TURBULENT TRANSPORT

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PASSIVE SCALAR TRANSPORT

Consider scalar C obeying

$$\partial_t C + \boldsymbol{\nabla} \cdot (\boldsymbol{U}\, C) - \kappa \Delta C = 0\,.$$

Split C and $\boldsymbol{U}$ into mean and fluctuating parts,

$$C = \overline{C} + c \qquad \boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}\,.$$

Then

$$\partial_t \overline{C} + \boldsymbol{\nabla} \cdot (\overline{\boldsymbol{U}}\, \overline{C} + \overline{\boldsymbol{\mathcal{F}}}) - \kappa \Delta \overline{C} = 0 \quad \text{where} \quad \overline{\boldsymbol{\mathcal{F}}} = \overline{\boldsymbol{u}\, c}$$

and $\quad \partial_t c + \boldsymbol{\nabla} \cdot (\overline{\boldsymbol{U}} c) - \kappa \Delta c = -\boldsymbol{\nabla} \cdot (\boldsymbol{u} \overline{C} + (\boldsymbol{u} c)')\,.$

$\overline{\boldsymbol{\mathcal{F}}}$ is a functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{C}$, linear in $\overline{C}$.

In the case of perfect scale separation

$$\mathcal{F}_i = \gamma_i \overline{C} - \kappa_{ij} \boldsymbol{\nabla}_j \overline{C} \, .$$

In general however non-local and non-instantaneous connection between $\overline{\boldsymbol{\mathcal{F}}}$ and $\overline{C}$,

$$\mathcal{F}_i(\boldsymbol{x}, t) = \iint \Big(\gamma_i(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \overline{C}(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \\ - \kappa_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \boldsymbol{\nabla}_j \overline{C}(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \Big) \, \mathrm{d}^3\xi \, \mathrm{d}\tau \, .$$

Introduce Fourier transformation

$$Q(\boldsymbol{\xi}, \tau) = (2\pi)^{-4} \iint \tilde{Q}(\boldsymbol{k}, \omega) \, \exp(\mathrm{i}(\boldsymbol{k}\boldsymbol{\xi} - \omega\tau)) \, \mathrm{d}^3 k \, \mathrm{d}\omega \, .$$

Then according to convolution theorem

$$\mathcal{F}_i(\boldsymbol{x}, t) = \int \int \Big(\tilde{\gamma}_i(\boldsymbol{x}, t; \boldsymbol{k}, \omega) - \mathrm{i} \tilde{\kappa}_{ij}(\boldsymbol{x}, t; \boldsymbol{k}, \omega) k_j \Big) \tilde{\overline{C}}(\boldsymbol{k}, \omega) \\ \exp(\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \, \mathrm{d}^3 k \, \mathrm{d}\omega \, .$$

Non-locality and non-instantaneity
occur as dependence of $\tilde{\gamma}_i$ and $\tilde{\kappa}_{ij}$ on $\boldsymbol{k}$ and ω.

Perfect scale separation

$$\mathcal{F}_i = \gamma_i \overline{C} - \kappa_{ij} \boldsymbol{\nabla}_i \overline{C}\,.$$

Non-local and non-instantaneous connection between $\overline{\boldsymbol{\mathcal{F}}}$ and $\overline{C}$

$$\mathcal{F}_i(\boldsymbol{x},t) = \int\int \Big(\tilde{\gamma}_i(\boldsymbol{x},t;\boldsymbol{k},\omega) - \mathrm{i}\tilde{\kappa}_{ij}(\boldsymbol{x},t;\boldsymbol{k},\omega)k_j\Big)\tilde{\overline{C}}(\boldsymbol{k},\omega)$$
$$\exp(\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\,\mathrm{d}^3k\,\mathrm{d}\omega\,.$$

$$\tilde{\gamma}_i(\boldsymbol{k}=\boldsymbol{0},\omega=0) = \gamma_i\,, \quad \tilde{\kappa}_{ij}(\boldsymbol{k}=\boldsymbol{0},\omega=0) = \kappa_{ij}$$

Determination of γ_i and κ_{ij}, or $\tilde{\gamma}_i$ and $\tilde{\kappa}_{ij}$:

- Second-order correlation approximation (SOCA)
  Ignore term with $(\boldsymbol{u}c)'$ on the rhs of the equation for c

  so that $\quad \partial_t c + \boldsymbol{\nabla} \cdot (\overline{\boldsymbol{U}} c) - \kappa \Delta c = -\boldsymbol{\nabla} \cdot (\boldsymbol{u} \overline{C})$.

  Applies to small $\boldsymbol{u}$ only.

- Test-scalar method
  Calculate $\overline{\boldsymbol{\mathcal{F}}}$ for "test-scalars" $\overline{C}^{(1)}, \overline{C}^{(2)}, \ldots$.
  Solve

$$\mathcal{F}_i^{(1)} = \gamma_i \overline{C}^{(1)} - \kappa_{ij} \boldsymbol{\nabla}_j \overline{C}^{(1)}, \; \mathcal{F}_i^{(2)} = \gamma_i \overline{C}^{(2)} - \kappa_{ij} \boldsymbol{\nabla}_j \overline{C}^{(2)}, \ldots$$

for γ_i and κ_{ij}

  Method can be extended to $\tilde{\gamma}_i$ and $\tilde{\kappa}_{ij}$.

MEAN–FIELD ELECTRODYNAMICS

Assume that $\boldsymbol{B}$ satisfies induction equation

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) - \eta \nabla^2 \boldsymbol{B} = 0 \,, \quad \nabla \cdot \boldsymbol{B} = 0 \,.$$

Split $\boldsymbol{B}$ and $\boldsymbol{U}$ into mean and fluctuating parts,

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b} \quad \boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u} \,.$$

Then

$$\partial_t \overline{\boldsymbol{B}} - \nabla \times (\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{\mathcal{E}}}) - \eta \nabla^2 \overline{\boldsymbol{B}} = 0 \,, \quad \nabla \cdot \overline{\boldsymbol{B}} = 0 \,,$$

where $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$,

$$\partial_t \boldsymbol{b} - \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b}) - \eta \nabla^2 \boldsymbol{b} = -\nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}} + (\boldsymbol{u} \times \boldsymbol{b})') \,, \; \nabla \cdot \boldsymbol{b} = 0 \,.$$

$\overline{\boldsymbol{\mathcal{E}}}$ is a functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$, linear in $\overline{\boldsymbol{B}}$.

If perfect scale separation

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\boldsymbol{\nabla}_k\overline{B}_j \, .$$

E.g.,

$$\overline{\boldsymbol{\mathcal{E}}} = \alpha_1(\boldsymbol{g}\cdot\boldsymbol{\Omega})\,\overline{\boldsymbol{B}} + \alpha_2(\boldsymbol{g}\cdot\overline{\boldsymbol{B}})\,\boldsymbol{\Omega} + \alpha_3(\boldsymbol{\Omega}\cdot\overline{\boldsymbol{B}})\,\boldsymbol{g} - \gamma\,\boldsymbol{g}\times\overline{\boldsymbol{B}}$$

$$-\beta\,\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} - \delta\,\boldsymbol{\Omega}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) - \cdots$$

Extension to non-local and non-instantaneous connections between $\overline{\boldsymbol{\mathcal{E}}}$ and $\overline{\boldsymbol{B}}$ analogous to passive scalar case.

Test-field method for determination of coefficients like α_1, α_2, ... analogous to test-scalar method.

EXAMPLE (1)

$\overline{\boldsymbol{U}} = \mathbf{0}$, $\boldsymbol{u}$ homogeneous isotropic non-helical turbulence.
Local and instantaneous connection between $\overline{\boldsymbol{\mathcal{F}}}$ and $\overline{C}$ and between $\overline{\boldsymbol{\mathcal{E}}}$ and $\overline{\boldsymbol{B}}$.
Then $\kappa_{ij} = \kappa_{\mathrm{t}}\delta_{ij}$ and $\eta_{ij} = \eta_{t}\delta_{ij}$,

$$\partial_t \overline{C} - (\kappa + \kappa_{\mathrm{t}})\Delta \overline{C} = 0$$

$$\partial_t \overline{\boldsymbol{B}} - (\eta + \eta_{\mathrm{t}})\boldsymbol{\nabla}^2 \overline{\boldsymbol{B}} = \mathbf{0}\,.$$

κ_{t} and η_{t} not always positive !!!

....

κ_t and η_t not always positive !!!

Put $\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{\psi} + \boldsymbol{\nabla}\phi$.

Assume $\boldsymbol{u}$ so small that SOCA applies.

Assume further $\tau_c \gg \lambda_c^2/\eta$.

Then straightforward calculation delivers

$$\eta_\mathrm{t} = \frac{1}{3\eta}\left(\overline{\psi^2} - \overline{\phi^2}\right) \quad \text{(Krause \& Rädler 1980)}$$

Analogously, with $\tau_c \gg \lambda_c^2/\kappa$, $\quad \kappa_\mathrm{t} = \frac{1}{3\kappa}\left(\overline{\psi^2} - \overline{\phi^2}\right)$.

A simple model

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi\,, \qquad \phi = \frac{u_0}{k_0}\cos k_0(x+\tilde{\xi})\,\cos k_0(y+\tilde{\eta})\,\cos k_0(z+\tilde{\zeta})$$

Define mean fields by averaging over x, y and $\tilde{\zeta}$.

SOCA calculation yields

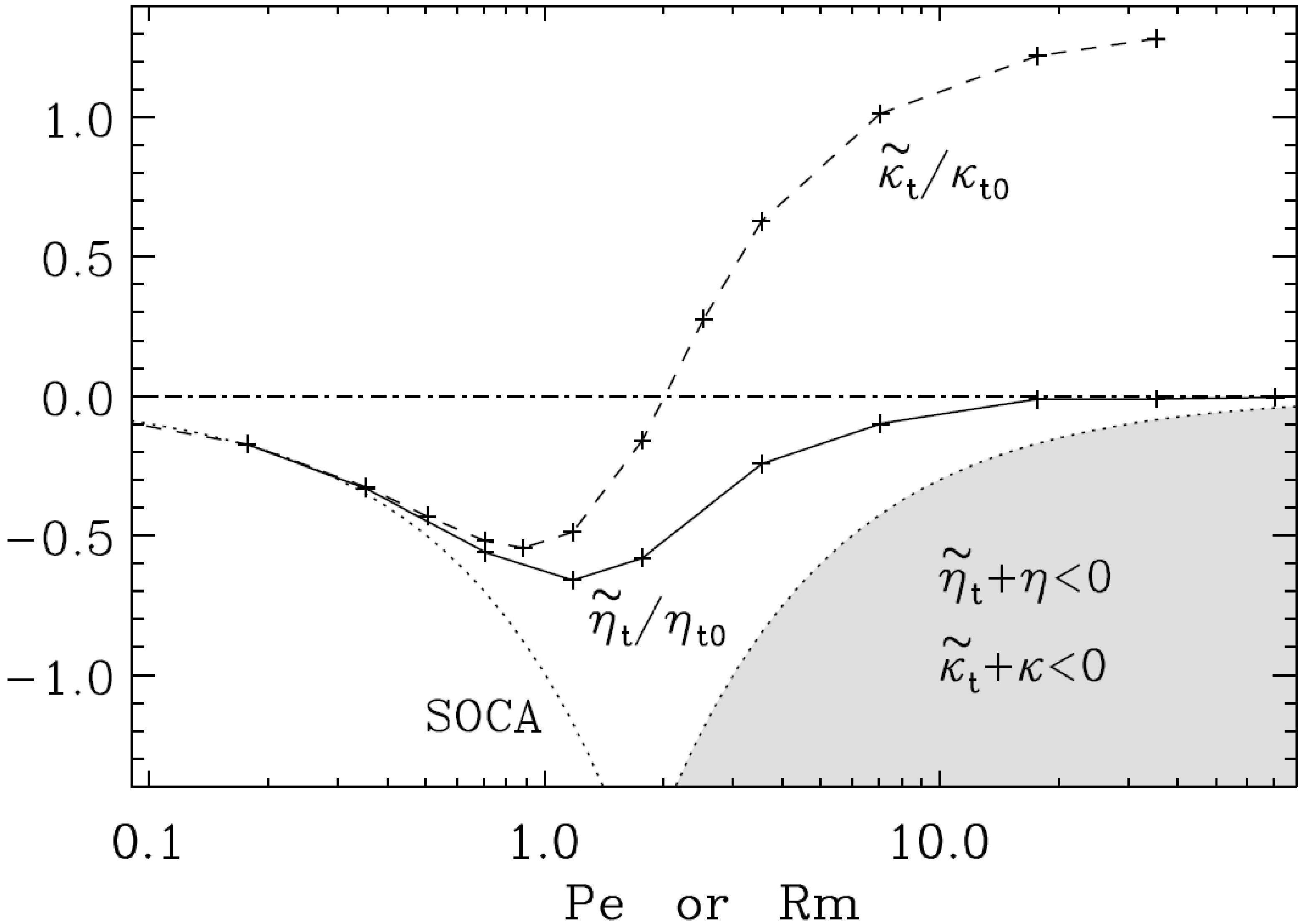
$$\tilde{\kappa}_t = -\kappa_{t0}\,Pe\,f(k/k_f) \qquad \tilde{\eta}_t = -\eta_{t0}\,Rm\,f(k/k_f)$$

$$\kappa_{t0} = \eta_{t0} = u_{rms}/3k_f\,, \quad u_{rms} = (\sqrt{3}/2\sqrt{2})\,u_0\,, \quad k_f = \sqrt{3}\,k_0$$

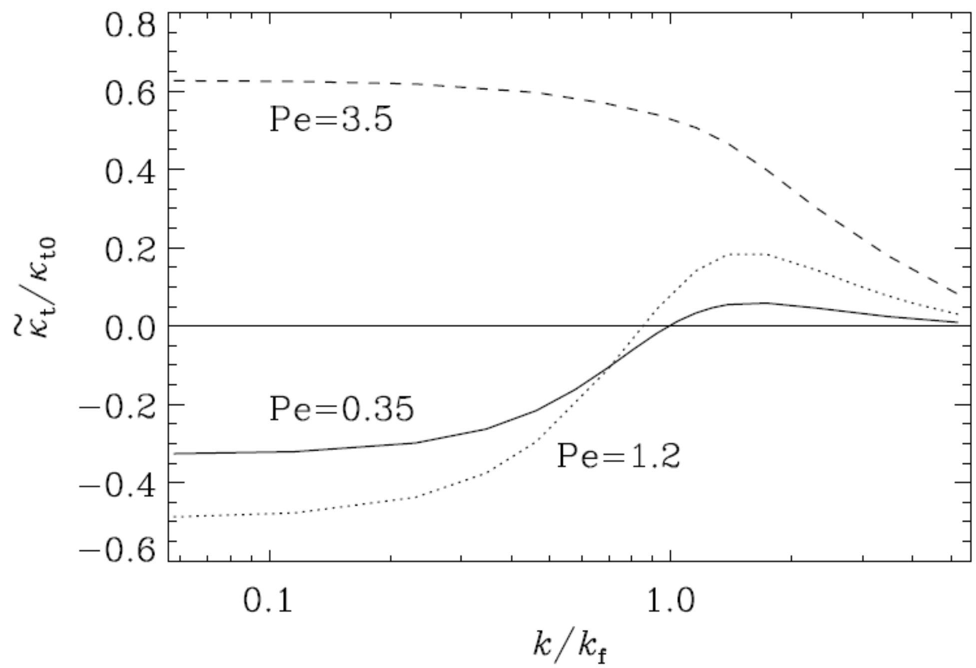
$$Pe = u_{rms}/\kappa k_f \qquad Rm = u_{rms}/\eta k_f$$

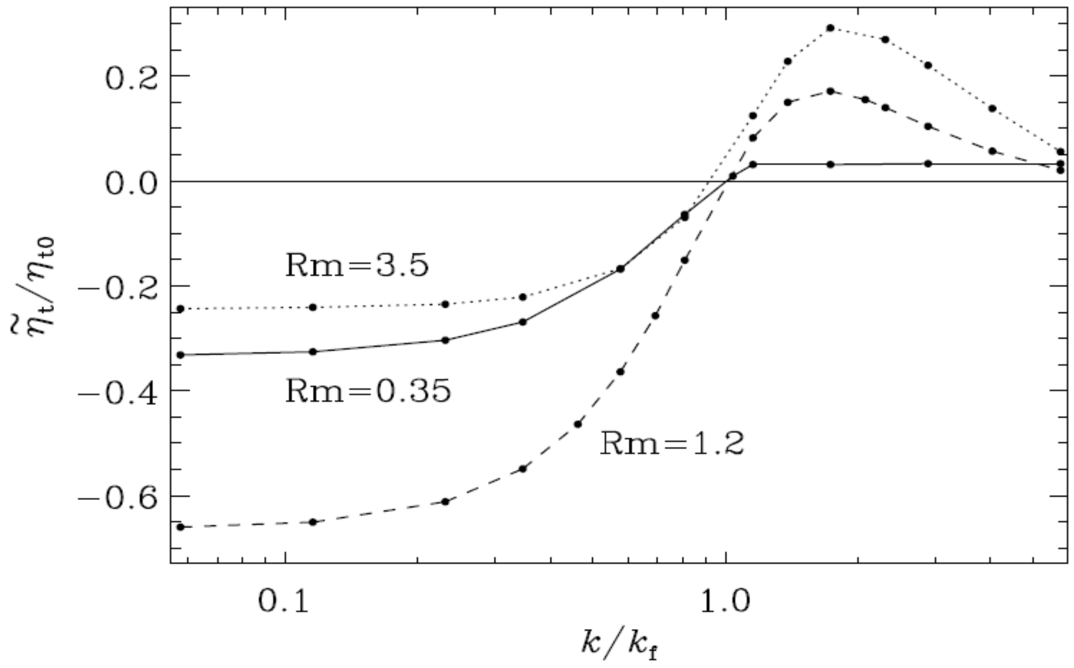
Peclet number   magnetic Reynolds number

$$f(v) = \frac{1-v^2}{1+(2/3)v^2+v^4}$$



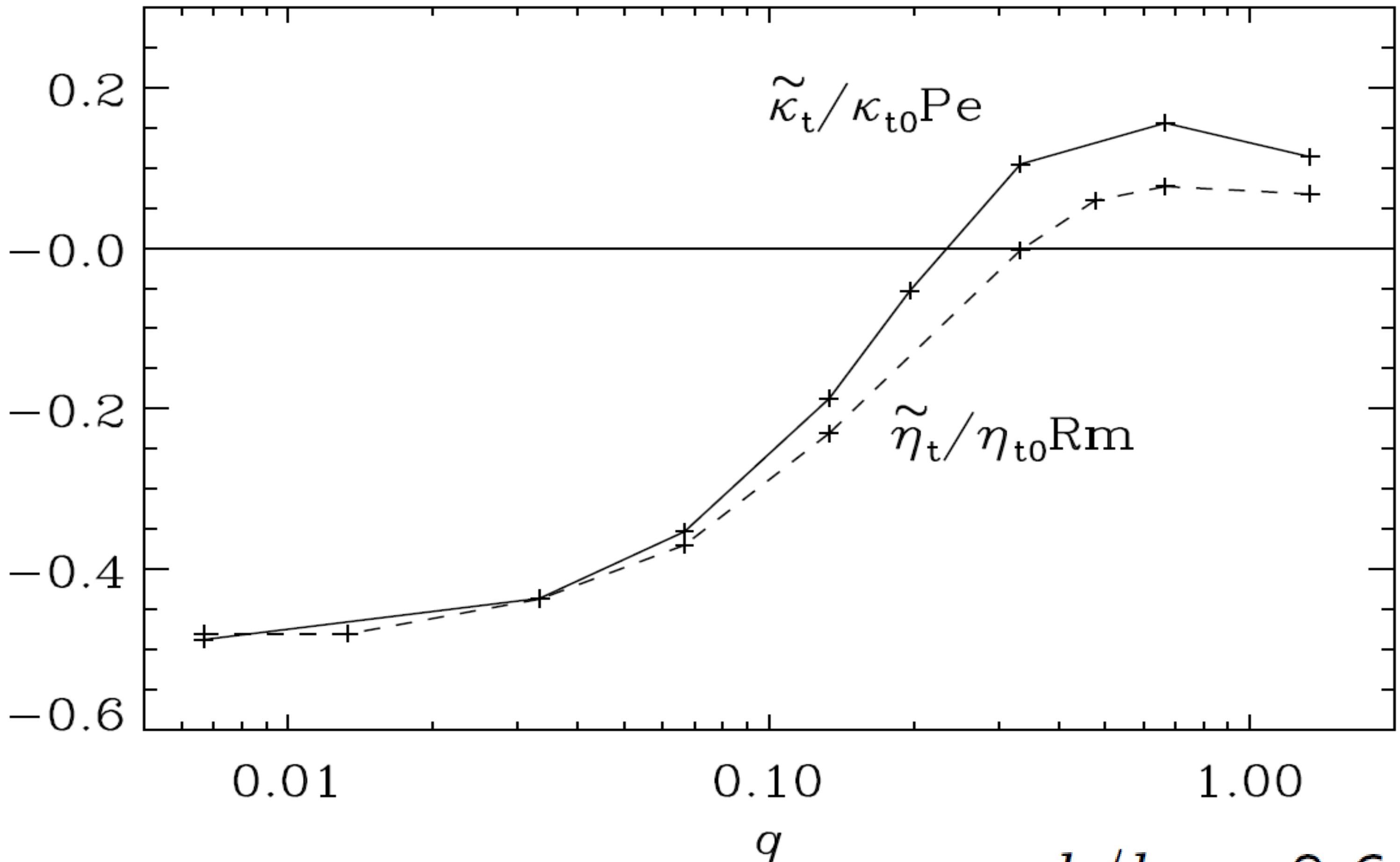
$k/k_f \approx 0.06$


Pe=3.5
Pe=0.35
Pe=1.2
$\tilde{\kappa}_t/\kappa_{t0}$
k/k_f


Rm=3.5
Rm=0.35
Rm=1.2
$\tilde{\eta}_t/\eta_{t0}$
k/k_f

"Renovating flow"

$q = (\kappa k_\mathrm{f}^2 \tau)^{-1}$ or $q = (\eta k_\mathrm{f}^2 \tau)^{-1}$, τ time interval of renovation



$k/k_\mathrm{f} \approx 0.6$

EXAMPLE (2)

$$\overline{\boldsymbol{U}} = \boldsymbol{0}\,, \quad \boldsymbol{u} = \boldsymbol{\nabla}\phi\,, \quad \phi = \frac{u_0}{k_0}\cos(k_0(\epsilon x + z) - \omega_0 t - \chi)$$

Sound wave in $(\epsilon, 0, 1)$ direction.
Define mean fields by averaging over x, y and χ.
For simplicity $\omega_0 = 0$, i.e. standing wave.

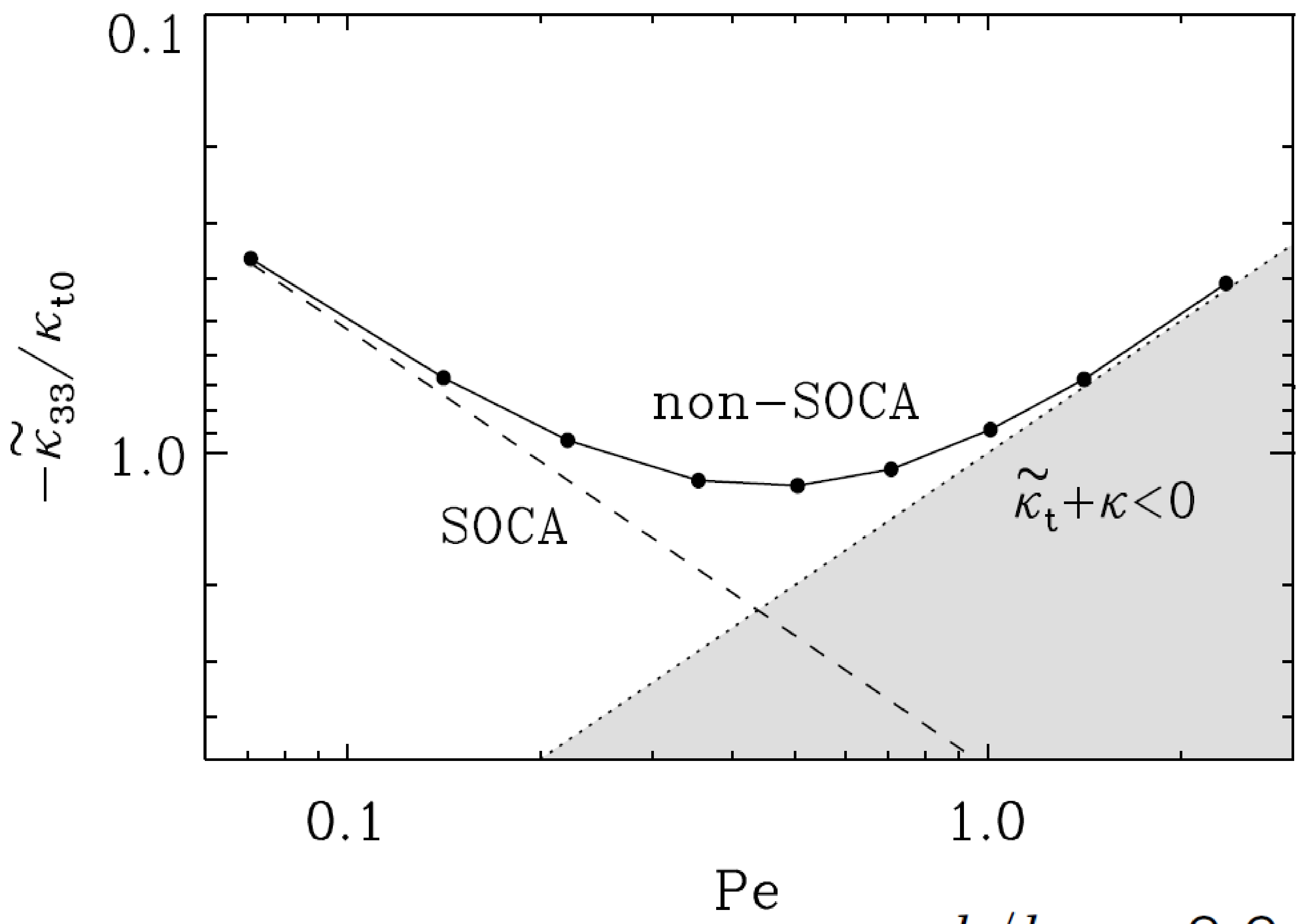
Then $\overline{\mathcal{F}}_z(z) = -\int \kappa_{zz}(\zeta)\partial_z\overline{C}(z-\zeta)\mathrm{d}\zeta\,.$

SOCA calculation
assuming that $\overline{C}$ varies like $\exp(\lambda t)$ in time yields

$$\tilde{\kappa}_{zz} = -\kappa_{\mathrm{t0}}\,Pe\,h(k/k_0, \lambda/\kappa k_0^2, \epsilon)\,, \ \kappa_{\mathrm{t0}} = u_{\mathrm{rms}}/k_0\,, \ Pe = u_{\mathrm{rms}}/\kappa k_0$$

$$h(v, w, \epsilon) = -\frac{1}{2v}\left(\frac{1+\epsilon^2+v}{(1+v)^2+\epsilon^2+w} - \frac{1+\epsilon^2-v}{(1-v)^2+\epsilon^2+w}\right)$$

For $\epsilon = w = 0$ singularity at $k/k_0 = 1$!!!


0.1
1.0
$-\tilde{\kappa}_{33}/\kappa_{t0}$
non−SOCA
SOCA
$\tilde{\kappa}_t+\kappa<0$
0.1
1.0
Pe

$k/k_0 = 0.9$

Decay rates $(\kappa + \tilde{\kappa}_{zz})k^2$
(with $\tilde{\kappa}_{zz}$ derived under the assumption $\lambda = 0$)
by a factor of about 20 greater
than those observed in direct numerical simulations,
i.e., decay observed in simulations much slower
than predicted by the calculated $\tilde{\kappa}_{zz}$.

Discrepancy can be explained by the "memory effect"
(Hubbard and Brandenburg 2009).
It vanishes if the dependence of $\tilde{\kappa}_{zz}$ on λ is considered.

- In both the passive scalar and the magnetic case mean–field diffusivities in compressible turbulent fluids can be markedly smaller than the molecular diffusivities, that is, the decay of mean fields can be markedly slower than that in the absence of turbulent motions.

- Negative contributions to the mean–field diffusivities occur for not too large Peclet or magnetic Reynolds numbers and slow variations of the fluid velocity in time.

- Mean–field diffusivities for passive scalar and magnetic case agree in SOCA but are different beyond this approximation.

- In isotropic turbulence the considered effect of compressibility is larger in the magnetic rather than the passive scalar case.