

Motivation: zonal jets slow evolution and rare transitions

**ZONAL JETS VELOCITY PROFILES**

Zonal jets are observed in upper planetary atmospheres (Earth, Jupiter, Saturn, ...).

![Left: snapshot of Jovian upper atmosphere. Right: zonally averaged velocity profile obtained from the data analysis of the fly-by of Cassini (2000) and Voyager (1979). Figures are taken from [1], see also [2].](image)

They evolve very slowly with respect to turbulent fluctuations and most of the energy is in zonal degrees of freedoms (about 95% on Jupiter, [2]).

How to predict these velocity profiles?

**MULTIPLE ATTRAJECTORS AND BISTABILITY**

Real data, experiments and DNS reveal the possibility of **sporadic transitions** among different attractors after a long period of apparent stability. This behaviour is observed in many geophysical flows and turbulence problems, for example the Earth magnetic field reversal, paths of oceanic currents, atmospheric flows.

Experimental idealisation [3] of the Earth polar jet stream in a rotating tank with topography. Left: blocked and zonal states. Right: Rare transitions showing bistability with different average residence times and fluctuations properties around the attractors.

How to predict the statistical properties of rare transitions?

**average residence time, fluctuations properties**

Stochastic barotropic equation on a beta-plane

Simplest model for mid-latitude large-scale dynamics

\[ \partial_t \mathbf{q} + \mathbf{v} \cdot \nabla \mathbf{q} = -f \mathbf{v} + \sqrt{\mathcal{L}} \mathbf{q}, \quad D = \mathbb{R}^2 \times [0,2\pi], \mathcal{L} = \partial_t^2 - \partial_y^2 \]

The stochastic forcing \( \sqrt{\mathcal{L}} \) represents the effect of unresolved features of the model (radiation, baroclinic instability, effect of different layers, ...). \( \mathcal{L} \) is Gaussian with correlation \( \mathbb{E}(\mathbf{q}(t), \mathbf{q}(t')) = \mathcal{C}(t-t') \).

**NON-DIMENSIONAL UNITS**

Denoting by \( \mathbb{E} \) the energy injection per unit of mass, we have the following typical values:

\( v = \mathcal{L}^{-1/2}, \mathcal{T} = 1/\mathcal{L}^{-1} \), time-scale of the zonal jet: \( \mathcal{T}_m = U/L \).

We rescale barotropic equations setting to unity \( \mathcal{T}_m \) and \( L \):

\[ \partial_t \mathbf{q} + \mathbf{v} \cdot \nabla \mathbf{q} = -f \mathbf{v} + \sqrt{\mathcal{L}} \mathbf{q}, \quad \mathcal{L} = \partial_t^2 - \partial_y^2 \]

**INERTIAL LIMIT:** \( \mathcal{T} \ll \mathcal{T}_m \). Slow evolution of the zonal jet with respect to the evolution time-scale of turbulent fluctuations.

Zonal degrees of freedom, stochastic averaging

To extract zonal degrees of freedom: \( f_s(y,t) = \langle f \rangle = \int \frac{d^2 y}{2\pi} f(y,t) \)

\( q_s(y,t) = \langle q(y,t) + \sqrt{\omega_m} \rangle = U/L + \sqrt{\omega_m} \mathbb{E} \)

**EXACT DYNAMICS IN TERMS OF** \( \mathcal{U} \) **AND** \( \omega_m \)

\[ \partial_t \mathcal{U} + \mathbb{F}(\mathcal{U}) = \mathcal{U} - \alpha \mathcal{U} + \sqrt{\omega_m} \mathbb{E} \]

\[ \alpha \mathbb{F}(\mathcal{U}) + \mathbb{L}[\omega_m] = 0 \]

advection operator linearized around \( \mathcal{U} \)

**STOCHASTIC AVERAGING**

We aim at a description of the evolution of \( \mathcal{U} \) tracing out the turbulent fluctuations contained in \( \omega_m \). We employ a perturbative expansion in the INERTIAL LIMIT called in literature stochastic averaging. This is a well known technique in mathematics [4] and very similar to kinetic theories for plasma and self-gravitating systems [5,6]. The technique is applied to the functional Fokker-Planck associated to the equations for \( \mathcal{U} \) and \( \omega_m \) above [7].

**Effective dynamics at order** \( \alpha \)

\[ \begin{cases} \partial_t \mathcal{U} + \alpha \mathbb{F}(\mathcal{U}) = \mathcal{U} - \alpha \mathcal{U} + \sqrt{\omega_m} \mathbb{E} \\ \alpha \mathbb{F}(\mathcal{U}) + \mathbb{L}[\omega_m] = 0 \end{cases} \]

- absence of eddy-eddy interactions (quasi-linear theory)
- Reynolds’ stresses suitably averaged through \( \mathbb{F} \):

\[ \mathbb{F}(\mathcal{U}) = \mathbb{E}(\mathcal{U}^2 - \langle \mathcal{U} \rangle^2) = \mathbb{E} \]

Our effective equations are similar to others already introduced phenomenologically [8,9,10] going under the names of S3T (Farrell, Ioannou, et al.), CE2 (Marston, Schneider, Young, et al.), quasi-linear theory. However:

- our equations are obtained from first principles
- the difference mainly stands in how the average of Reynolds’ stresses is defined

The theory is self-consistent

A perturbative expansion in an infinite dimensional system may suffer of ultraviolet divergences which would invalidate the development.

\[ \lim_{\mathcal{T} \to \infty} \mathbb{E}(\mathcal{U}(\mathcal{T})) < \infty \quad \text{YESS if} \quad \mathcal{U} \] has no unstable nor neutral modes

The average Reynolds’ stresses can be written through the deterministic evolution of \( \omega_m \) with appropriate initial conditions (that depends on \( C(r-t') \)):

\[ \mathbb{E}(\mathcal{U}(\mathcal{T})) = \int_{\mathbb{R}} \mathbb{E}(\mathcal{U}(\mathcal{T})) d\mathcal{T} \]

**ORR MECHANISM**

For large times, vorticity creates finer and finer structures, but velocity decays to zero!

Vorticity evolution: \( \mathcal{U}(\mathcal{T}) = \alpha \mathcal{U} + \beta \mathcal{U} \)

Proven in [11] for any base flow \( \mathcal{U}(\mathcal{T}) \) and \( \beta = 0 \)

Comparison with DNS (figures taken from [9])

The theory should exactly predict the profile \( \mathcal{U}(\mathcal{T}) \) in the inertial limit. In literature [8,9] there are good hints. Further tests are necessary

**Bistability: effective dynamics at order** \( \alpha^2 \)

Multiple attractors are sometimes found between different jets configurations

• same physical parameters, different initial condition.
  Figure taken from [12]

A single long run will put in evidence bistability between these two attractors

**EFFECTIVE DYNAMICS AT ORDER** \( \alpha^2 \)

The stochastic averaging can be carried on at order \( \alpha^2 \)

\[ \partial_t \mathbb{E}(\mathcal{U}) = \alpha \mathbb{F}(\mathcal{U}) = \mathbb{E}(\mathcal{U}) + \mathbb{E}(\mathcal{F}(\mathcal{U})) \]

Deterministic \( \sim \alpha \)

Stochastic \( \sim \alpha^2 \)

\( \mathbb{F}(\mathcal{U}) \) is a zero average Gaussian noise whose correlation can be explicitly computed

\[ \mathbb{E}(\mathcal{F}(\mathcal{U})) = \mathbb{E}(\mathcal{U}) - \alpha \mathbb{E}(\mathcal{F}(\mathcal{U})) \]

\[ \mathbb{E}(\mathcal{F}(\mathcal{U})) = \alpha^2 \mathbb{E}(\mathcal{U}) \]

Work in progress: apply large deviations techniques to the effective evolution to explain statistical properties of bistability

CONCLUSIONS

- effective dynamics of zonal jet velocity profile through a perturbative approach
- very good (exact!) predictions in the inertial limit \( \alpha \ll 1 \)

PERSPECTIVES

- bistability: large deviations applied to the effective evolution
- effective evolution in presence of neutral modes
- precise numerical validation of the theory

References


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STOCHASTIC AVERAGING, JET FORMATION AND BISTABILITY IN TURBULENT PLANETARY ATMOSPHERES

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