Direct Statistical Simulation of Flows by Expansions in Cumulants

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ABSTRACT

Low-order statistics of model geophysical and astrophysical fluids may be directly accessed by solving the equations of motion for the statistics themselves. We implement such Direct Statistical Simulation by systematic expansion in equal-time cumulants. The approach is illustrated on the sphere by a barotropic model of jet formation.

The first cumulant is the zonally averaged vorticity as a function of latitude, and the second and higher cumulants encode information about nonlocal teleconnections. Closure of the equations of motion at second order (CE2) is realizable and retains the eddy -- mean-flow interactions, but neglects eddy-eddy interactions. Eddy-eddy interactions appear at third (CE3) order, but care must be taken to maintain realizability with a non-negative probability distribution function. The cumulant expansion is conservative, order-by-order, in the absence of forcing and dissipation: CE2 conserves the total angular momentum, total energy, and enstrophy; CE3 further conserves the cubic vorticity Casimir, and CE-N conserves N Casimirs. An intermediate approximation, CE2.5, is related to the Eddy-Damped Quasi-Normal Markovian (EDQNM) approximation and maintains realizability at the expense of the introduction of a phenomenological timescale for eddy damping. First and second cumulants accumulated by direct numerical simulation are compared with those obtained at fixed points found at CE2, CE2.5, and CE3 levels of approximation. We demonstrate that CE2 reproduces qualitative features of the zonal jets, and CE2.5 and CE3 further improve quantitative agreement in both the zonal means, and in the teleconnections.

Direct Numerical Simulation (DNS) EOM for a Barotropic Model of Jet Formation

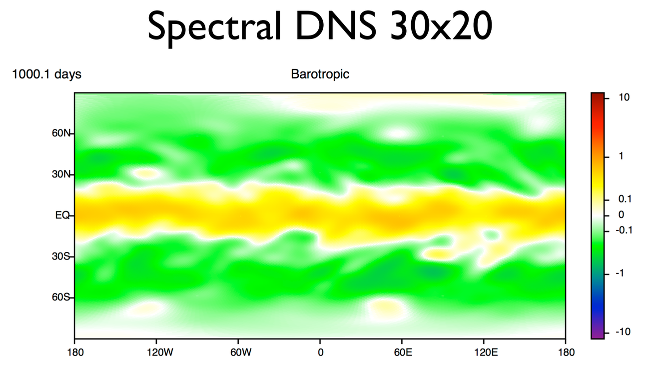
$$\dot{\zeta} = J[\zeta + f,\ \psi] - (\kappa + \nu_2 \nabla^4)\zeta + \eta(t)$$

Coriolis term $f \equiv 2\Omega \sin\phi$.

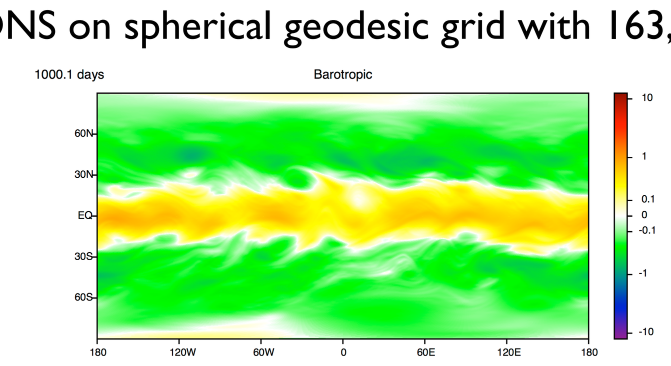
The fluid is driven by stochastic forcing $\eta(t)$ and spontaneously forms zonal jets.

Late-time instantaneous snapshots of the zonal velocity

Spectral DNS 30x20

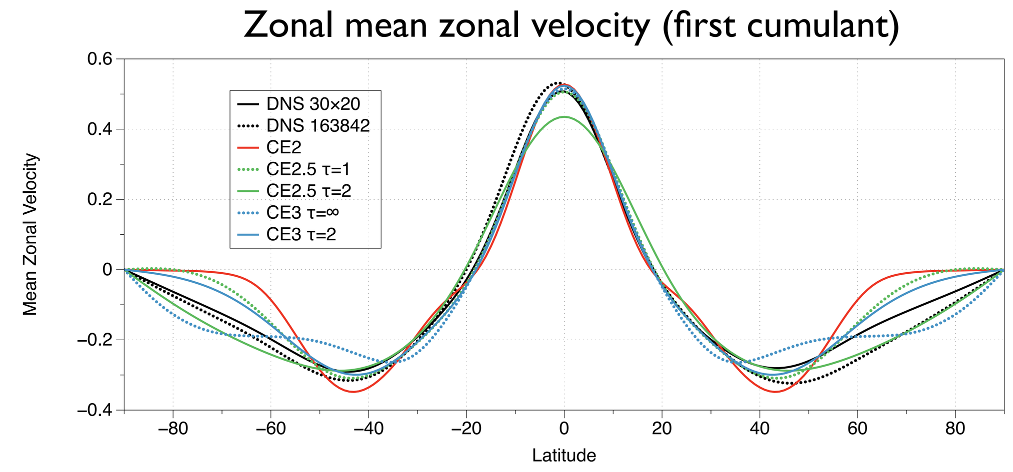


DNS on spherical geodesic grid with 163,842 cells



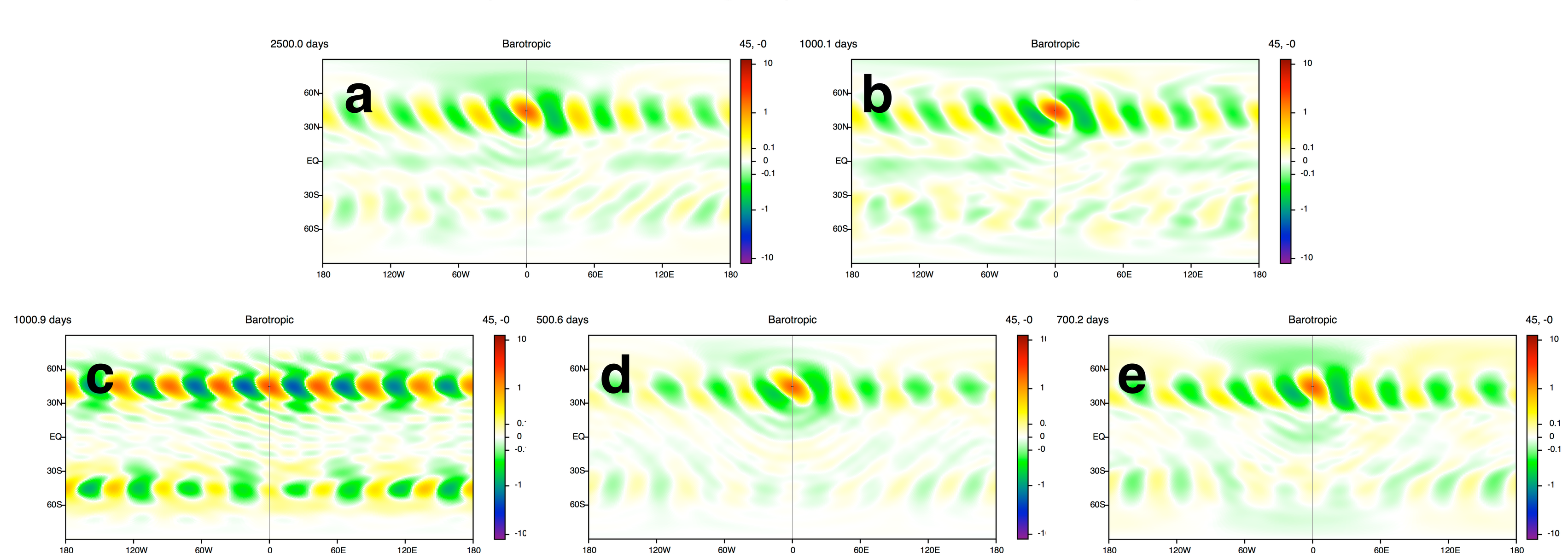
Comparison between DNS (spectral and real-space) and DSS (CE2, CE2.5 and CE3)

Zonal mean zonal velocity (first cumulant)

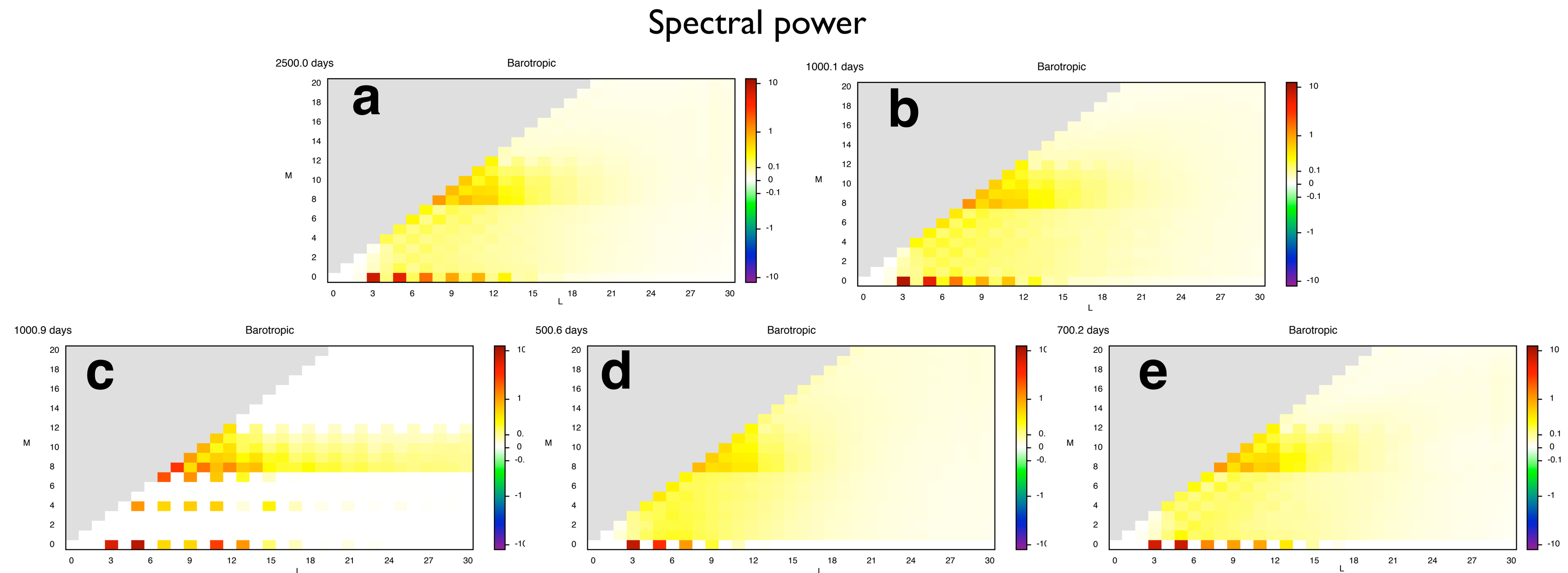


Two-point correlation function of the vorticity (second cumulant)

One point is centered along the prime meridian at latitude 45 degrees.



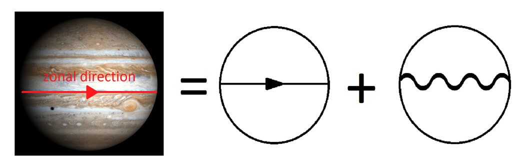
Spectral power



(a) Spectral DNS 30x30 (b) DNS 163842

(c) CE2 (d) CE2.5 with τ = 2 days (e) CE3 with τ = 2 days

Mean Flow and Eddy



Reynolds decomposition into mean flow $\langle\zeta\rangle$ and eddy ζ':

$$\zeta(\mathbf{r}, t) = \langle\zeta(\mathbf{r}, t)\rangle + \zeta'(\mathbf{r}, t),$$

with $\langle\zeta'(\mathbf{r}, t)\rangle = 0$.

$\langle\cdots\rangle$: spatial averaging over the coordinate along which the jet forms (the zonal direction).

Low-order Cumulants:

$$\begin{aligned} c_1(\mathbf{r}, t) &= \langle\zeta(\mathbf{r}, t)\rangle \\ c_2(\mathbf{r}_1, \mathbf{r}_2, t) &= \langle\zeta'(\mathbf{r}_1, t)\zeta'(\mathbf{r}_2, t)\rangle \\ c_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t) &= \langle\zeta'(\mathbf{r}_1, t)\zeta'(\mathbf{r}_2, t)\zeta'(\mathbf{r}_3, t)\rangle \end{aligned}$$

Direct Statistical Simulation (DSS) EOM and Closures

Closure Problem: EOM of cumulants involves an infinite hierarchy of coupled equations due to the nonlinear interaction.

$$\begin{aligned} \frac{\partial c_1}{\partial t} &= F_1(c_1, c_2), \\ \frac{\partial c_2}{\partial t} &= F_2(c_1, c_2, c_3), \\ \frac{\partial c_3}{\partial t} &= F_3(c_1, c_2, c_3, c_4), \\ &\cdots \end{aligned}$$

Closure assumptions are used to close the set of equations:

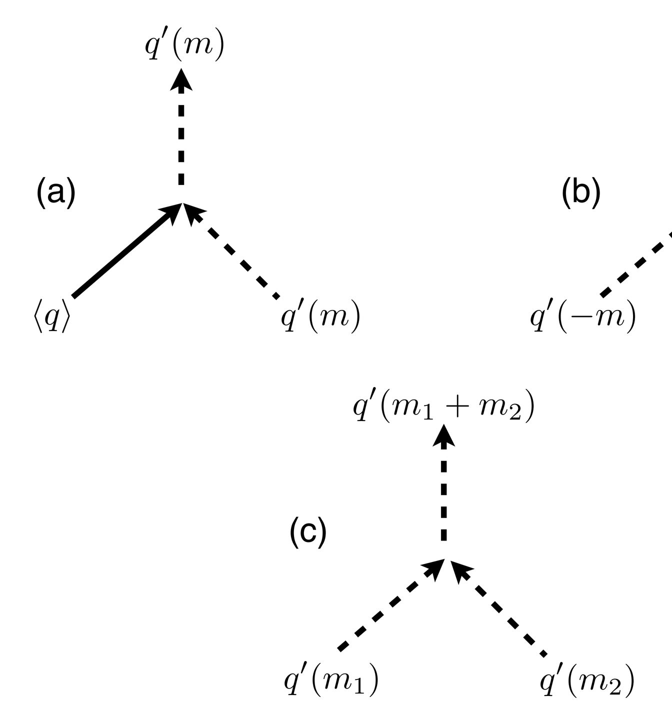
CE2: $c_3 = c_4 = \cdots = 0$;

CE3: $c_4 = c_5 = \cdots = 0$, and use a phenomenological eddy-damping term for $\dot{c}_3$ with timescale τ;

CE2.5: $c_4 = c_5 = \cdots = 0$, and use a diagnostic equation for $c_3 = \tau \cdot F(c_2)$ with a phenomenological eddy-damping timescale τ.

CE closures are nonlocal in space and make no assumption about the homogeneity or isotropy of the nonlinear interactions.

CE Closures and Triads



CE2 = Quasilinear Approximation = (a) + (b)

CE2.5 or CE3 = (a) + (b) + some (c)

CE Closures and Conservation Laws

No forcing, no dissipation: exact dynamics conserves total angular momentum L_z, total energy E, and an infinite hierarchy of Casimirs:

$$\Gamma_n \equiv \int d^2\mathbf{r}\ q^n,\ \ n = 1,\ 2,\ 3, \cdots,$$

where $q \equiv \zeta + f$.

Closure	Conservation Laws
Exact (No Closure)	$L_z,\ E,\ \Gamma_1,\ \Gamma_2, \cdots$
CE2, CE2.5, and CE3 with finite τ	$L_z,\ E,\ \Gamma_1,\ \Gamma_2$
CE3 (with infinite τ)	$L_z,\ E,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3$
CE-N	$L_z,\ E,\ \Gamma_1,\ \Gamma_2, \cdots,\ \Gamma_N$

Enstrophy $\Gamma_2 = F(c_1,\ c_2)$ (quadratic), so $d\Gamma_2/dt$ only depends on $\partial c_1/\partial t$ and $\partial c_2/\partial t$, not the higher ones. CE3 approximation does not modify the first two equations in the hierarchy of DSS EOM, so CE3 will not violate the enstrophy conservation. This reasoning directly shows that CE-N at least conserves up to Γ_{N-1}. Non-trivial calculation further shows that CE-N also conserves Γ_N.

DISCUSSION

The DSS approach focuses only on the slow modes of low-order statistics by integrating out the fast modes of high-order statistics. Compared to DNS where the statistics are accumulated, DSS can be significantly faster and can lead to improved understanding. The fixed point of the DSS EOM describes the steady-state equal-time statistics and may be accessed by other methods as well.

DSS only works for systems with non-trivial mean flows. These inhomogeneous and anisotropic flows are ubiquitous in planetary atmospheres, ocean, stars and accretion disks. CE-N Closures are nonlocal and are conservative order by order.

The accuracy of a CE-N description varies with the system. In the above example, DSS truncated at CE2 suffices to reproduce the qualitative feature of the zonal jets, but this depends on the values of the rotation rate, friction and the rms velocity. Tobias and Marston [1] demonstrated on the β-plane that CE2 gives an accurate description of the statistics when the system is close to equilibrium. However as the system is driven further from equilibrium, the jets are more intermittent and meander more, and the eddy-eddy interaction becomes more important. In that case CE2 even fails to predict the number and strength of the zonal jets. How to achieve the desired accuracy of DSS for certain types of systems is the topic of current research.

[1] S. M. Tobias and J. B. Marston, *Direct Statistical Simulation of Out-of-Equilibrium Jets*, PRL **110**, 104502 (2013)