Spectral Reduction: A Statistical Description of Turbulence

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2D Turbulence

2D Navier–Stokes vorticity equation:

$$\frac{\partial \omega_k}{\partial t} + \nu_k \omega_k = \int dp \int dq \frac{\epsilon_{kpq}}{q^2} \omega_p^* \omega_q^*, \quad \text{where } \nu_k \equiv \nu k^2$$

is antisymmetric under permutation of any two indices.

Energy $E_0$ and enstrophy $Z_0$ on the fine grid:

$$E_0 \equiv \frac{1}{2} \int dk \frac{\left| \omega_k \right|^2}{k^2}, \quad Z_0 \equiv \frac{1}{2} \int dk \left| \omega_k \right|^2$$

First consider $\nu_k = 0$. Conservation of $E_0$ and $Z_0$ follow from:

$$\frac{1}{k^2} \frac{\epsilon_{kpq}}{q^2} \quad \text{antisymmetric in } \ k \leftrightarrow q,$$

$$\frac{\epsilon_{kpq}}{q^2} \quad \text{antisymmetric in } \ k \leftrightarrow p.$$
Spectral Reduction

- Introduce a coarse-grained grid indexed by $K$.
- Define new variables
  \[
  \Omega_K = \langle \omega_k \rangle_K = \frac{1}{\Delta K} \int_{\Delta K} \omega_k \, dk,
  \]
  where $\Delta K$ is the area of bin $K$.
- Evolution of $\Omega_K$:
  \[
  \frac{\partial \Omega_K}{\partial t} + \langle \nu_k \omega_k \rangle_K = \sum_{P,Q} \Delta_P \Delta_Q \left\langle \frac{\epsilon_{kpq}}{q^2} \Omega^*_p \Omega^*_q \right\rangle_{KPQ},
  \]
  where $\langle f \rangle_{KPQ} = \frac{1}{\Delta K \Delta_P \Delta_Q} \int_{\Delta K} \, dk \int_{\Delta P} \, dp \int_{\Delta Q} \, dq \, f$.
- Approximate $\omega_p$ and $\omega_q$ by bin-averaged values $\Omega_P$ and $\Omega_Q$:
  \[
  \frac{\partial \Omega_K}{\partial t} + \langle \nu_k \rangle_K \Omega_K = \sum_{P,Q} \Delta_P \Delta_Q \left\langle \frac{\epsilon_{kpq}}{q^2} \right\rangle_{KPQ} \Omega^*_P \Omega^*_Q.
  \]
Wavenumber Bin Geometry (3 x 8 bins)
On the coarse grid, define the energy $E$ and enstrophy $Z$

$$E = \frac{1}{2} \sum_K \frac{|\Omega_K|^2}{K^2} \Delta_K, \quad Z = \frac{1}{2} \sum_K |\Omega_K|^2 \Delta_K.$$ 

Enstrophy is still conserved since

$$\left\langle \frac{\epsilon_{kpq}}{q^2} \right\rangle_{KPQ} \text{ antisymmetric in } K \leftrightarrow P.$$ 

But energy conservation has been lost!

$$\frac{1}{K^2} \left\langle \frac{\epsilon_{kpq}}{q^2} \right\rangle_{KPQ} \text{ NOT antisymmetric in } K \leftrightarrow Q.$$ 

Reinstate both desired symmetries with the modified coefficient

$$\frac{\left\langle \epsilon_{kpq} \right\rangle_{KPQ}}{Q^2}.$$ 

Energy and enstrophy are now simultaneously conserved.
Properties

- We call the forced-dissipative version of this approximation *Spectral Reduction (SR):*

\[
\frac{\partial \Omega_K}{\partial t} + \langle \nu_k \rangle_K \Omega_K = \sum_{P,Q} \Delta P \Delta Q \frac{\langle \epsilon_{kpq} \rangle_{KPQ}}{Q^2} \Omega_P^* \Omega_Q^*.
\]

- SR conserves both energy and enstrophy and reduces to the exact dynamics in the limit of small bin size.

- It has the same general structure and symmetries as the original equation and in this sense may be considered a *renormalization.*

- SR obeys a Liouville Theorem; in the inviscid limit, it yields statistical-mechanical (equipartition) solutions.
Moments

**Q.** How accurate is Spectral Reduction?

**A.** For large bins, the *instantaneous* dynamics of SR is inaccurate.

However: the equations for the *time-averaged* (or ensemble-averaged) moments predicted by SR closely approximate those of the exact bin-averaged statistics. *Eg.*, time average the exact bin-averaged enstrophy equation:

\[
\frac{\partial}{\partial t} \left\langle |\omega_k|^2 \right\rangle_K + 2 \text{Re} \left\langle \nu_k |\omega_k|^2 \right\rangle_K = 2 \text{Re} \sum_{P,Q} \Delta P \Delta Q \left\langle \frac{\epsilon_{kpq}}{q^2} \omega_k^* \omega_p^* \omega_q^* \right\rangle_{K PQ},
\]

where the bar means time average and \( \langle \cdot \rangle_K \) means bin average.

Time-averaged quantities such as \( |\omega_k|^2 \) and \( \omega_k^* \omega_p^* \omega_q^* \) are generally *smooth* functions of \( k, p, q \) on the four-dimensional surface defined by the triad condition \( k + p + q = 0 \).
Mean Value Theorem for integrals: for some $\xi \in K$,

$$|\Omega_K|^2 = |\omega_\xi|^2 \approx |\omega_k|^2 \quad \forall k \in K.$$ 

To good accuracy these statistical moments may therefore be evaluated at the characteristic wavenumbers $K, P, Q$:

$$\frac{\partial}{\partial t} |\Omega_K|^2 + 2 \text{Re} \langle \nu_k \rangle_K |\Omega_K|^2 = 2 \text{Re} \sum_{P,Q} \Delta P \Delta Q \left\langle \frac{\epsilon_{kpq}}{q^2} \right\rangle_{K P Q} \Omega_K^* \Omega_P^* \Omega_Q^*.$$ 

To the extent that the wavenumber magnitude $q$ varies slowly over a bin:

$$\frac{\partial}{\partial t} |\Omega_K|^2 + 2 \text{Re} \langle \nu_k \rangle_K |\Omega_K|^2 = 2 \text{Re} \sum_{P,Q} \Delta P \Delta Q \left\langle \frac{\epsilon_{kpq}}{Q^2} \right\rangle_{K P Q} \Omega_K^* \Omega_P^* \Omega_Q^*.$$ 

But this is precisely the time-average of the SR equation!
Convergence

- The previous argument suggests that Spectral Reduction can indeed provide an accurate statistical description of turbulence, even when each bin contains many statistically independent modes.

- As the wavenumber partition is refined, one expects the solutions of the time-averaged SR equations to converge to the exact statistical solution.

- An object-oriented C++ program (Triad) has been developed to implement and test Spectral Reduction.
Convergence of Partition

\[ E(k) \]

- 16\times8 \text{ bins}
- 32\times8 \text{ bins}
- 64\times8 \text{ bins}
- 16\times16 \text{ bins}
- 683\times683 \text{ modes}
- RTFM

\[ k \]

\[ E(k) \] vs. \[ k \] for different bin configurations and modes.
Noncanonical Hamiltonian Formulation

- Underlying noncanonical Hamiltonian formulation for inviscid 2D vorticity equation:

\[
\omega_k = \int dq \ J_{kq} \frac{\delta H}{\delta \omega_q},
\]

where

\[
H = \frac{1}{2} \int dk \ \frac{\left| \omega_k \right|^2}{k^2},
\]

\[
J_{kq} = \int dp \ \epsilon_{kpq} \omega_p^*.
\]

- Leads to inviscid Navier–Stokes equation:

\[
\frac{\partial \omega_k}{\partial t} + \nu_k \omega_k = \int dp \ \int dq \ \frac{\epsilon_{kpq}}{q^2} \omega_p^* \omega_q^*.
\]
Liouville Theorem

- **Navier–Stokes:**
  \[ J_{kq} = \int d\mathbf{p} \epsilon_{kpq} \omega_p^* \]
  \[ \Rightarrow \int dk \frac{\delta \omega_k}{\delta \omega_k} = \int dk \int dq \left( \frac{\delta J_{kq}}{\delta \omega_k} \frac{\delta H}{\delta \omega_q} + J_{kq} \frac{\delta^2 H}{\delta \omega_k \delta \omega_q} \right) = 0. \]
  \[ \epsilon_k(-k)q = 0 \]

- **Spectral Reduction:**
  \[ J_{KQ} = \sum_P \Delta_P \langle \epsilon_{kpq} \rangle_{KPQ} \Omega_P^* \]
  \[ \Rightarrow \sum_K \frac{\partial \dot{\Omega}_K}{\partial \Omega_K} = \sum_{K,Q} \left( \frac{\partial J_{KQ}}{\partial \Omega_K} \frac{\partial H}{\partial \Omega_Q} + J_{KQ} \frac{\partial^2 H}{\partial \Omega_K \partial \Omega_Q} \right) = 0. \]
  \[ \langle \epsilon_{kpq} \rangle_{K(-K)Q} = 0 \]
Statistical Equipartition

- If the dynamics are mixing, the Liouville Theorem and the coarse-grained invariants

\[ E = \frac{1}{2} \sum_K \left( \frac{\Omega_K}{K^2} \right)^2 \Delta_K, \quad Z = \frac{1}{2} \sum_K |\Omega_K|^2 \Delta_K, \]

lead to statistical equipartition of \((\alpha/K^2 + \beta) |\Omega_K|^2 \Delta_K\).

- This is the correct equipartition only for uniform bins. However, for nonuniform bins, a rescaling of time by \(\Delta_K\):

\[
\frac{1}{\Delta_K} \frac{\partial \Omega_K}{\partial t} + \langle \nu_k \rangle_K \Omega_K = \sum_{P,Q} \Delta_P \Delta_Q \frac{\langle \epsilon_{kpq} \rangle_{KPQ}}{Q^2} \Omega_P^* \Omega_Q^*. 
\]

yields the correct inviscid equipartition:

\[
\langle |\Omega_k|^2 \rangle = \frac{1}{\alpha/K^2 + \beta}.
\]
Relaxation to equipartition

\[ E(k) \]

- **16x8 bins**
- **21x21 modes**
- **\( \pi k / (\alpha + \beta k^2) \)**

Relation to equipartition:

\[ k \]

Graph showing the relaxation to equipartition with different bin and mode counts.
Stiffness Problem

- The rescaling of time does not change the steady-state moment equations.
- It does affect the statistical trajectory of the system and the resulting statistical solution.
- However, the resulting system becomes numerically very stiff.
- **Unsolved Problem:** given an efficient numerical method for evolving the system of equations

\[
\frac{dy}{dt} = S(y),
\]

find an efficient numerical method to evolve

\[
\frac{dy}{dt} = \Lambda S(y),
\]

where \( \Lambda \) is a constant real diagonal matrix.
The logarithmic slope of $E(k)$ is shown in the graph. The y-axis represents the logarithmic slope of $E(k)$, and the x-axis represents the wavenumber $k$. The graph includes multiple curves labeled as:

- Red curve: cascade/sr
- Blue curve: dns/cascade/ps
- Green curve: dns/cascade/ps3

The horizontal dotted lines indicate the range of logarithmic slopes from $-1$ to $-3$. The graph is divided into two main sections, highlighting the differences in the behavior of $E(k)$ across different wavenumber ranges.
The logarithmic slope of $E(k)$ is shown in the graph. The red line represents the cascade/inv process, while the blue line represents the dns/cascade/inv process. The graph displays the relationship between the logarithmic slope of $E(k)$ and the wave number $k$. The y-axis represents the logarithmic slope of $E(k)$, and the x-axis represents $k$. The graph includes horizontal lines at $\log_{10}(E(k)) = -2$ and $\log_{10}(E(k)) = -3$ for reference.