BPS, DUALITY, AND THE HYDROGEN ATOM

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HWS

KITP
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BPS

Bogomol'nyi
Prasad
Sommerfield

TOPOLOGICAL FIELD CONFIG.

BPS bound: $E \geq |G|$ (2nd order eqn)

$E = G$: 1st order eqn
BPS eqn.

[Z Hlousek & DS, Nuc Phys B397]

DUALITY

$g \leftrightarrow 1/g$

$R \leftrightarrow 1/R$

electric $\leftrightarrow$ magnetic

QFT
Strings
Branes

H ATOM

needs no introduction

stand-in for exactly solvable models

TO ANSWER THESE QUESTIONS:

SUPERSYMMETRIC QUANTUM MECHANICS with CENTRAL CHARGE

(SQMCC)

WHAT IS SIMPLEST EXAMPLE

OF TARGET SPACE DUALITY?

WHAT IS SIMPLEST EXAMPLE

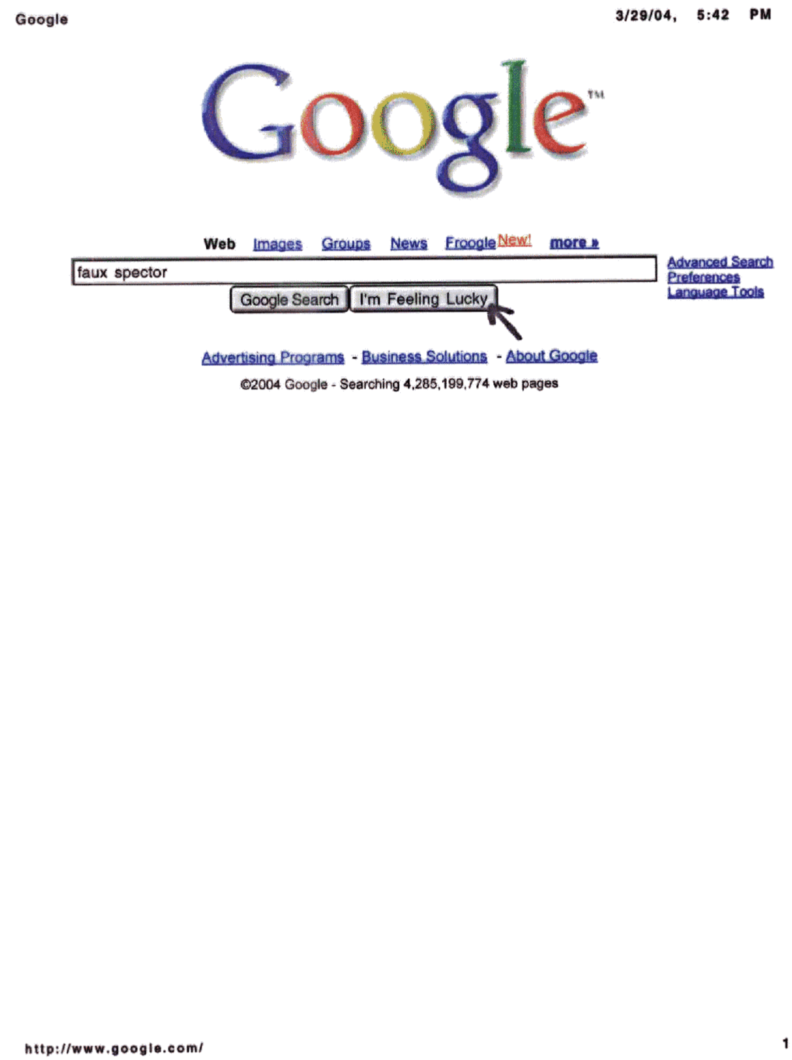
OF BPS?

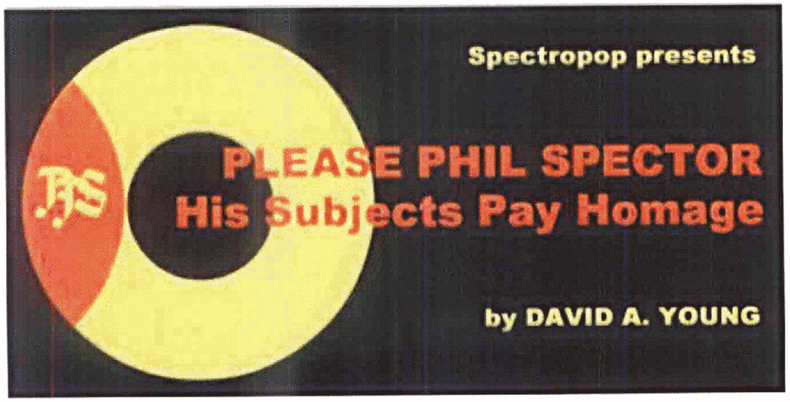
Michael Faux + DS - Phys Rev D70 (2004) 085014

J Phys A 37 (2004) 10397

Michael Faux, David Kagan, DS hep-th/0406152

D.S. arXiv:0707.1028





The case that **PHIL SPECTOR** has made a deep and permanent mark on pop music scarcely needs to be made. It's a given, as evidenced every time someone records a cover version of one of his classic songs or tries to duplicate his trademark Wall Of Sound in the studio. His legacy earned him an induction into the Rock and Roll Hall of Fame (in the 'non-performer' category) in the Hall's fourth year, and it's probably safe to say that he has greater name recognition, among musical cognoscenti and the general public alike, than any other record producer. In addition, his reputation as an eccentric recluse is almost as legendary as his indelible stamp on the world of rock.

Another measure of the impact he's had, on music in particular and on pop culture in general, is the number of times other artists have included references to him and his music in their own works and/or the packaging of same. This work is intended as a guide to such tributes, and is sorted by category.

In order to limit the scope of the project to a manageable size, I have chosen not to include cover versions of Spector songs. Parodies and answer records *do* qualify by virtue of the extent to which the lyrics are altered, although 'gender-switched' covers, consisting only of pronoun substitution, are not considered answer records for the purposes of this discussion.

Also not included here are the many faux-Spector productions that, with wildly varying degrees of success, have attempted through the years to capture his signature sound. Either of the above excluded categories would require an entire in-depth Web site unto itself, each with literally thousands of citations.

Part 1: HONORABLE MENTION
Songs about or including lyrical references to Phil Spector

Part 2: QUOTE UNQUOTE
Songs that include lyrics originally found in Spector records

Part 3: IT SAYS HERE . . .
The category for printed allusions to Spector

SUPERSYMMETRY

$\uparrow Q^\dagger$

$\downarrow Q$

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \{Q^\dagger, Q\}$$

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

$$Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}$$

EXACTLY SOLVABLE MODELS & SQM C.C.

SUSY QM

$H_1(g) = A^\dagger(g)A(g)$

$d=1$, NRQM

$E_{\text{ground}} = 0$

- $H_1 \geq 0$
- $H_1\psi = 0 \Leftrightarrow A(g)\psi(x;g) = 0$

Suppose $H_1\tilde{\psi} = E\tilde{\psi}$, $E > 0$

Define $H_2(g) = A(g)A^\dagger(g)$

Then

$H_2(A\tilde{\psi}) = A[A^\dagger A\tilde{\psi}] = E(A\tilde{\psi})$

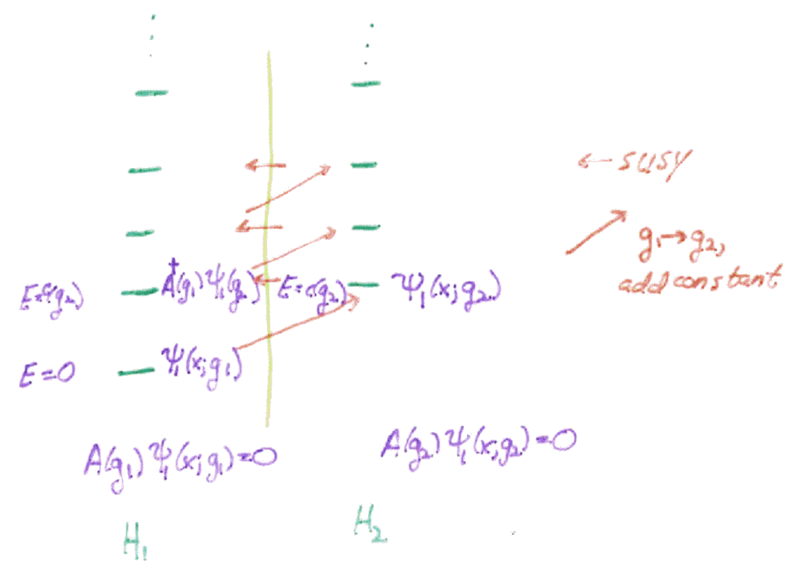
SHAPE INVARIANCE

$H_1 = A^\dagger(g_1)A(g_1)$

$H_2 = A(g_1)A^\dagger(g_1) = A^\dagger(g_2)A(g_2) + c(g_2)$ ← c-number

$f: g_1 \to g_2$

H_1 & H_2 connected two ways: (1) susy
(2) $g_1 \to g_2$, add $c(g_2)$



EXAMPLE: $V(x) = \dfrac{b}{\cosh^2 x}$

$$A = \frac{d}{dx} + g \tanh x$$

$$H_1(g) = A^\dagger A = -\frac{d^2}{dx^2} - g(g+1) \cdot \frac{1}{\cosh^2 x} + g^2$$

$$H_2(g) = A A^\dagger = -\frac{d^2}{dx^2} - g(g-1) \cdot \frac{1}{\cosh^2 x} + g^2$$

so!

$$H_2(g) = H_1(g-1) + 2g - 1$$

Ground state of H_j:

$A(g_j)\,\psi = 0$

$\vdots$

ψ_3, E_3

$\uparrow A^\dagger(g_2)$

ψ_2, E_2

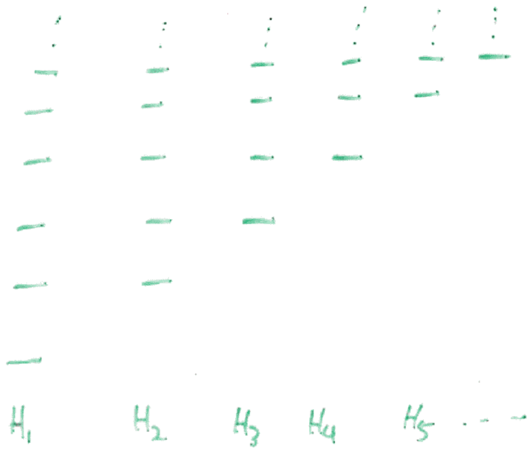
$\uparrow A^\dagger(g_1)$

ψ_1, E_1

STATES OF H_1: $E_j = c(g_j) + c(g_{j-1}) + \cdots + c(g_2) + c(g_1)$

$\psi_j = A^\dagger(g_1) A^\dagger(g_2) \cdots A^\dagger(g_{j-1})\, \psi_1(g_j)$

ITERATE!



$$H_1 = A^\dagger(g_1)A(g_1)$$
$$H_2 = A(g_1)A^\dagger(g_1) = A^\dagger(g_2)A(g_2) + c(g_2)$$
$$H_3 = A(g_2)A^\dagger(g_2) + c(g_2) = A^\dagger(g_3)A(g_3) + c(g_2) + c(g_3)$$
$$\vdots$$

SUSY Q.M. WITH CENTRAL CHARGE

$$\{Q^\dagger, Q\} = H \qquad \{Q,Q\} = Z = \{Q^\dagger, Q^\dagger\}$$
$$[H,Q] = [H,Q^\dagger] = 0 \qquad (Z \text{ real})$$
$$\Rightarrow [Q,Z] = [Q^\dagger, Z] = 0$$
$$H \geq |Z|$$

A realization:

$$Q = \begin{pmatrix} -\eta & 0 \\ A & \eta \end{pmatrix} \quad Q^\dagger = \begin{pmatrix} -\eta & A^\dagger \\ 0 & \eta \end{pmatrix} \quad \eta \in \mathbb{R}$$

$$H = \begin{pmatrix} A^\dagger A + 2\eta^2 & 0 \\ 0 & AA^\dagger + 2\eta^2 \end{pmatrix} \quad Z = \begin{pmatrix} 2\eta^2 & 0 \\ 0 & 2\eta^2 \end{pmatrix}$$

SWAP SECTORS $\tilde{Q} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$ not Q!

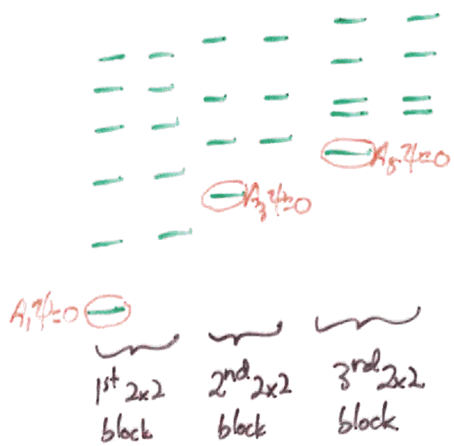
$$H = \{\tilde{Q}^\dagger, \tilde{Q}\} + Z$$

GENERALIZE TO MULTIPLE SECTORS

• PARTNERSHIPS

$$Q = \begin{pmatrix} -\eta_1 & 0 & & & & \\ A_1 & \eta_1 & & & & \\ & & -\eta_3 & 0 & & \\ & & A_3 & \eta_3 & & \\ & & & & -\eta_5 & 0 \\ & & & & A_5 & \eta_5 \end{pmatrix}$$

each 2x2 block is as in first realization



CARICATURE OF SHAPE INVAR.

PROPOSAL: SHAPE INV. is SQM C.C. plus...

How to Align Sectors Across Partnerships?

Recall: $\begin{pmatrix} 0 & 0 & & \\ A_1 & 0 & & \\ & & 0 & 0 \\ & & A_3 & 0 \end{pmatrix} = \tilde{Q}$ for swapping within partnership

so: Require $S = \begin{pmatrix} 0 & 0 & & \\ A_1 & 0 & 0 & \\ & C & 0 & 0 \\ & & A_3 & 0 \end{pmatrix}$ to satisfy $[H, S] = 0$

If this can be done, then the SQM cc spectrum $\longrightarrow$ Shape Invariance spectrum

When does such an S exist?

$S \equiv$ "shift operator"

$$S = \begin{pmatrix} 0 & & & \\ A_1 & 0 & & \\ & C & 0 & \\ & & A_3 & 0 \end{pmatrix} \qquad [H, S] = 0$$

ACHIEVABLE WHEN:

- A_3 IS UNITARY TRANSF. OF A_1 ↳ in parameter space
- Choose C to be "unitary mean" of A_1 & A_3

$$C = \mathcal{U}^{\dagger} A_1 \mathcal{U}, \quad A_3 = \mathcal{U}^{\dagger} C \mathcal{U}$$

Let $\tilde{A}_1 = A_1 \mathcal{U}$, $\tilde{\eta} = \mathcal{U} \eta \mathcal{U}^{\dagger}$

$$[H, S] = 0 \Rightarrow \{\tilde{A}_1, [\tilde{A}_1, \tilde{A}_1^{\dagger}]\} = 2(\tilde{\eta}_3^2 - \tilde{\eta}_1^2)\tilde{A}_1$$

Works for

- $[\tilde{A}_1, \tilde{A}_1^{\dagger}] = k$

or: $A_1(g) A_1^{\dagger}(g) = A_1^{\dagger}(g') A_1(g') + k$ SHAPE INV. CONDITION!

WHEN A_1 satisfies Shape Inv condition, it can be embedded in SQM C.C. Model with shift operator!

$$2\eta_3^2 = 2\eta_1^2 + k + \mathcal{U}^{\dagger} k \mathcal{U}$$

BPS plus

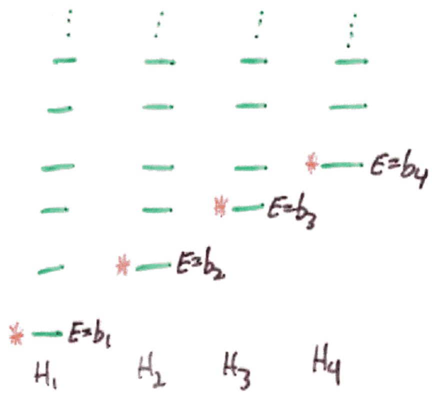
. . . . $E_4 = b_4$ $E_3 = b_3$ $E_2 = b_2$ $E_1 = b_1$

$$S = \begin{pmatrix} 0 & & & \\ A_1 & 0 & & \\ & U^\dagger A_1 U & 0 & \\ & & U^{\dagger 2} A_1 U^2 & 0 \\ & & & & \ddots \end{pmatrix}$$

conserved Shift Operator

Then: $H = S^\dagger S + B$, $B = \begin{pmatrix} 2m_1^2 & & & \\ & 2m_1^2 + k & & \\ & & 2m_1^2 + k + U^\dagger k U & \\ & & & \ddots \end{pmatrix}$

$H \geq B$, $H = B \Leftrightarrow S\psi = 0$ BPS



$* \; S\psi = 0$

$$H = \begin{pmatrix} H_1 & & \\ & H_2 & \\ & & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & & \\ & b_2 & \\ & & \ddots \end{pmatrix}$$

SHAPE INVARIANCE:

- SQM with Central Charge plus conserved Shift Operator
- Shift operator is like Runge-Lenz
- BPS interpretation of Shape Inv.
- Beyond BPS: Every state degenerate with a BPS state

? Topology ?

? Geometry ?

? Shape Invariance elsewhere?

RADIAL EQN FOR H ATOM VIA SHAPE INVARIANCE

(Angmom is $\ell-1$)

BAD (Theorists' Units)

Let $A = -\frac{d}{dr} + \frac{\ell}{r} + \frac{b}{\ell}$, $A^\dagger = \frac{d}{dr} + \frac{\ell}{r} + \frac{b}{\ell}$

$$H_1(\ell,b) = A^\dagger A = -\frac{d^2}{dr^2} + \frac{\ell(\ell-1)}{r^2} + \frac{2b}{r} + \frac{b^2}{\ell^2}$$

$$H_2(\ell,b) = AA^\dagger = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + \frac{2b}{r} + \frac{b^2}{\ell^2}$$

SO:

$$H_2(\ell,b) = H_1(\ell+1,b) + \underbrace{\frac{b^2}{\ell^2} - \frac{b^2}{(\ell+1)^2}}$$

Lyman, Balmer, Paschen, Brackett, etc.

SIMPLEST CASE OF BPS (in fact, enhanced BPS)!

SUPERFIELD APPROACH

QUANTUM MECHANICS AS 0+1 DIM. FIELD THEORY

SUPERSYMMETRIC Q.M.

$V(t,\theta,\bar\theta)$ superfield

T	bosonic	$\delta T = i\epsilon\chi + i\epsilon^\dagger\chi^\dagger$
$\chi, \chi^\dagger$	fermionic	$\delta\chi = \epsilon^\dagger(\dot T + iB)$
B	auxiliary	$\delta B = \epsilon\dot\chi - \epsilon^\dagger\dot\chi^\dagger$

$$S = \int dt\, d\bar\theta\, d\theta\, G_{ij}(V) D^\dagger V^i D V^j$$

$$D = \frac{\partial}{\partial\theta} + i\theta^\dagger\partial_t$$

This realizes susy

$\{Q^\dagger, Q\} = H \quad \{Q,Q\} = 0 = \{Q^\dagger, Q^\dagger\}$

$\boxed{\text{CENTRAL CHARGE?}}$

$\{Q^\dagger, Q\} = H \qquad \{Q, Q\} = Z = \{Q^\dagger, Q^\dagger\}$

MODIFIED SUSY TRANSF:

$\delta T_1 = i\epsilon\chi_1 + i\epsilon^\dagger\chi_1^\dagger \qquad \delta T_2 = i\epsilon\chi_2 + i\epsilon^\dagger\chi_2^\dagger$

$\delta\chi_1 = \epsilon^\dagger(\dot{T}_1 + iB_1) \qquad \delta\chi_2 = \epsilon^\dagger(\dot{T}_2 + iB_2) + \mu\epsilon$

$\delta B_1 = \epsilon\dot{\chi}_1 - \epsilon^\dagger\dot{\chi}_1^\dagger \qquad \delta B_2 = \epsilon\dot{\chi}_2 - \epsilon^\dagger\dot{\chi}_2^\dagger$

μ: real parameter

$\delta_z T_1 = 0 \ , \ \delta_z T_2 = \mu$

WE'VE REALIZED SQM with central charge

BUT:

$S = \int dt\, d\theta^\dagger d\theta\, G_{ij}(V) D^\dagger V^i D V^j$

IS NO LONGER SUSY-INVARIANT!

TO APPROACH THIS:

$\delta S_0 = 0$ UNDER ORDINARY SUSY

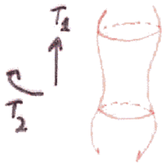
$= \mu \cdot (\ldots)$ UNDER MODIFIED SUSY

find S_1:

$\delta[S_0 + \mu S_1] = \mu^2 \times (\ldots)$ UNDER MODIFIED SUSY

$S_0 + \mu S_1 + \mu^2 S_2$ etc. till $\delta[S_0 + \cdots + \mu^k S_k] = 0$.

SPECIAL CASE: $G_{ij} = (1, h(T_1))$



Radius $R(T_1) = \sqrt{h(T_1)}$

$$L = L_0 + \mu L_1 + \mu^2 L_2 \quad \text{suffices!}$$

$$L_0 = \tfrac{1}{2}\dot{T}_1^2 - \tfrac{i}{2}\chi_1^\dagger \overleftrightarrow{\partial_t}\chi_1 + \tfrac{1}{2}B_1^2 + h(T_1)\left\{\tfrac{1}{2}\dot{T}_2^2 - \tfrac{i}{2}\chi_2^\dagger\overleftrightarrow{\partial_t}\chi_2 + \tfrac{1}{2}B_2^2\right\}$$
$$+ i h'(T_1)\chi_1^\dagger \chi_2 \dot{T}_2 - \tfrac{1}{2}h''(T_1)\chi_2^\dagger\chi_2\chi_1^\dagger\chi_1$$
$$+ i h'(T_1)(\chi_1^\dagger\chi_2 B_2 + \chi_2^\dagger\chi_1 B_2 - \chi_2^\dagger\chi_2 B_1)$$

$$L_1 = -\tfrac{i}{2}h'(T_1)(\chi_1\chi_2 + \chi_1^\dagger\chi_2^\dagger)$$

$$L_2 = -\tfrac{1}{2}h(T_1)$$

NOT SO BAD!

(Superfield approach to centrally extended superalgebras)

NOW: QUANTIZE

$$Q = P_1\chi_1 + P_2\chi_2 + \tfrac{i}{2}h' \underbrace{:\chi_2\chi_2^\dagger\chi_1:}_{\text{ordering}} + \mu h \chi_2^\dagger$$

$$Z = \mu P_2$$

$$H = \tfrac{1}{2}P_1^2 + \tfrac{1}{2h}P_2^2 - i\tfrac{h'}{h}P_2(\chi_1^\dagger\chi_2 - \chi_2^\dagger\chi_1) + \tfrac{1}{2}:h\mathcal{R}\chi_1^\dagger\chi_1\chi_2^\dagger\chi_2:$$
$$+ \tfrac{i}{2}\mu h'(\chi_1\chi_2 + \chi_1^\dagger\chi_2^\dagger) + \tfrac{1}{2}\mu^2 h$$

$$\mathcal{R} = \tfrac{1}{2}\left(\tfrac{h'}{h}\right)^2 - \tfrac{h''}{h}$$

SO WHAT?

ANSWER: DUALITY!

- Fix SECTOR $P_2 = \nu \quad \nu \in \mathbb{Z}$
- FIND

$$H = \begin{pmatrix} A_+^\dagger A_+ + \mu\nu & & & \\ & A_+ A_+^\dagger + \mu\nu & & \\ & & A_-^\dagger A_- + \mu\nu & \\ & & & A_- A_-^\dagger + \mu\nu \end{pmatrix}$$

where $A_\pm = \frac{\partial}{\partial T_1} + \omega_\pm(T_1)$

familiar form!

$$\omega_\pm = -\frac{1}{2}\frac{R'}{R} \pm \left(\frac{\nu}{R} - \mu R\right)$$

INVARIANCE:

$R \to \frac{1}{R}$

$\mu \leftrightarrow \nu$

$T_1 \to -T_1$

$R(T_1)$ even: $A_\pm \to -A_\pm$

$R(T_1)$ odd: $A_\pm \to A_\mp$

(Generally $H \to \Omega^\dagger H \Omega$)

TARGET SPACE DUALITY!

MORE PRECISELY!

$P_2 = \nu$ sector at given μ $\leftrightarrow$ $P_2 = \mu$ sector at given ν

- μ is arbitrary parameter
- $\nu \in \mathbb{Z}$, summed over

$\Rightarrow$ MASTER THEORY WITH μ SECTORS

QUANTIZED μ — ORIGIN?

TARGET SPACE DUALITY IN Q.M.

INTRIGUING ASPECTS:

- SHAPE INVARIANCE AS RESTRICTION ON GEOMETRY
- INDICATIONS OF "ELECTRIC/MAGNETIC"-TYPE DUALITIES

$$\delta_Q(\epsilon) X^n = i\epsilon\psi^n + i\epsilon^\dagger\psi_n^\dagger$$

$$\delta_Q(\epsilon)\psi^n = \epsilon^\dagger(\dot{X}^n + iB^n) + \epsilon f^n(X)$$

$$\delta_Q(\epsilon)\psi^{n\dagger} = \epsilon(\dot{X}^n - iB^n) + \epsilon^\dagger f^n(X)$$

$$\delta_Q(\epsilon) B^n = \epsilon\dot{\psi}^n - \epsilon^\dagger\dot{\psi}^{n\dagger} - \epsilon\partial_m f^n(X)\psi^{m\dagger} + \epsilon^\dagger\partial_m f^n(X)\psi^m$$

$$\boxed{\text{PHYSICS \& } f^n(X)}$$

Take $S_0 = \frac{1}{2}\int dt\, d\theta\, d\theta^\dagger g_{mn}(V)\, D^\dagger V^m D V^n$

$$\delta_Q(\epsilon) S_0 = \delta^0_{Q(\epsilon)} S_0 + \underline{\delta'_{Q(\epsilon)} S_0}$$

$$\delta'_Q(\epsilon) S_0 = \int dt\, d\theta\, d\theta^\dagger\, \tfrac{i}{2} g_{mn}(V)\,(i\epsilon^\dagger f^m(V) D V^n + h.c.)$$

$$+ \int dt\, d\theta\, d\theta^\dagger\,(i\theta\epsilon + i\theta^\dagger\epsilon^\dagger)\,\Omega_{mn}(V)\, D^\dagger V^m D V^n$$

where $\Omega_{mn} = \nabla_{(m} f_{n)}$ ∇_m: target space cov. derivative

To make this invariant under susy w/central charge:

$\Omega_{mn} = 0$ $\left[\Omega_{mn} = \nabla_{(m} f_{n)}\right]$ Killing Eqn!

so: $f^n(X)$ from $\delta_Z X^n = f^n(X)$

corresponds to an isometry!

Then can construct $S = S_0 + S_1 + S_2$ that is invariant!

Appears that SQM with Central Charge

requires Central Charge $\Longleftrightarrow$ Isometry of target space

Conclusions, Prospects, etc.

- Susy QM with Central Charge is a <u>RICH</u> subject
- Underlying Structure of Shape Invariance (SQM with CC plus shift operator)
- Enhanced BPS Structure & Interpretation
- SQM with C.C as Field Theory
- Prescription for generating models
- Emergence of Target Space Duality in QM!
- $\mathbb{R}\times T^p$ models including an $\mathbb{R}\times T^2$ with Modular Transf.
- Isometry $\leftrightarrow$ Central Charges

To Do:
- Understand geometry/topology of Shape Invariance
- Explore other field content, target spaces
- More fundamental picture of dualities
- Connection to higher-dim, string, M-Theory ...