Turbulent Mixing of Passive Tracers on Small Scales in a Channel Flow

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In collaboration with
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Transport of passive scalar

\[ \frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = D \nabla^2 \theta \]

**Definitions:**
- \( \varepsilon_S \): rate of scalar fluctuations dissipation;
- \( \varepsilon \): rate of turbulent energy dissipation.

**Kolmogorov (dissipation) wave number:**

\[ k_K = \left( \frac{\varepsilon}{\nu^3} \right)^{1/4} \]

**Batchelor (dissipation) wave number:**

\[ k_B = \left( \frac{\varepsilon}{\nu D^2} \right)^{1/4} \]

**Control parameters of the problem:**  \( Sc = \nu/D >> 1 \) and \( Pe = VL/D >> 1 \)
$\log E(k)$

or

$\log E_s(k)$

Inertial-convective region, $k^{-5/3}$

Viscous-convective region, $k^{-1}$ (Batchelor region)

Viscous region

Diffusive region

$K_0 (1/d)$

$k_k (1/\eta_k)$

$k_B (1/\eta_B)$

Batchelor (1959), Kraichnan (1968)
Batchelor regime of mixing

- velocity field is statistically homogeneous, isotropic, and stationary
- random in time and spatially smooth velocity field
- energy spectrum is steeper than $k^{-3}$
Batchelor regime of mixing at $S >> 1$

- Distribution of small blob of scalar by repeated stretching and folding, so that the scale reduces as

$$l(t) \approx L \exp(-\lambda_1 t),$$

where $\lambda_1$ is the Lyapunov exponent, and $\lambda_1 \approx \dot{\gamma}$

Then velocity gradient is smeared out, when the diffusion rate becomes comparable with the rate of stretching: $D / l(t_{mix})^2 \approx \dot{\gamma}$

**This condition defines the Batchelor dissipation scale:**

$$\eta_B \approx L Pe^{-1/2}, \quad Pe = \dot{\gamma} L^2 / D, \quad \text{and} \quad \dot{\gamma} \approx \frac{\Delta V}{L} - \text{shear rate}$$

Then mixing time due to advection in the Batchelor regime is

$$t_{mix} = \frac{1}{\dot{\gamma}} \cdot \ln(\text{Pe})$$

or mixing length

$$l_{mix} = L \cdot \ln(\text{Pe})$$
Theoretical predictions on turbulent mixing in Batchelor decay regime for unbounded systems


1. Power spectrum of passive scalar is $E_s \propto k^{-1}$ at $k \ll k_B$
or for passive scalar correlation function: $\langle \theta(0)\theta(r) \rangle \propto \ln r$

2. PDFs of passive scalar concentration exhibit exponential tails, which range increases with Pe

3. Moments of PDFs decay exponentially along a channel

4. Exponents of the moments decay $\gamma_i$ saturate at $i>6$. It means that PDFs are intermittent but not self-similar

5. PDFs of scalar concentration increments at different scales at the same bend have exponential tails and are self-similar

6. Structure functions of passive scalar concentration increments have logarithmic dependence on increments $\delta r$
Turbulent mixing in the Batchelor decay regime with no-slip boundaries

Influence of the boundaries on mixing:

\[ V_r \propto (d - r)^2 \quad \text{or} \quad V_z \propto d \]

\[ \langle V_r \rangle = 0 \]

Boundary layer is a sink for a passive tracer

As the result the mixing is less effective than in an unbounded system, and the mixing time (or length) varies as:

\[ t_{mix} = \dot{\gamma}^{-1} \cdot \text{Pe}^{1/4} \quad \text{or} \quad l_{mix} = d \cdot \text{Pe}^{1/4} \]

A. Chernykh and V. Lebedev, JETP Lett. 87, 682 (2008)
Predictions of the decay theory in peripheral region (finite width channel flow with no-slip boundaries)

1. Decay of the $i$-th moment $M_i \equiv \langle |\theta - \bar{\theta}|^i \rangle / \bar{\theta}^i$

   \[ M_i \propto \exp(-\gamma_i z / V z), \quad \gamma_i \propto \lambda_1 \cdot Pe^{-1/4}, \quad \lambda_1 \text{-Lyapunov exponent} \]

2. Decay becomes algebraic in transient regime, when boundary layer width $\delta$ is still wider than diffusive $r_{bl}$ (two-stage mixing process)

3. Non-uniform spatial distribution of passive scalar: peripheral region and diffusive boundary layer $r_{bl}$

   \[ \Delta r \propto d \cdot Pe^{-1/4} \]

4. Correlation function

   \[ F_2(r) \propto M_2(r)^{-\beta}, \quad \beta \propto Pe^{-1/4} \ll 1 \quad \text{at } Pr \gg 1 \]

   and close to logarithmic dependence

5. Quantitative predictions about mixing length in a channel flow
Quantitative predictions on mixing length in the passive scalar decay


In the Batchelor decay regime of mixing in a channel flow, scalar dynamics is given as

\[ \partial_t \theta + V_\theta \partial_z \theta + \delta v_r \nabla_i \theta = D \nabla^2 \theta, \]  

(1)

where \( V_\theta \) is average velocity along channel; \( \delta v_r \equiv V_r - \langle V_r \rangle \) is fluctuating part of transversal velocity.

In 2D flow in channel \( \langle V_\theta \rangle \) is independent of \( z \), \( \langle V_\theta \rangle \propto r \) and \( \delta v_r \propto r^2 \) near the wall. Then one gets

\[ sr \partial_z \langle \theta \rangle = \left[ \mu \partial_r r^4 \partial_r + D \partial^2_r \right] \langle \theta \rangle, \]

(2)

\[ \langle V_\theta \rangle = sr; D_{rr}(r) \equiv \mu r^4 = \int_0^\infty \langle \delta v_r(t, r) \delta v(0, r) \rangle dt \]

In the first stage peripheral region has width \( \delta = s/(\mu z) \gg r_{bl} \) and at the second stage \( \delta = r_{bl} = (D/\mu)^{1/4} \).

Then for the first moment of passive scalar PDF at large \( z \) one gets for \( \delta \)-correlated velocity

\[ \langle \theta \rangle \propto \exp \left( - l_{mix}^{-1} z \right), l_{mix}^{-1} \approx 3.72 D^{1/4} \mu^{3/4} s^{-1}, \]

(3)
Mixing in a channel flow: experiment

Coordinate along the channel - N, the number of bend
Curved flow trajectories are crucial for the elastic turbulence

**Solution:** 80 ppm PAAm, (M=18Mda), 65% sucrose – water solvent with viscosities

\[ \eta_s = 113 \cdot 8 \text{mPas}, \eta = 137 \cdot 6 \text{mPas} \]

The longest polymer relaxation time \( \lambda = 11.5 \text{s} \)

<table>
<thead>
<tr>
<th>Dye (2 ppm)</th>
<th>D (mm(^2)/s)</th>
<th>Wi</th>
<th>Pe=((\partial V_r/\partial r))(^{\text{rms}})d(^2)/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluorescein</td>
<td>3.1x10(^{-6})</td>
<td>354</td>
<td>2.5x10(^6)</td>
</tr>
<tr>
<td>Fluorescein</td>
<td>3.1x10(^{-6})</td>
<td>1006</td>
<td>3.6x10(^6)</td>
</tr>
<tr>
<td>FITCD 40Kda</td>
<td>3.1x10(^{-7})</td>
<td>354</td>
<td>2.4x10(^7)</td>
</tr>
<tr>
<td>FITCD 40Kda</td>
<td>3.1x10(^{-7})</td>
<td>1224</td>
<td>4x10(^7)</td>
</tr>
<tr>
<td>FITCD 2Mda</td>
<td>5.4x10(^{-8})</td>
<td>1224</td>
<td>2.3x10(^8)</td>
</tr>
</tbody>
</table>

\(v=150 \text{mm}^2/\text{s}\)  \(\text{Sc up to } 3\times10^9\)

T. Burghulea, E. Segre, and V. Steinberg, *PRL* **92**, 164501 (2004)-100 \(\mu\)m-confocal microscope
Spectra of velocity fluctuations, measured in the center of the channel with LDV (N=12) in 3mm channel

Average velocity - $V = 6.6 \text{ mm/s}$; fluctuations: longitudinal - $V_{\text{rms}} = 0.09V$, transversal - $V_{\text{rms}} = 0.04V$.

Taylor hypothesis: velocity fluctuations in time are due to fluctuations in space $\Rightarrow P \sim k^{-3.3}$

The Batchelor regime of mixing!
Photographs of the flow taken with laser sheet at different $N$. Bright regions correspond to high fluorescent dye concentration

(a) $N=29$-pure solvent
(b) $N=8$
(c) $N=29$
(d) $N=54$ (polymer solution)
As fluid advances downstream and gets mixed, the contrast decreases but the characteristic size of the structures does not seem to change very much.
Probability density functions for the passive scalar in a point at different positions, N, along the channel (3 mm channel)

Spatial correlation of the concentration across the channel at different N. Each line - statistics over about $10^7$ points, and over the total discharge of $2 \cdot 10^3 d^3$

$x_0 = 50 \, \mu m$; molecular diffusion scale $(D/(V_{rms}/d))^{1/2} = 25 \, \mu m$, illuminating light sheet - 40 \, \mu m thick
Dependence of the moments, $M_i$, of the passive scalar PDF on the position, $N$, along the channel (3 mm channel)

$$M_i \equiv \left\langle |\theta - \bar{\theta}|^i \rightangle / \bar{\theta}^i$$

Exponential decay of the moments with the position, $N$, along the channel and with the mixing time, $t = 7.6 \ 2\pi \ 4.5/V = 21.5$ sec.

Mixing time in the flow – 21.5 sec; diffusion time $d^2/D \approx 10^5$ sec. $M_i \sim \exp(-\gamma_i N)$ was also found for higher order moments.

**Dependence of the exponents, $\gamma_i$ on the order, $i$, of the momentum**

Linear growth of $\gamma_i$ at small $i$ and saturation at larger $i$, in complete agreement with the theoretical predictions.
Moments of the scalar PDF decay exponentially downstream (100 μm channel)


\[ Pe = 4.15 \times 10^5 \]

Inset: decay exponents of \( i \)-th moments at different \( Pe \)
Batchelor regime in the bounded system: boundary layers are a sink for the tracer. This affects scaling: theory—Chertkov and Lebedev (2003), $L_{mix} \propto Pe^{1/4}$ instead of $\ln Pe$ in unbounded system. We get in the experiment with 100 μm channel

$$N_{mix} \propto Pe^{0.26 \pm 0.01}$$
Schematic drawing of experimental set-up with 1 mm channel


PIV with 2 μm red fluorescent particles and passive scalar concentration measurements
Space-time plots of time series of mixing across the channel width for 6, 30, and 60 bends
First and second moments of PDfs of fluorescent intensities as a function of N

For $Wi=354$ and $Pe=3.62 \times 10^8$
Decay exponents versus the number of the moment $i$ (a) and normalized values (b) for 4 values of $Pe$.
Mixing length as a function of $Pe$ in the bulk ($r/d=0.2-0.08$) for three experiments.

Inset: the data only for 1 mm channel $N_{mix}$ vs $Pe=(\partial V_r/\partial r)_{mix}d^2/D$.

Linear fits have slopes of $0.27\pm0.02$ for $M_1$ and $0.27\pm0.01$ for $M_2$. 
at N=40, 50 and 60 bends and Pe=2.28x10^8 in the bulk (full lines) and peripheral region (dashed lines). Dash-dotted line is power-law fit $k^{-\alpha}$ with $\alpha=1.08$
PDFs of passive scalar increments at different \( \delta r \) for \( N=40 \) and \( \text{Pe}=2.28 \times 10^8 \)

(a) Second and fourth moments of PDFs
(b) Second moments at \( N=40, 50 \) and 60
Time correlation functions at $N=40$

And 5 values of $Pe$ in the bulk (a) and the same in peripheral region (b)

Space correlation functions at $N=40$ for 5 values of $Pe$ in linear-log coordinates in the bulk (a) and peripheral region (b)

Logarithmic decay is obvious and corresponds to $\sim k^{-1}$ decay in spatial power spectra for scalar

One cannot resolve difference between log – dependence from $C_{\delta r} \sim (\delta r)^{-\beta}$ with $\beta \sim Pe^{-1/4}$ at $Pe >> 1$ ($Pe \sim 10^8$), suggested by recent theory for bounded system
Quantitative experimental determination of mixing length in the decay Batchelor regime

Mean longitudinal velocity profiles across the channel

At high $Wi > Wi_c$ the boundary layer is obviously visible
Mean longitudinal velocity profiles in the narrow boundary layer region near the wall. Vertical dotted line indicates range of linear fitting to get the slope $s$.

Slope $s$ vs $Wi$. Linear fit is $s = -48 + 0.329$ $Wi$.
RMS fluctuations of transversal velocity profiles across the channel for various Wi.

Mean transversal velocity profiles across the channel for various Wi.

RMS fluctuations of transversal velocity profiles across the channel for various Wi.
Transverse component of turbulent diffusion tensor $D_{rr}$ vs $(r/d)^4$

Lines are linear fit with slope $\mu$. Vertical dotted line defines the range of fitting in the boundary layer.

$$D_{rr} (r) \equiv \mu r^4 = \int_{0}^{\infty} \left\langle \delta v_r (t, r) \delta v(0, r) \right\rangle dt$$

$$\delta v_r \equiv V_r - \left\langle V_r \right\rangle$$

Slope $\mu$ vs $Wi$. Linear fit is $\mu=-38.9+141.7\, Wi$
Inverse mixing length vs parameters of the problem

Linear fit to the data gives coefficient to compare with theory: $C=0.013$
Comparison with theoretical predictions:

Theory predicts $C=3.72$
Experiment gives $C=0.013\pm0.007$
Important difference between theory and the experiment is jets (plumes) observed in numerical simulations and experiment


Dimension of the image 1x0.768 mm$^2$

Correlation time of passive scalar vs N for Pe = $3.62 \times 10^6$ (a), and vs Pe at N=40 (b) in both bulk (solid) and peripheral (open).

The scalar correlation time is of the order of velocity correlation time! So δ-correlated in time velocity field approximation is not valid.
Conclusions

• All qualitative and functional dependencies of statistical and scaling properties of passive scalar behavior in the Batchelor decay regime were verified

• The main result of the quantitative test of the value of the mixing length shows huge discrepancy with theoretical predictions on the coefficient. It is of about 200 times smaller in the experiment that means that the experimental mixing length is about 200 times larger

• It was suggested two possible sources of this discrepancy:
  (i) assumption made in the theory about $\delta$-correlated velocity field is in odds with the experimental measurements
  (ii) probably the most relevant observation in this respect is jets, rare localized and vigorous ejection of scalar trapped near the wall into the peripheral and bulk, which drastically reduces mixing efficiency and increases mixing length