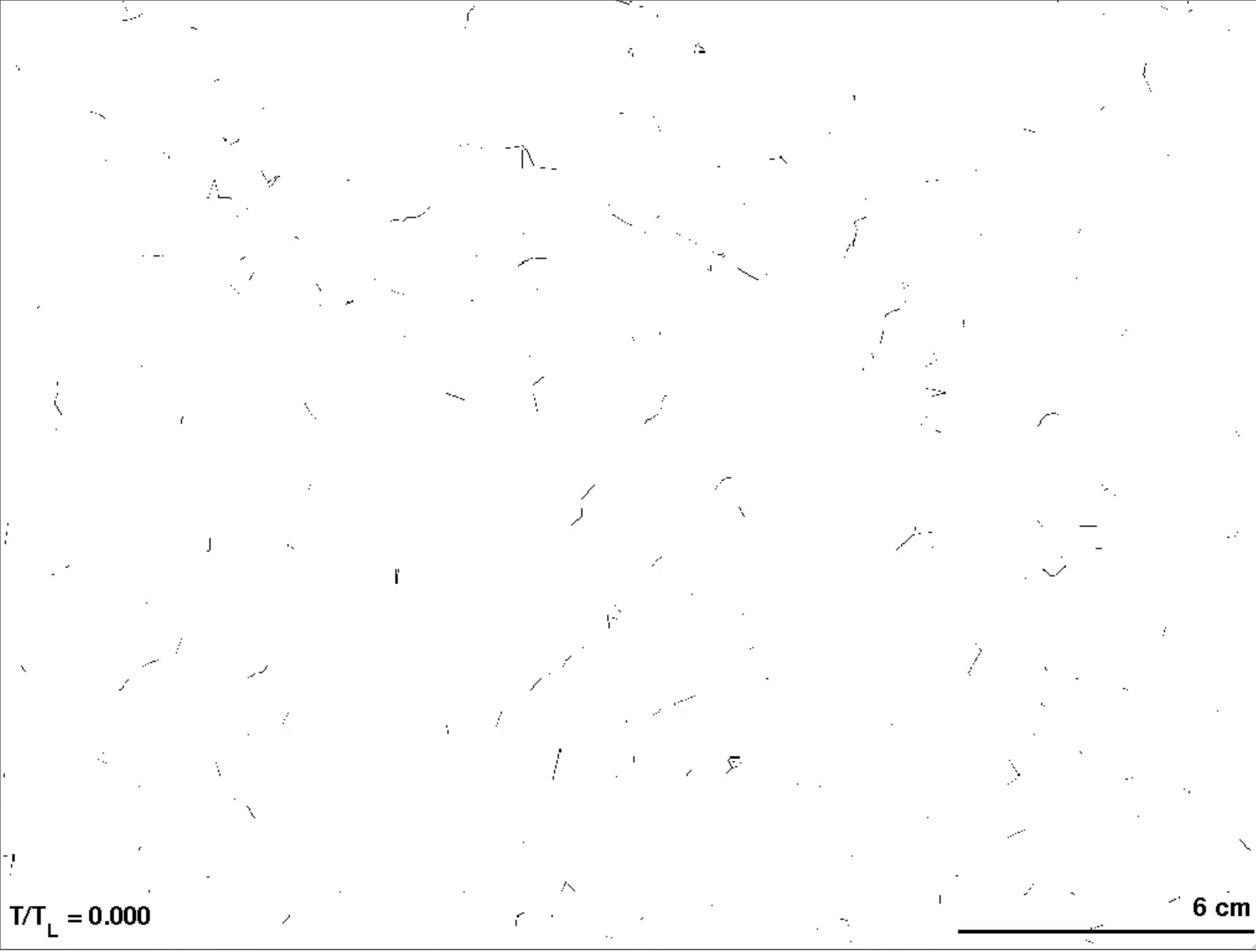


A^2

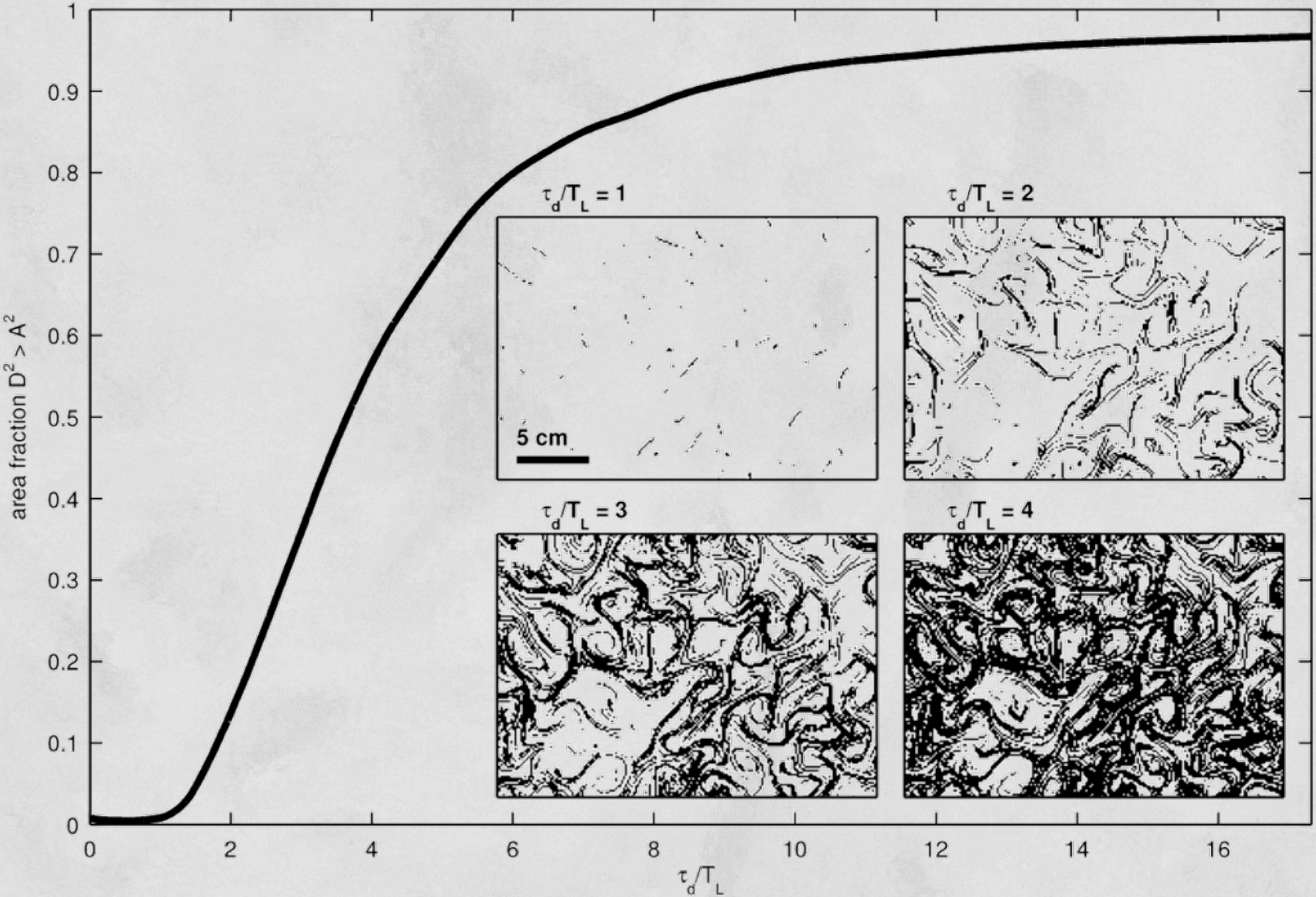
D^2



$\log_{10}(D^2/A^2)$
−3
−2
−1
0
1
A, T/T_L = 2
B, T/T_L = 0.25
C, T/T_L = 1
D, T/T_L = 3



T/T_L = 0.000
6 cm



area fraction $D^2 > A^2$
τ_d/T_L
$\tau_d/T_L = 1$
$\tau_d/T_L = 2$
$\tau_d/T_L = 3$
$\tau_d/T_L = 4$
5 cm
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
4
6
8
10
12
14
16

Lagrangian Nonlinearity:

Material areas deform nonlinearly at long times

Lagrangian nonlinearity is initially localized;
later grows to fill space

Stretching and folding occur on different time scales

Connections with Eulerian nonlinearity?

D. H. Kelley & NTO, Nature Phys. (2011)

Eulerian Nonlinearity

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u}$$

Eulerian Nonlinearity

$$\frac{\partial \boldsymbol{u}}{\partial t} + \underline{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u}$$

Triad Interactions:

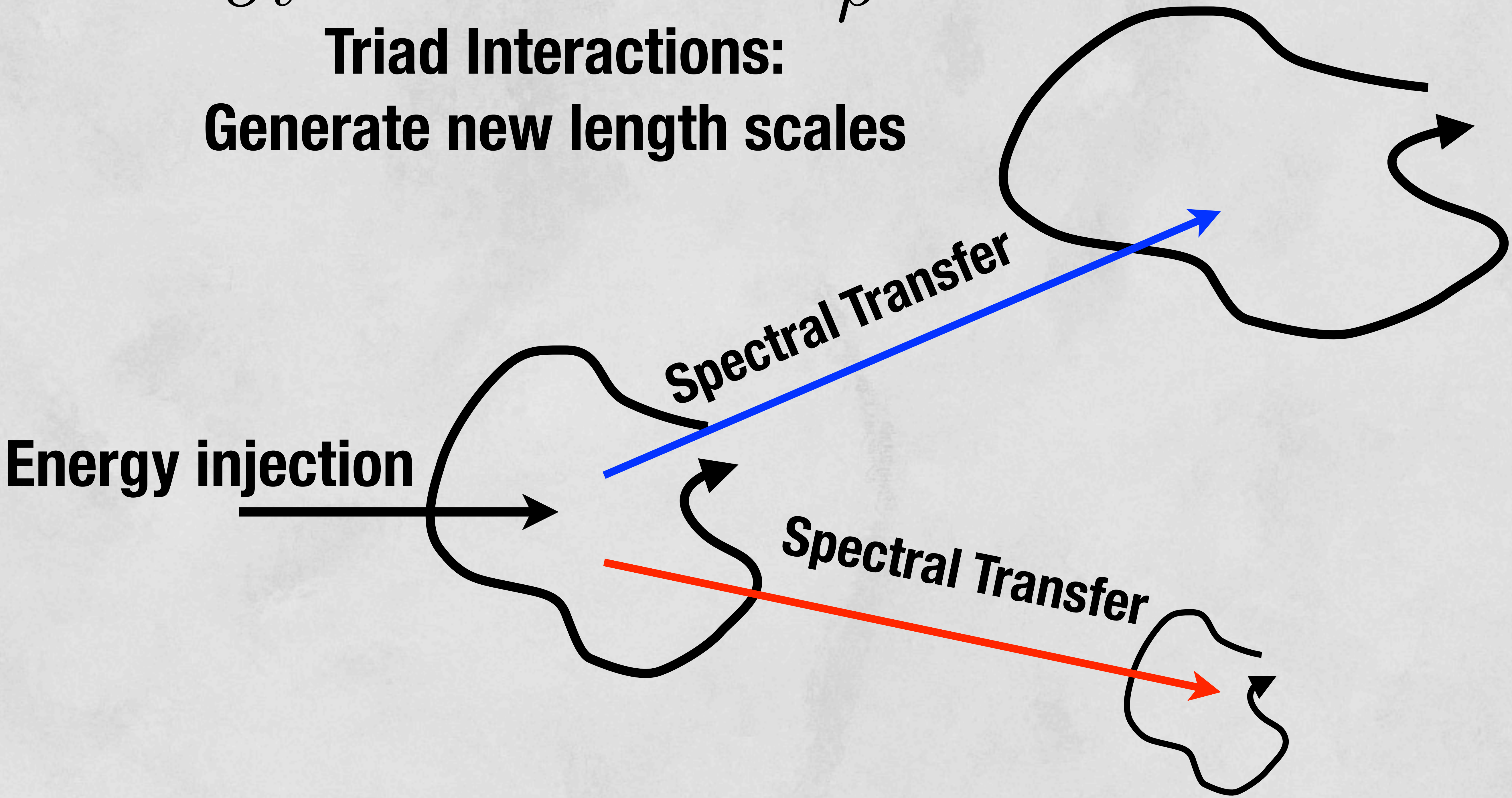
Generate new length scales

Eulerian Nonlinearity

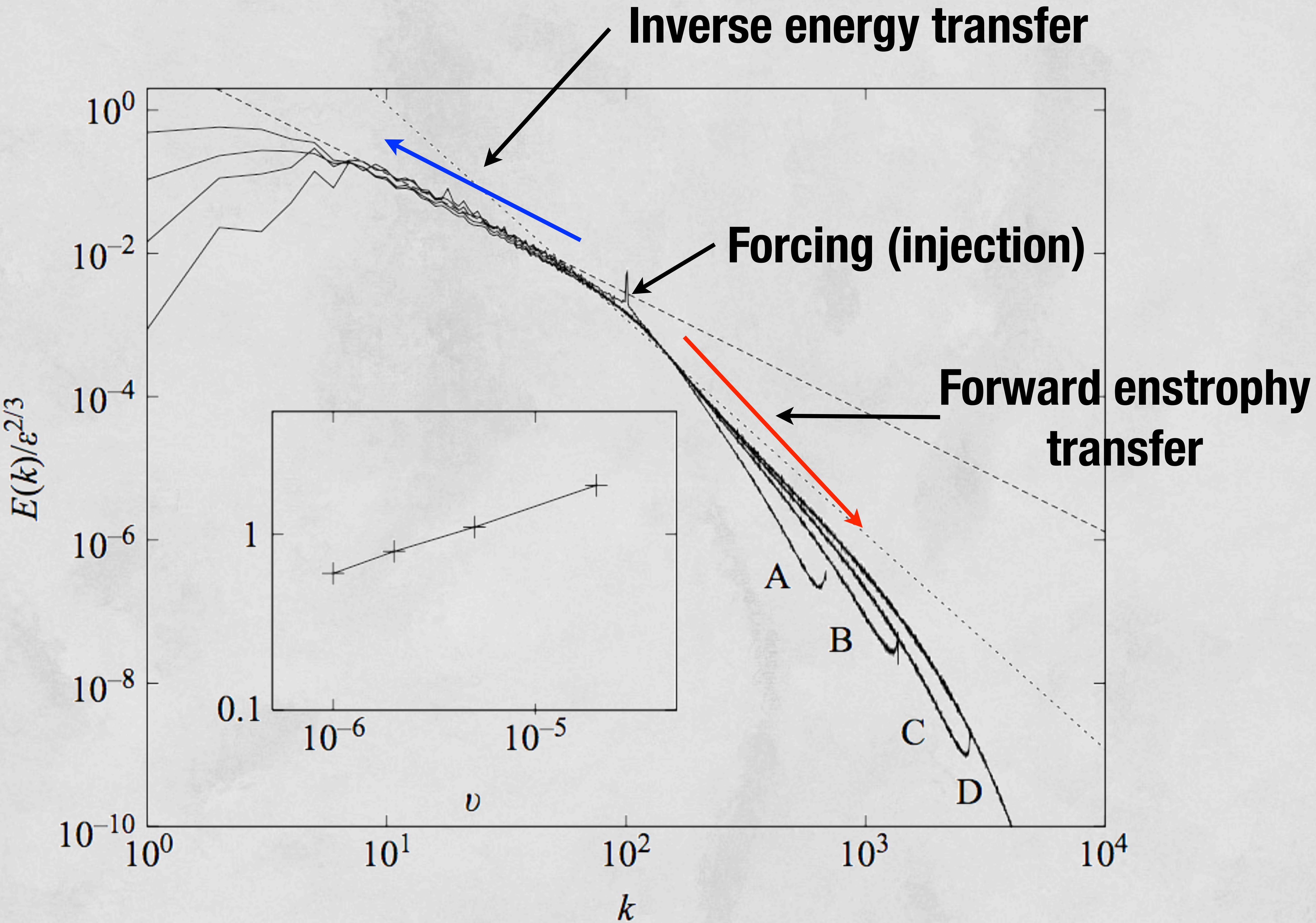
$$\frac{\partial \boldsymbol{u}}{\partial t} + \underline{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u}$$

Triad Interactions:

Generate new length scales



2D Turbulence



G. Boffetta, J. Fluid Mech. (2007)

Calculating spectral flux:

1. **Convolve velocity field with spectral low-pass filter:**

$$\boldsymbol{u}^{(r)} = \int G^{(r)}(\boldsymbol{x} - \boldsymbol{x}')\boldsymbol{u}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}'$$

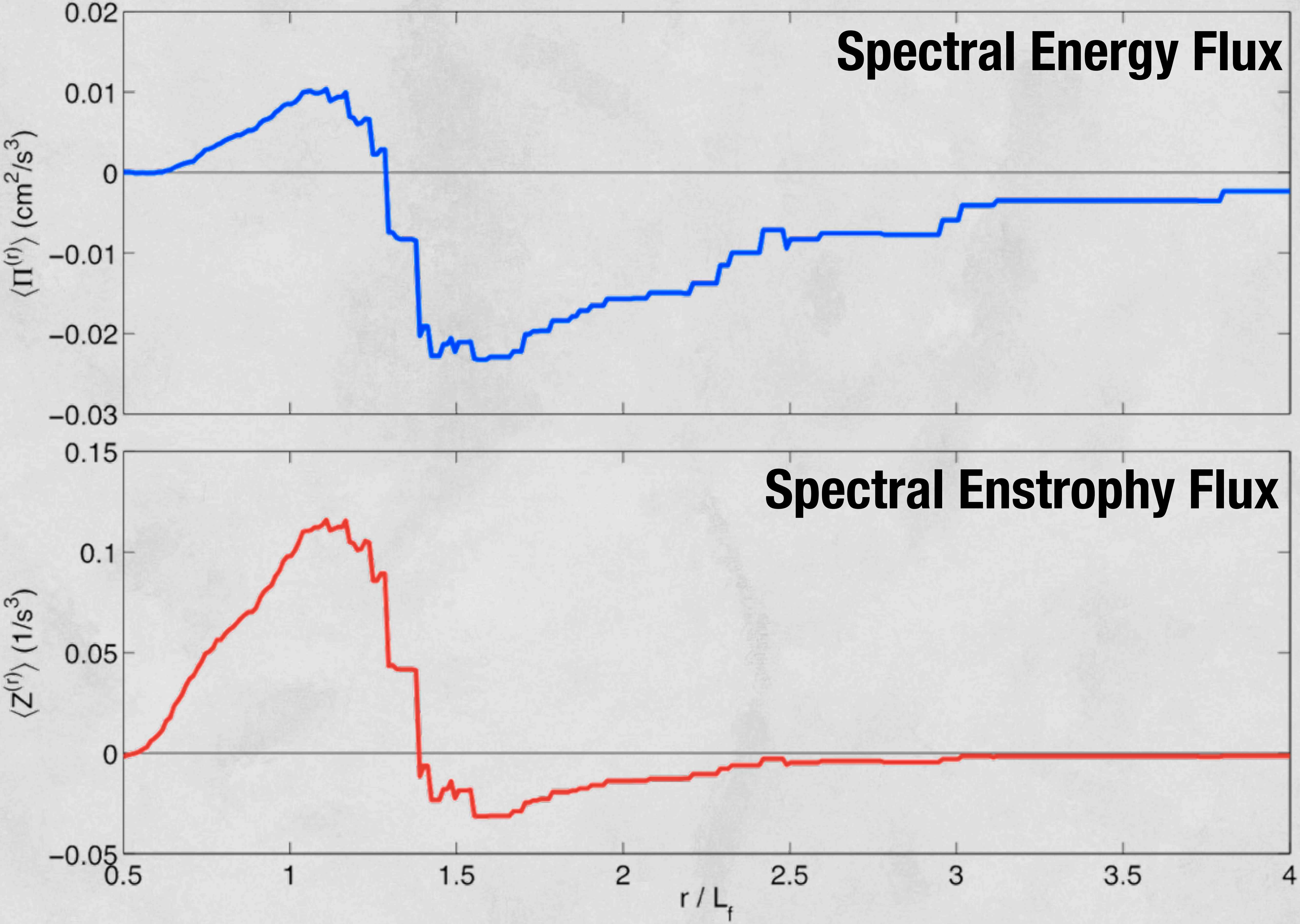
2. **Construct filtered strain rate, subgrid stress:**

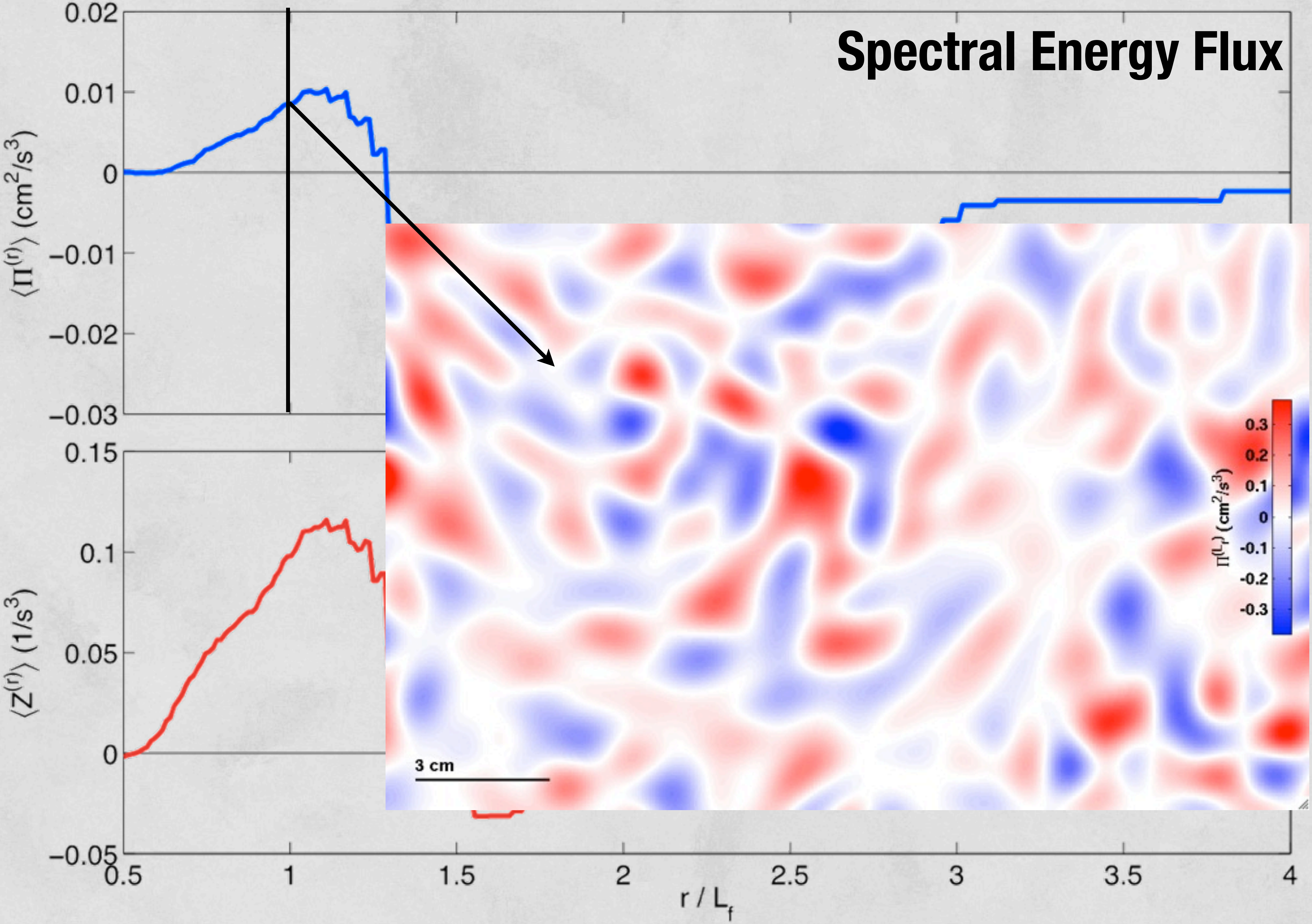
$$s_{ij}^{(r)} = \frac{1}{2}\left(\frac{\partial u_i^{(r)}}{\partial x_j} + \frac{\partial u_j^{(r)}}{\partial x_i}\right) \qquad \tau_{ij}^{(r)} = (u_i u_j)^{(r)} - u_i^{(r)} u_j^{(r)}$$

3. **Compute spectral flux:**

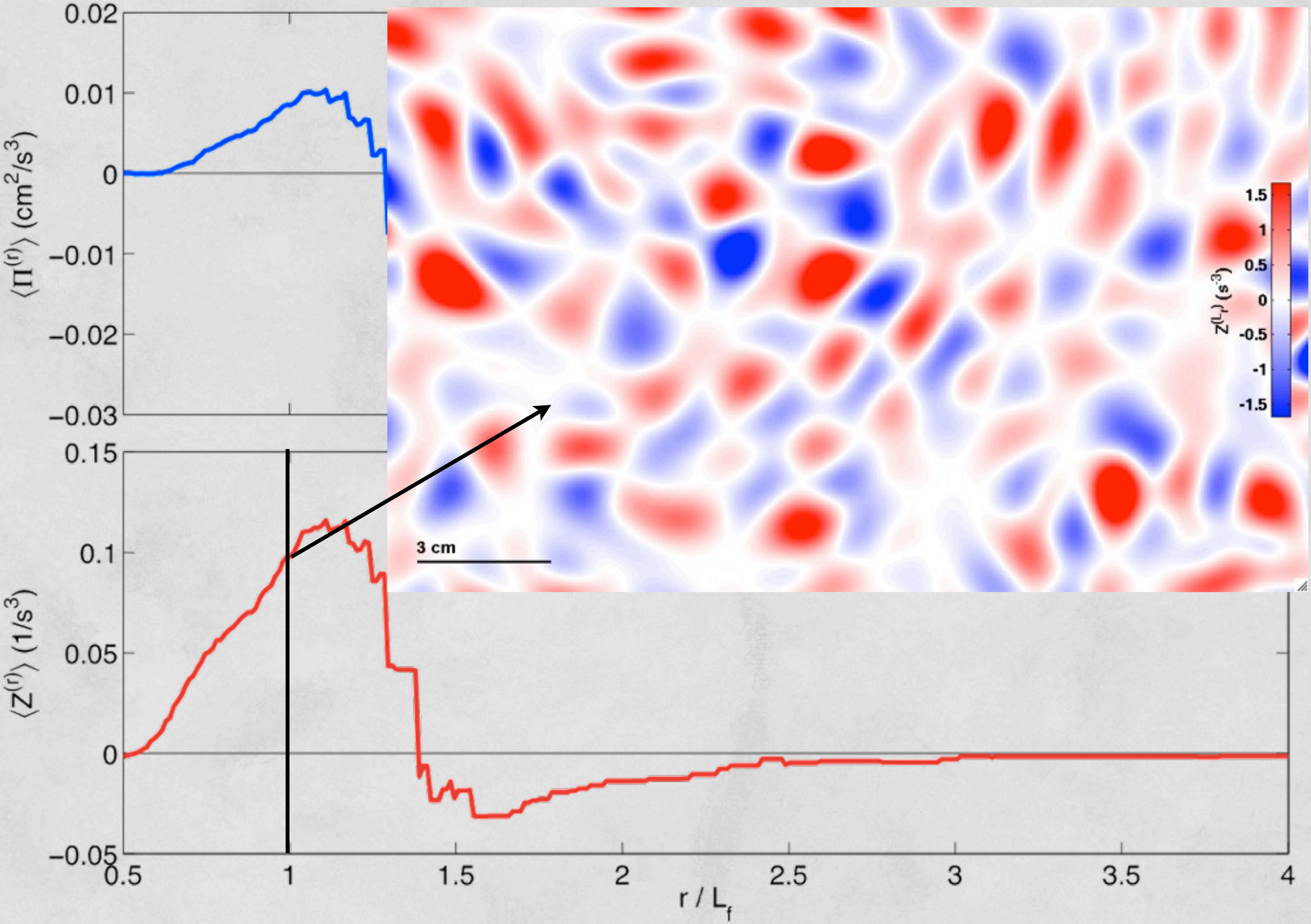
$$\Pi^{(r)} = -\tau_{ij}^{(r)} s_{ij}^{(r)}$$

M.K. Rivera et al., Phys. Rev. Lett. (2003)


Spectral Energy Flux
Spectral Enstrophy Flux
0.02
0.01
0
−0.01
−0.02
−0.03
$\langle \Pi^{(r)} \rangle$ (cm^2/s^3)
0.15
0.1
0.05
0
−0.05
$\langle Z^{(r)} \rangle$ ($1/s^3$)
0.5
1
1.5
2
2.5
3
3.5
4
r / L_f

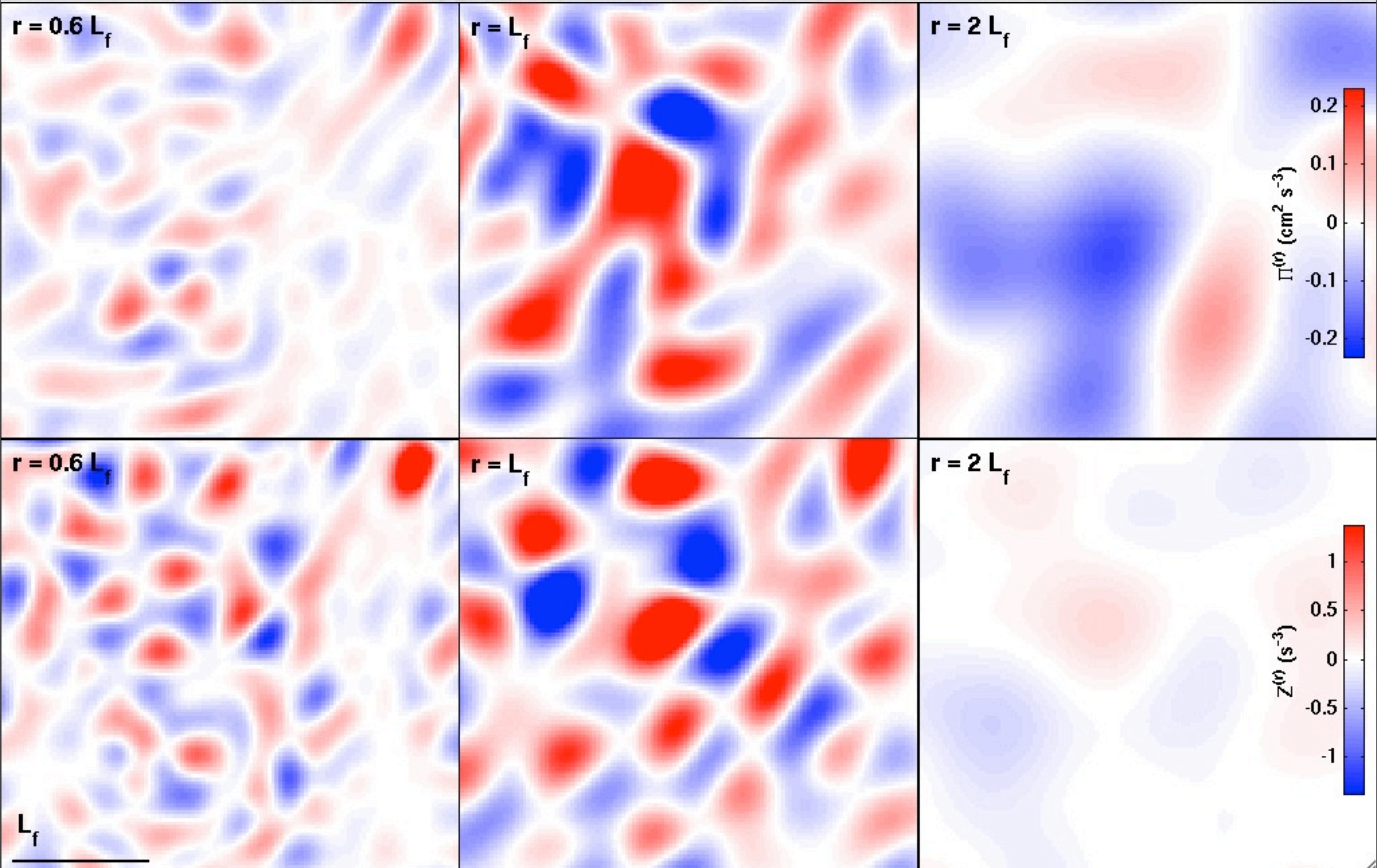


Spectral Energy Flux
$\langle \Pi^{(r)} \rangle$ (cm^2/s^3)
0.02
0.01
0
−0.01
−0.02
−0.03
$\langle Z^{(r)} \rangle$ (1/s^3)
0.15
0.1
0.05
0
−0.05
0.5
1
1.5
2
2.5
3
3.5
4
r / L_f
3 cm
$\Pi^{(L_r)}$ (cm^2/s^3)
0.3
0.2
0.1
0
-0.1
-0.2
-0.3



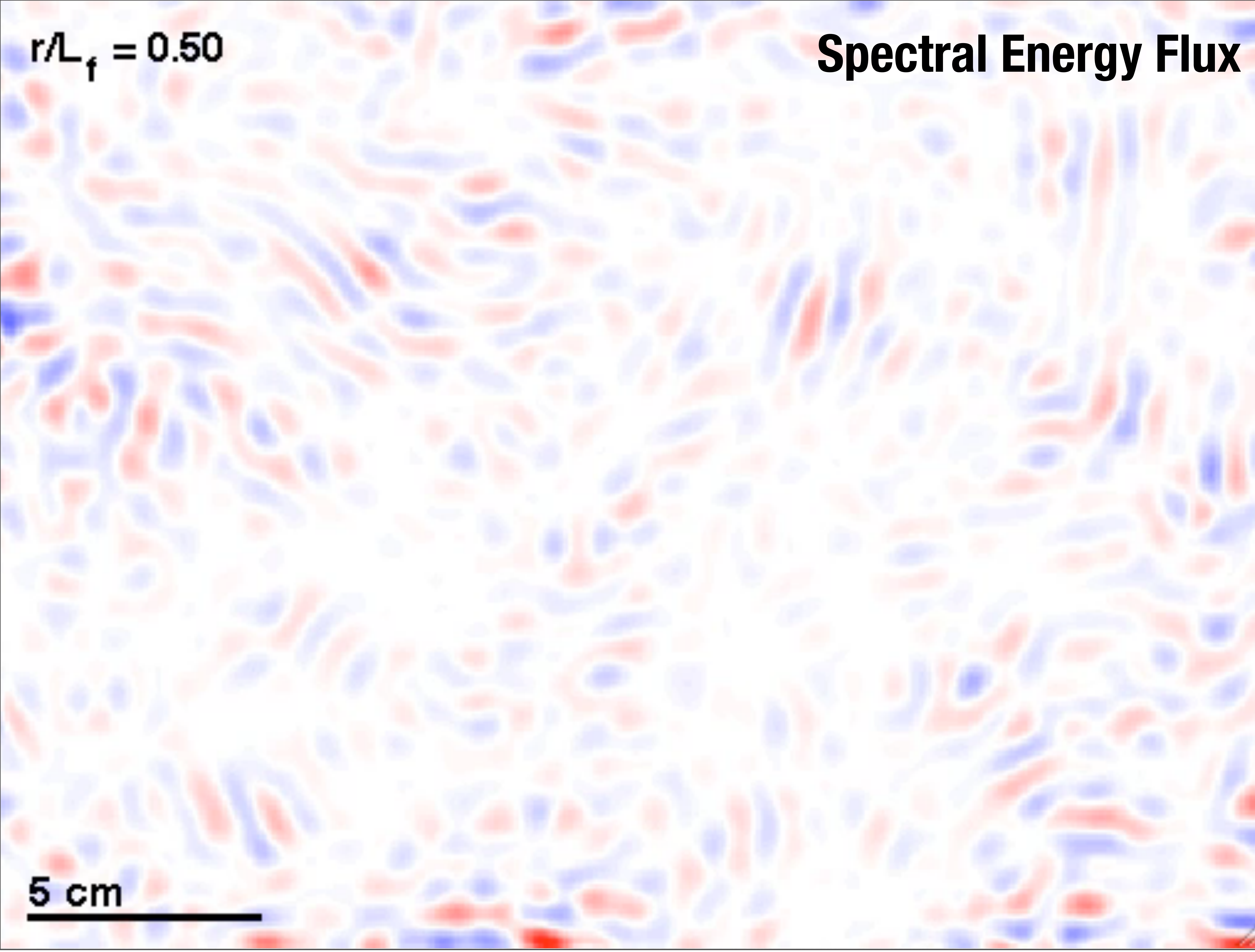
0.02
0.01
0
−0.01
−0.02
−0.03
⟨Π^(r)⟩ (cm^2/s^3)
0.15
0.1
0.05
0
−0.05
⟨Z^(r)⟩ (1/s^3)
0.5
1
1.5
2
2.5
3
3.5
4
r / L_f
3 cm
1.5
1
0.5
0
-0.5
-1
-1.5
Z^(L_f) (s^-3)

Energy →



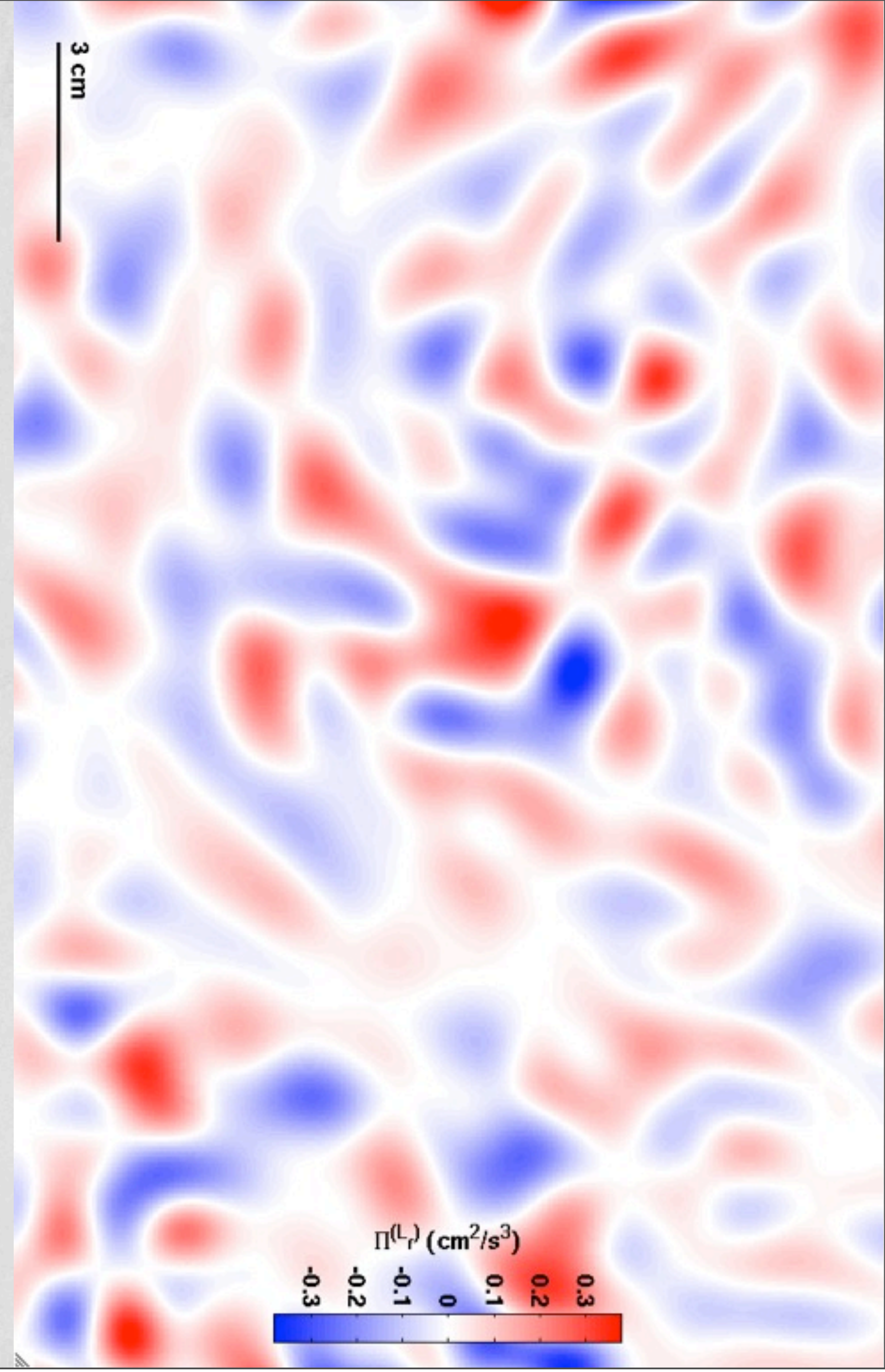
Enstrophy →

Spectral Energy Flux

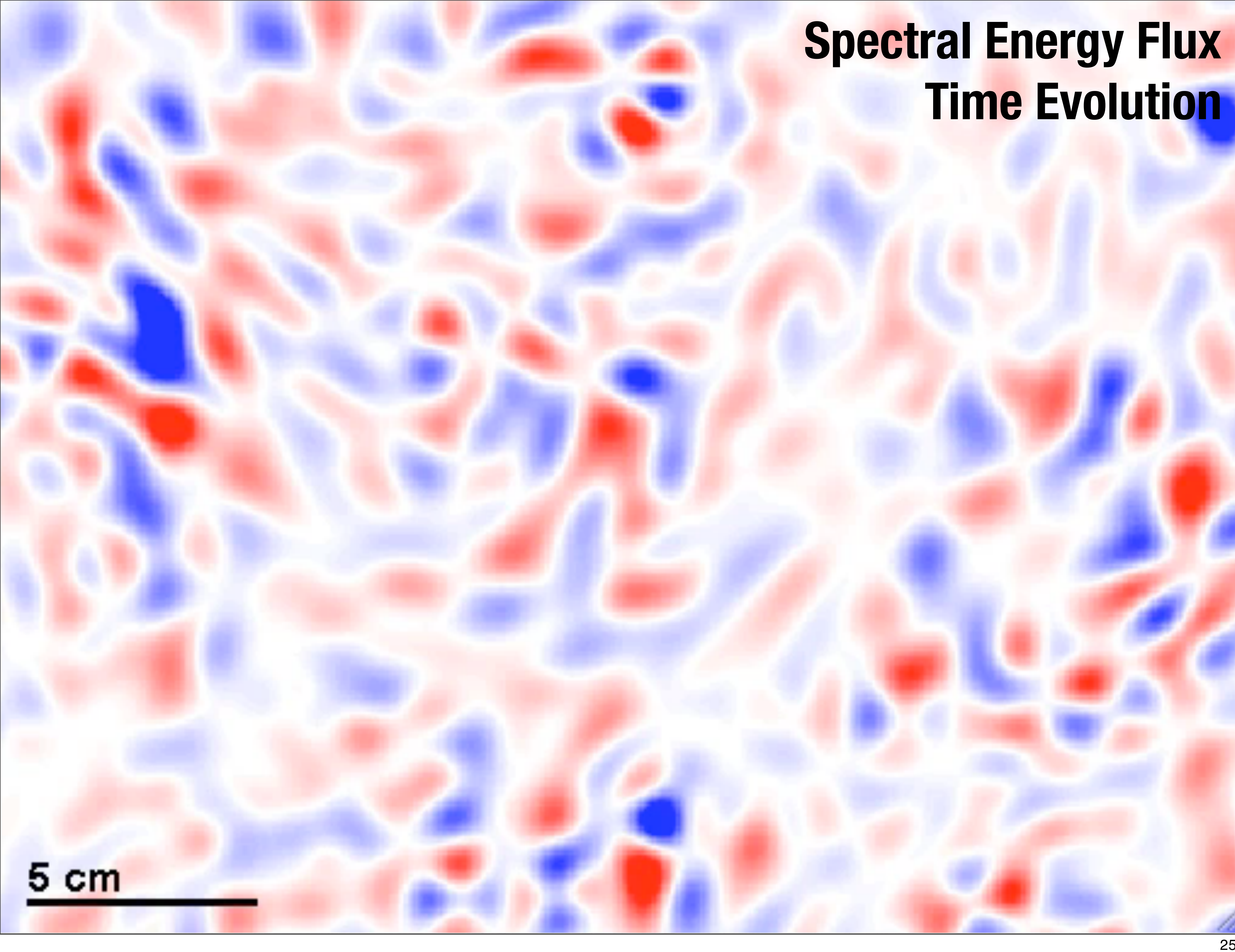

$r/L_f = 0.50$
Spectral Energy Flux
5 cm

Spectral transfer is not constant in time!

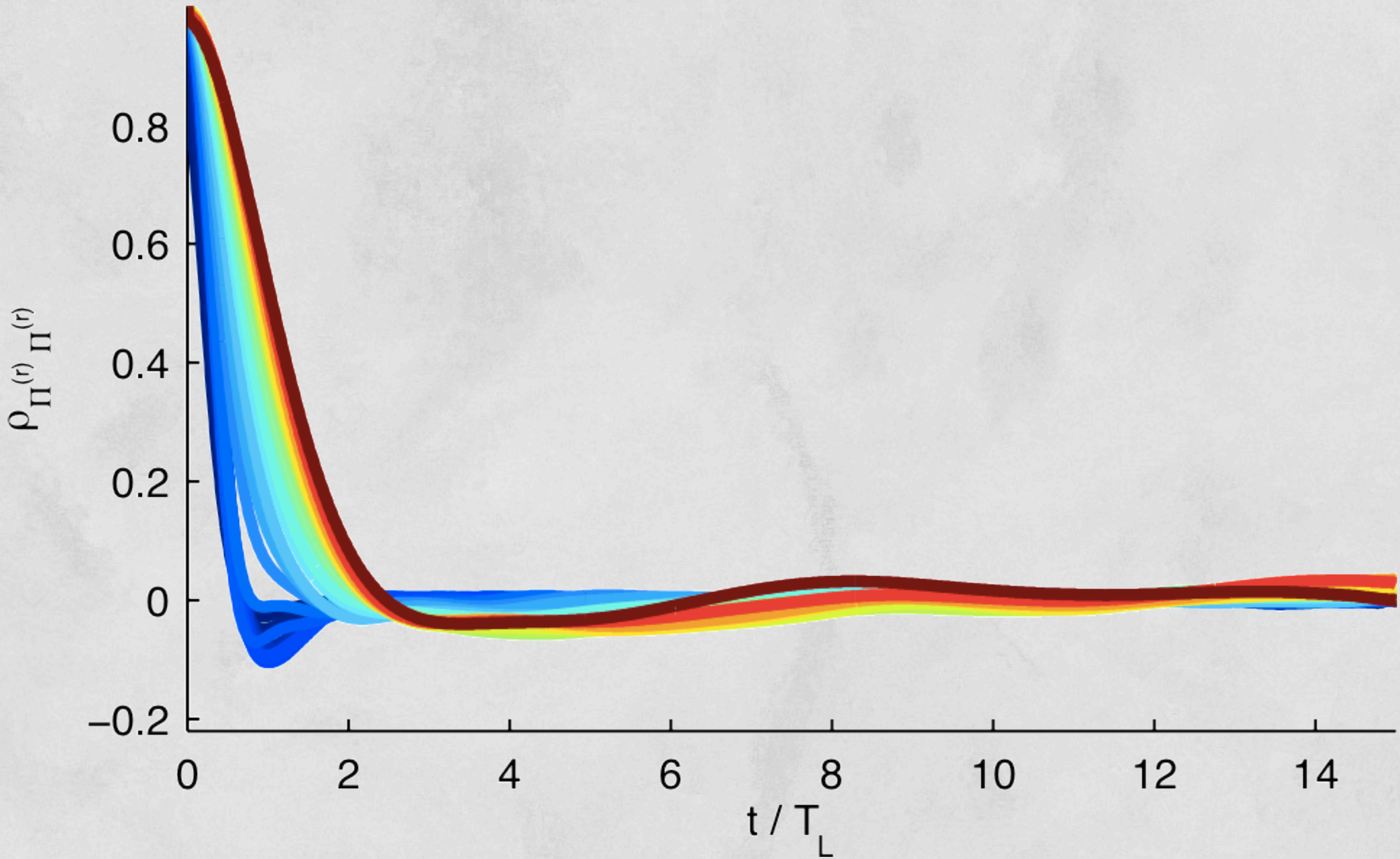
How does is change? What are its dynamics?



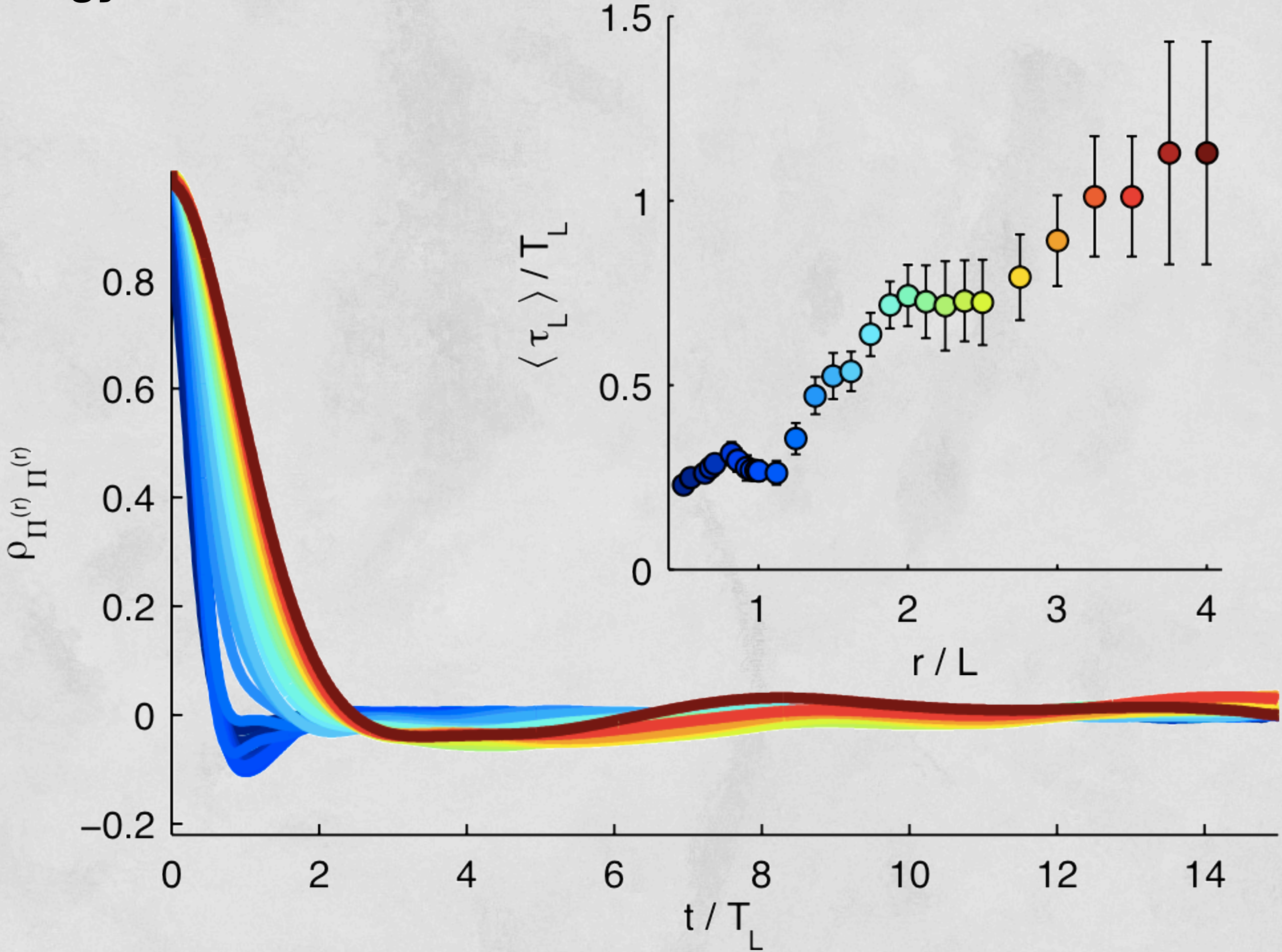
Spectral Energy Flux
Time Evolution


Spectral Energy Flux
Time Evolution
5 cm

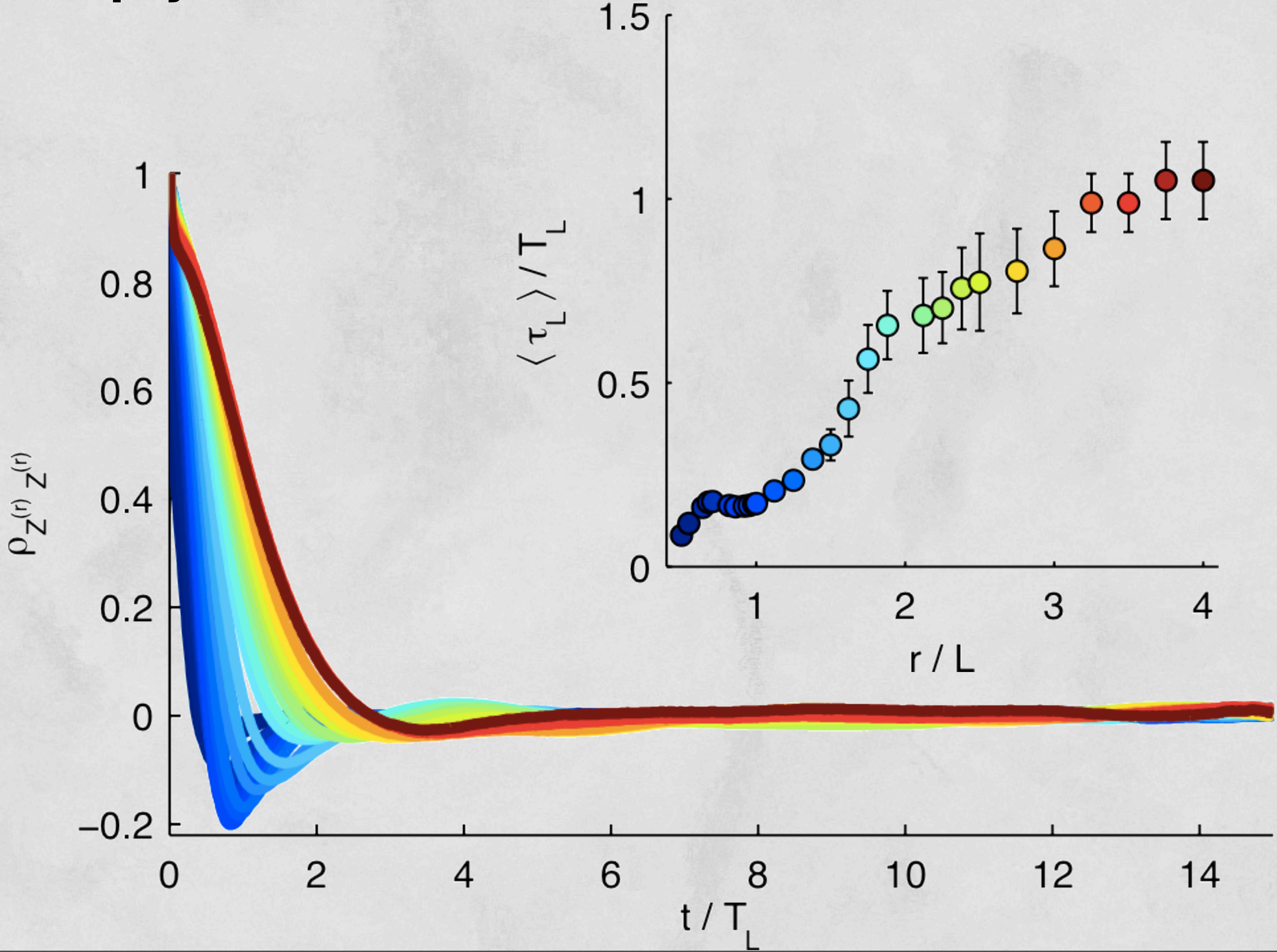
Energy Flux Correlations



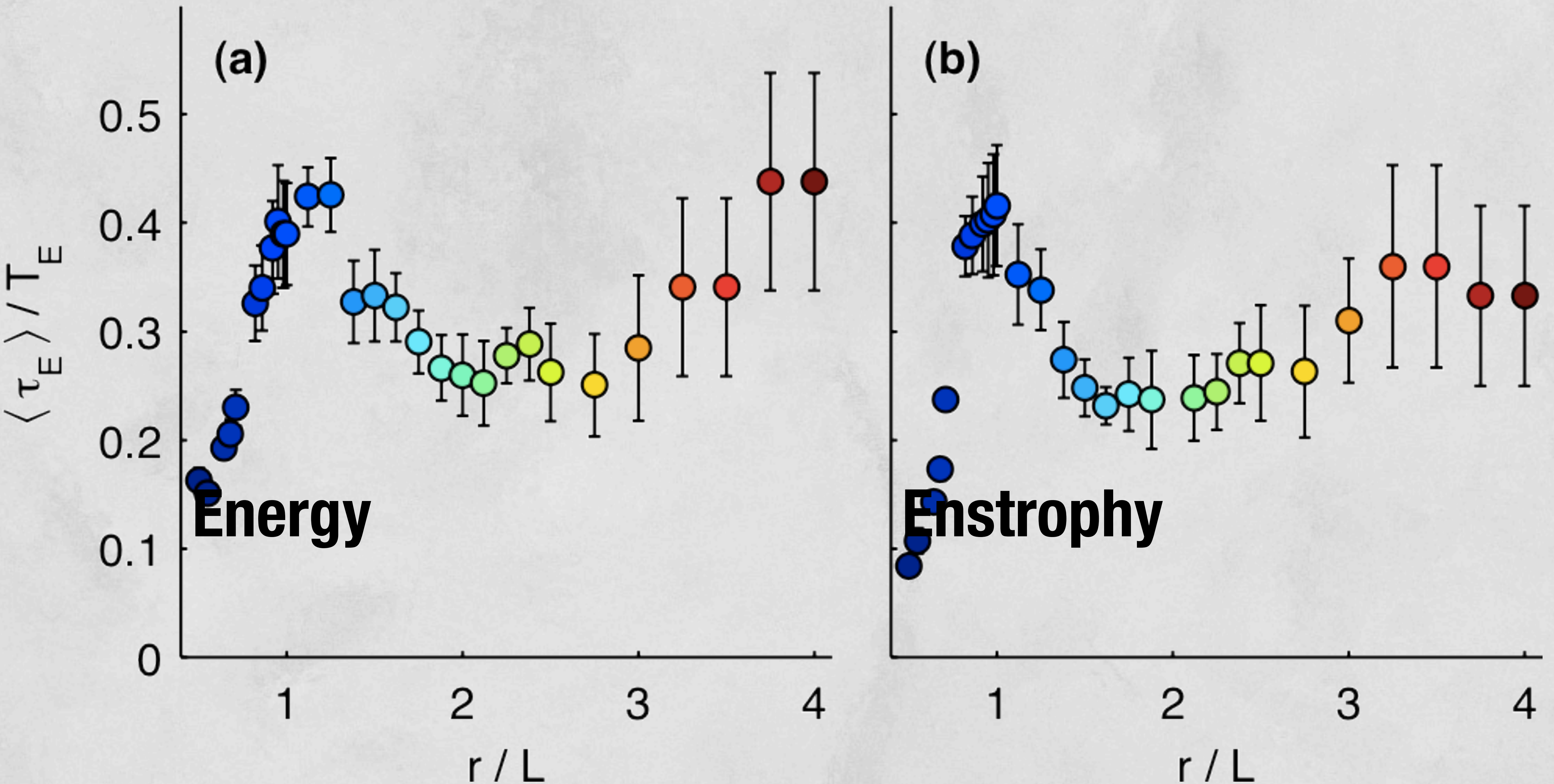
Energy Flux Correlations



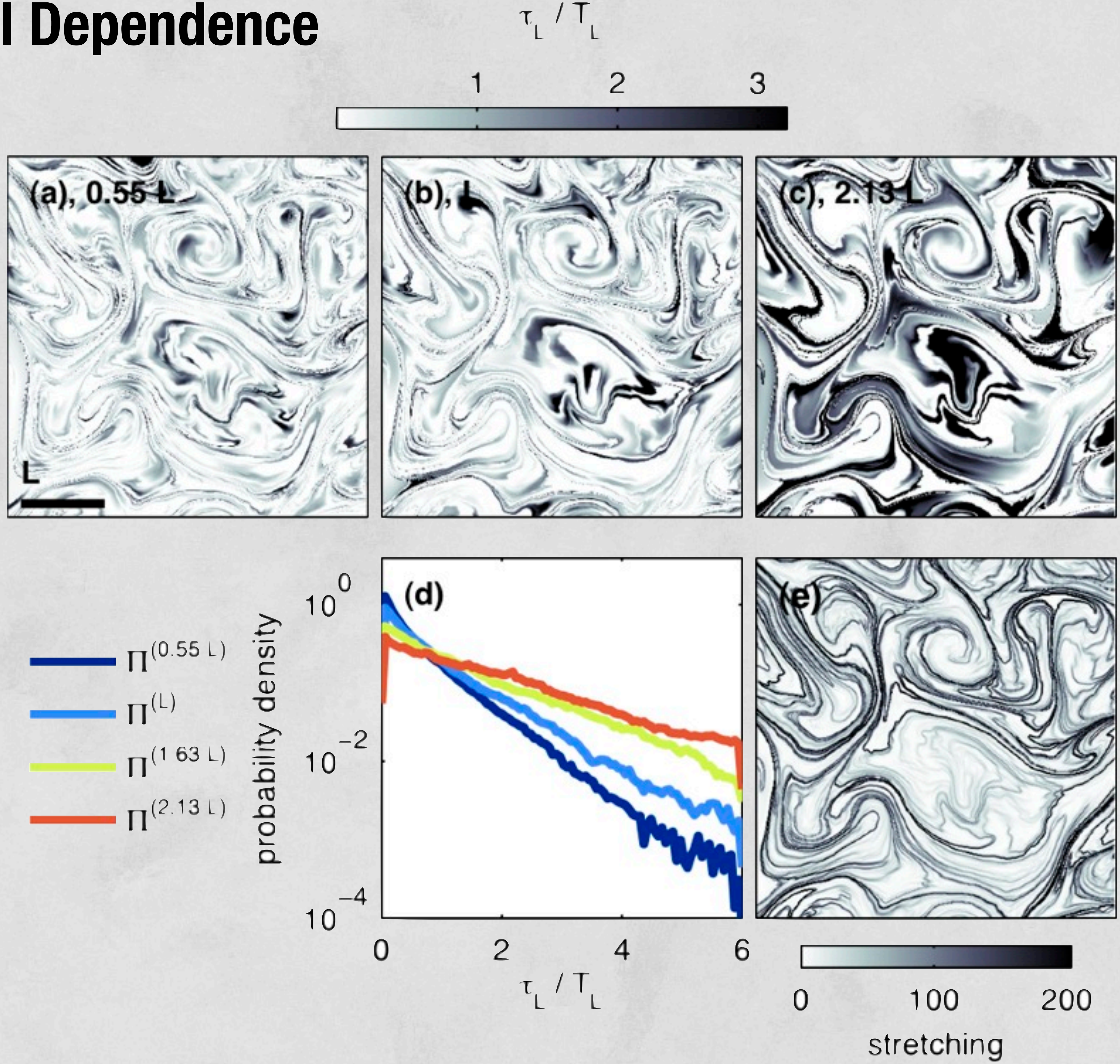
Enstrophy Flux Correlations



Eulerian Correlations

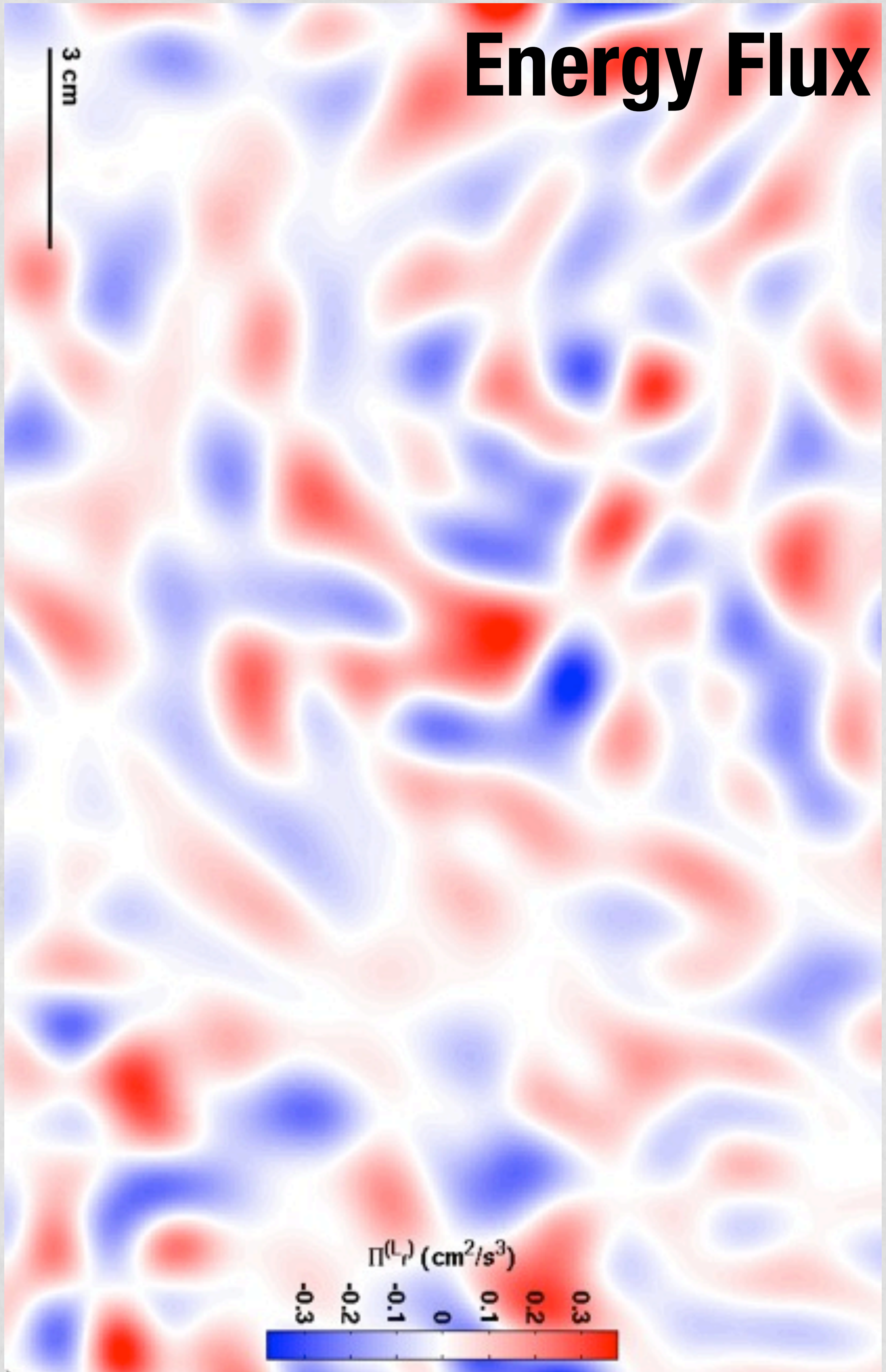


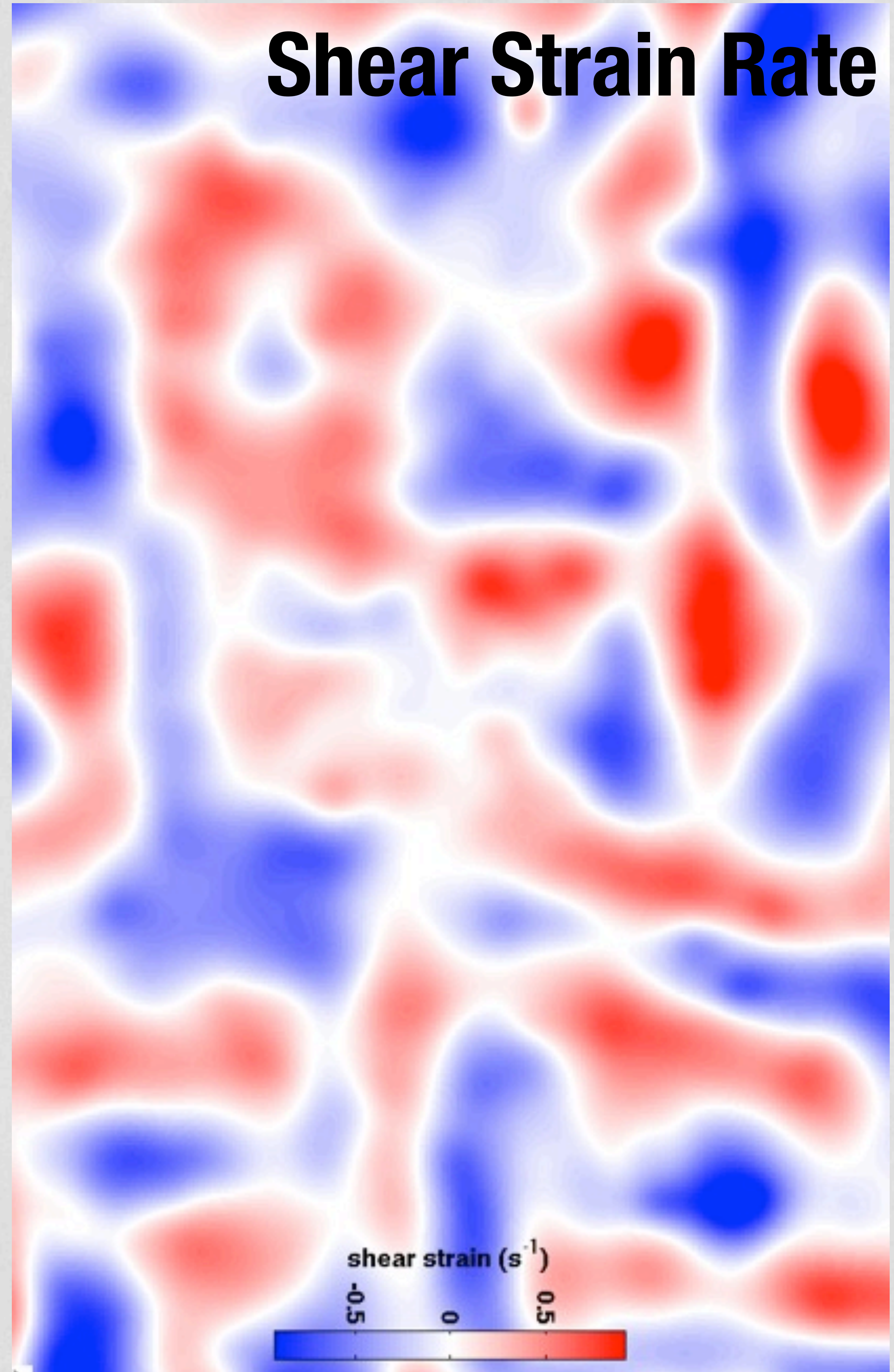
Spatial Dependence



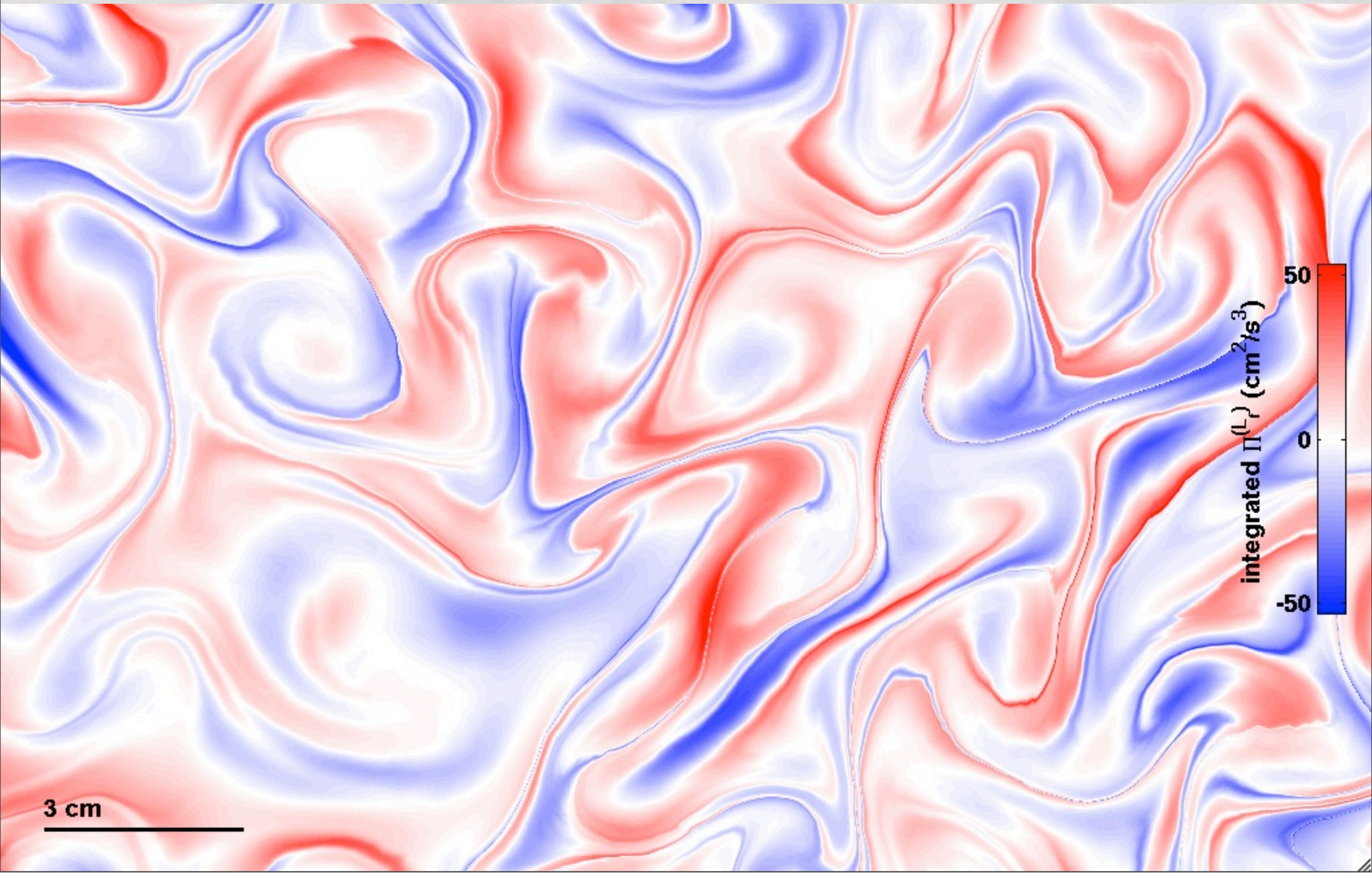
What determines instantaneous distribution of spectral flux?

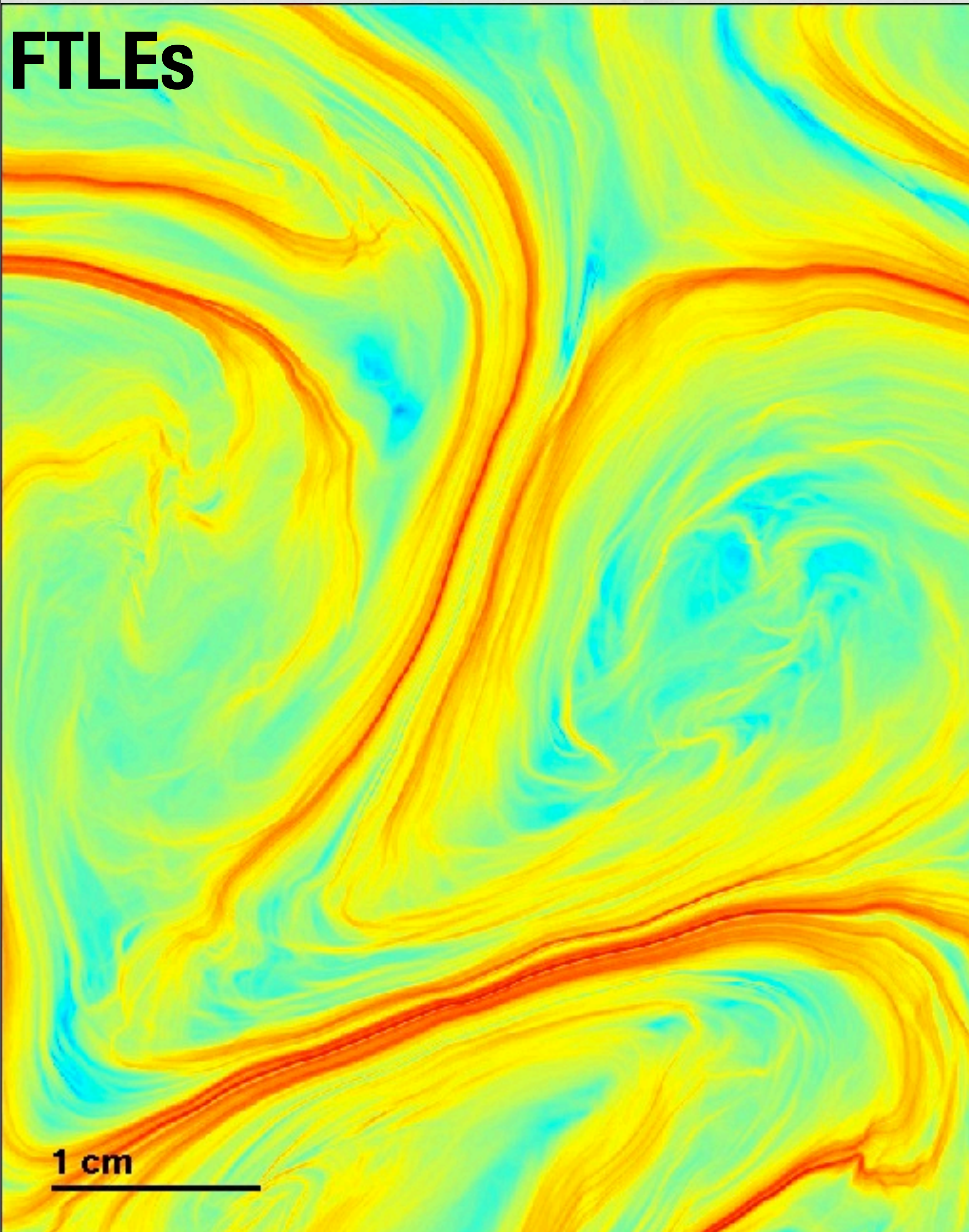
What determines instantaneous distribution of spectral flux?

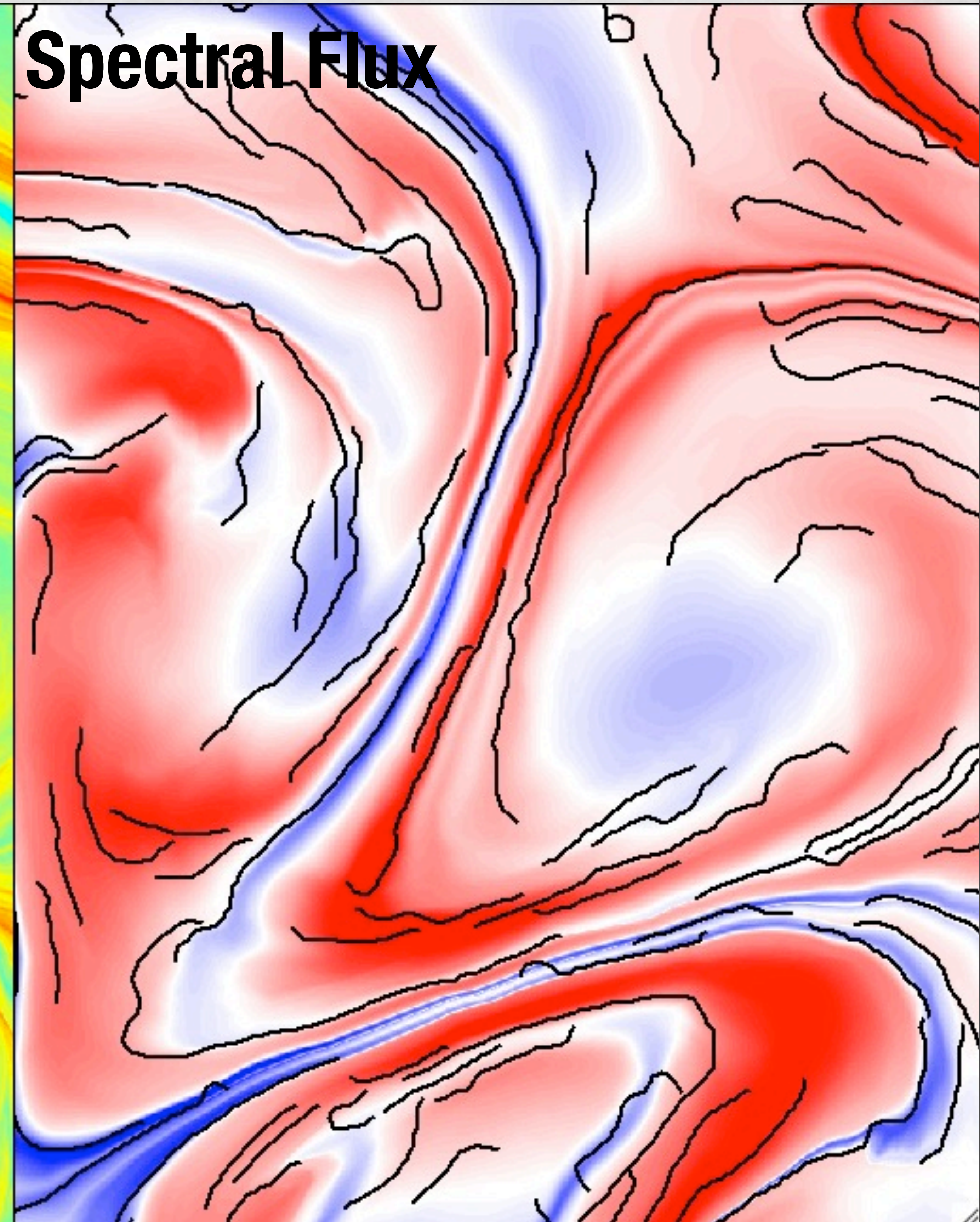




$$\int_{t}^{t+\tau} \Pi^{(r)}(t')\mathcal{D}\boldsymbol{x}(t')$$




FTLEs
1 cm


Spectral Flux

Eulerian Nonlinearity:

Spectral energy and enstrophy flux have strong spatiotemporal variation

Fluxes are advected at large scales

Small scale persistence in hyperbolic regions, large-scale persistence in elliptic regions

Correlations between energy flux and Lagrangian strain?

Future Directions & Open Questions:

What determines spatial distributions of spectral flux?

Can we connect Lagrangian nonlinearity (deformation) and Eulerian nonlinearity (spectral flux)?

New definitions of “coherent structures”?