Angular distribution of energy spectrum in two-dimensional β-plane turbulence in the long-wave limit
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Abstract
The time-evolution of two-dimensional decaying turbulence governed by the long-wave limit, in which $L/\lambda = 0$, of the quasi-geostrophic equation is investigated numerically. Here, $L_0$ is the Rossby radius of deformation, and $L$ is the characteristic length scale of the flow. As the degree of nonlinearity decreases, more energy accumulates in a wedge-shaped region [where $|l| > \sqrt{3}|k|$] in the two-dimensional wavenumber space. Here, $k$ and $l$ are the longitudinal and latitudinal wavenumbers, respectively. When the degree of nonlinearity is decreased further, energy concentrates on the lines of $l \approx \pm 3k$. These results are interpreted based on the conservation of zonostrophy, which is an extra invariant other than energy and enstrophy and was determined in a previous study. Considerations concerning the appropriate form of zonostrophy for the long-wave limit and a discussion of the possible relevance to Rossby waves in the ocean are also presented.

1. Two-dimensional turbulence on a β-plane

- governed by Quasi-geostrophic equation.
- $\psi$: stream function
- $\beta$: deformation radius
- $\gamma$: y-derivative of Coriolis parameter
- tends to have a zonally elongated structure.
  - Effect of β-term
  - prevents upscale energy cascade.
  - makes energy cascade anisotropic (Rhines effect).
  - becomes prominent when $L - L_0$.
- Zonostrophy
- used to explain anisotropic energy cascade (Balk, 2005).
- conserved when the nonlinearity is sufficiently weak.
- named by Nazarenko and Quinn (2009).

2. Purpose of this study

- Various parameter regimes
  - There are six combinations of the three length scales $L$, $L_0$, and $L_0$ (Table 1).
  - Regimes (5) and (6) ($L_0 < L_0$) are not well-studied.
- We consider the long-wave limit ($L/\lambda \rightarrow 0$) as an extreme case and:
  1. derive the asymptotic form of zonostrophy.
  2. investigate the anisotropy of energy cascade and check the conservation of zonostrophy (by numerical experiments).

3. Long-wave limit

- Long-wave limit of quasi-geostrophic equation
  - Consider the quasi-geostrophic equation:
  - In the long-wave limit ($L/\lambda \rightarrow 0$), the operator in the first term expands as follows:
  - After a proper Galilean transform and nondimensionalization, we get:
  - Long-wave limit of QG equation
  - Long-wave limit of zonostrophy

4. Numerical experiments

- Experimental setup
  - Governing equation:
    - Long-wave limit of QG eq. + hyper viscosity
    - (Freely decaying turbulence)
  - Results
    - 2D energy spectrum (average of 41 ensemble members) and their angular distribution.

5. Summary

- In the present study, we studied 2D turbulence governed by the long-wave limit of quasi-geostrophic equation and revealed:
  - (1) The asymptotic form of zonostrophy.
  - (by numerical experiments)
  - (2) The relationship between the parameter $\gamma$ and anisotropy of 2D energy spectrum.
  - $\gamma > 1$: Accumulation in the wedge-shaped region $W$.
  - $\gamma \geq 1$: Concentration around $\theta = 60^\circ$.

6. Discussion

- 1st baroclinic Rossby waves in the mid-latitude ocean
  - wave length: several hundreds – thousand (km)
  - characteristic velocity scale: $U = 1$ (cm/s)
  - observed in the mid-latitude ocean ($L_0$ is several tens (km))
  - their wavenumber vector typically deviate from the zonal by 50°– 80°
  - For example, for 25°N latitude, $|l/|U| = 5.8$ (cm/s) and we get:
  - $\beta \sim \frac{|l/|U|}{L_0} \sim \frac{|l/|U|}{L_0} = 0$, which satisfies the condition for anisotropy ($\gamma > 1$).

References


Table 1

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Regime</th>
<th>Dimension</th>
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<td>$0 &lt; \gamma &lt; 0.25$</td>
<td>L &lt; L_0 &lt; L_0</td>
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<td>$\gamma = 1$</td>
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<td>$\gamma = 5$</td>
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Time-evolution of zonostrophy (Z) for five $\gamma$.

Clear contrast across the boundary ($\theta = 60^\circ$).

<table>
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<tr>
<th>Parameter</th>
<th>Domain</th>
<th>Zonostrophy parameter</th>
<th>Quasi-geostrophic parameter</th>
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References