Angular distribution of energy spectrum in two-dimensional β-plane turbulence in the long-wave limit

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Abstract

The time-evolution of two-dimensional decaying turbulence governed by the long-wave limit, in which $L_D/L \to 0$, of the quasi-geostrophic equation is investigated numerically. Here, L_D is the Rossby radius of deformation, and L is the characteristic length scale of the flow. As the degree of nonlinearity decreases, more energy accumulates in a wedge-shaped region where $|l| > \sqrt{3}|k|$ in the two-dimensional wavenumber space. Here, k and l are the longitudinal and latitudinal wavenumbers, respectively. When the degree of nonlinearity is decreased further, energy concentrates on the lines of $l = \pm\sqrt{3}k$. These results are interpreted based on the conservation of zonostrophy, which is an extra invariant other than energy and enstrophy and was determined in a previous study. Considerations concerning the appropriate form of zonostrophy for the long-wave limit and a discussion of the possible relevance to Rossby waves in the ocean are also presented.

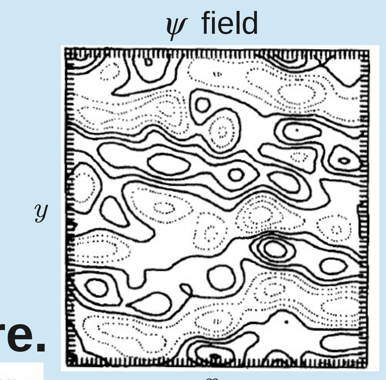
1. Two-dimensional turbulence on a β-plane

- governed by **Quasi-geostrophic equation.**

$$\frac{\partial}{\partial t}\left(\nabla^2\psi-\frac{1}{L_D^2}\psi\right)+\beta\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}=0$$

ψ : stream function L_D: deformation radius
β : y-derivative of Coriolis parameter

- tends to have a **zonally elongated structure.**

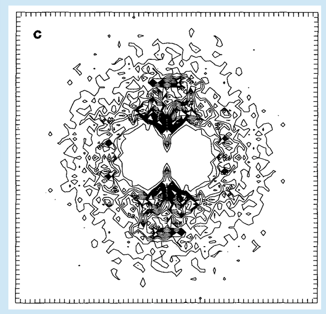


(Rhines, 1975)

$$\psi(x,y,t)=\frac{1}{2\pi}\int\hat{\psi}_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}\qquad \mathbf{k}=(k,l),\ \mathbf{x}=(x,y)$$

  - Effect of β-term
    - prevents upscale energy cascade.
    - makes energy cascade anisotropic (Rhines effect).
    - becomes prominent when $L \sim L_\beta$.
      L : length scale of the flow (U: velocity scale of the flow)
      L_β : **Rhines scale** $L_\beta = \sqrt{U/\beta}$

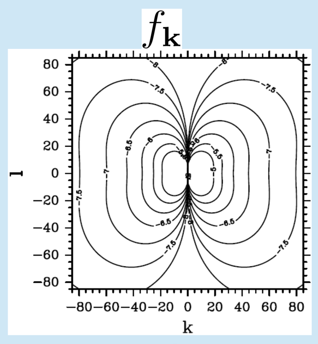
2D energy spectrum



(Vallis and Maltrud, 1993)

- **Zonostrophy**

$$Z=\int f_{\mathbf{k}}\varepsilon_{\mathbf{k}}\,d\mathbf{k}\qquad \begin{cases}\varepsilon_{\mathbf{k}}=\frac{1}{2}(|\mathbf{k}|^2+L_D^{-2})|\hat{\psi}_{\mathbf{k}}|^2 & \text{(: 2D energy spectrum)}\\ f_{\mathbf{k}}=\frac{|\mathbf{k}|^2+L_D^{-2}}{\beta k}\left(\tan^{-1}\left[\frac{l-k\sqrt{3}}{L_D|\mathbf{k}|^2}\right]-\tan^{-1}\left[\frac{l+k\sqrt{3}}{L_D|\mathbf{k}|^2}\right]\right)\end{cases}$$



Label: 10^x

  - quasi-invariant for QG equation (Balk, 1991).
  - used to explain anisotropic energy cascade (Balk, 2005).
  - conserved when the nonlinearity is sufficiently weak.
  - named by Nazarenko and Quinn (2009).

2. Purpose of this study

- Various parameter regimes
  - There are six combinations of the three length scales L, L_D and L_β (Table 1)
  - Regimes (5) and (6) ($L>L_D$, L_β) are not well-studied.

No.	Regime	Anisotropy	MEMO
(1)	$L < L_\beta < L_D$		・Move to regime (2) due to upscale energy cascade
(2)	$L_\beta < L < L_D$	YES	・Predominance of zonal flow
(3)	$L < L_D < L_\beta$	NO	・Checked by Okuno and Masuda (2003)
(4)	$L_D < L < L_\beta$	NO	・Checked by Okuno and Masuda (2003)
(5)	$L_\beta < L_D < L$		Target of this study
(6)	$L_D < L_\beta < L$		Target of this study

Table 1

- We consider the **long-wave limit** ($L_D/L \to 0$) as an extreme case and:
  1. derive the asymptotic form of zonostrophy.
  2. investigate the anisotropy of energy cascade and check the conservation of zonostrophy (by numerical experiments).

3. Long-wave limit

- **Long-wave limit of quasi-geostrophic equation**

Consider the quasi-geostrophic equation:

$$\frac{\partial}{\partial t}\left(\nabla^2-\frac{1}{L_D^2}\right)\psi+\beta\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}=0$$

$$x = Lx_*,\quad y = Ly_*,\quad \psi = UL\psi_*,\quad t=\frac{L^3}{L_D^2 U}t_*,\quad \nabla^2 = L^{-2}\nabla_*^2$$

Nondimensionalization

In the long-wave limit ($L_D/L \to 0$),
the operator in the first term expands as follows:

$$(\nabla^2 - L_D^{-2}) = -L_D^{-2}(1 - L_D^2\nabla^2) = -L_D^{-2}(1 + (L_D^2\nabla^2) + (L_D^2\nabla^2)^2 + \cdots)^{-1}$$

After a proper Galilean transform
and nondimensionalization, we get:

$$-\frac{\partial\psi_*}{\partial t_*}+\gamma\frac{\partial}{\partial x_*}\nabla_*^2\psi_*+\frac{\partial\psi_*}{\partial x_*}\frac{\partial\nabla_*^2\psi_*}{\partial y_*}-\frac{\partial\psi_*}{\partial y_*}\frac{\partial\nabla_*^2\psi_*}{\partial x_*}=0$$

: Long-wave limit of QG equation

Dimensionless number

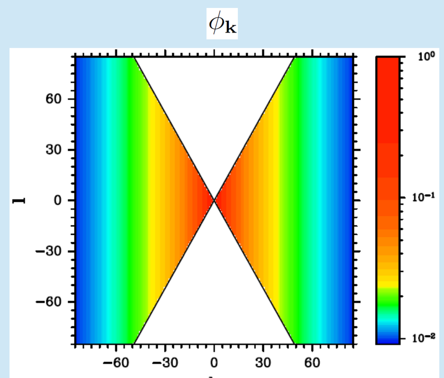
$$\gamma = \frac{\beta L_D^2}{U}$$

- (square of) the ratio of L_D to L_β
- does not depend on L
- considered as constant in time (since U is constant in time)
- introduced by Okuno and Masuda (2003)

- **Long-wave limit of zonostrophy**

We found the following form:

$$Z=\iint \phi_{\mathbf{k}}\varepsilon_{\mathbf{k}}\,d\mathbf{k}\qquad \begin{cases}\varepsilon_{\mathbf{k}}=\frac{1}{2}|\hat{\psi}_{\mathbf{k}}|^2 & \text{(: 2D energy spectrum)}\\ \phi_{\mathbf{k}}=\begin{cases}1/|k| & (|l| < \sqrt{3}|k|)\\ 0 & (|l| > \sqrt{3}|k| \text{ or } |\mathbf{k}| = 0)\\ 1/(2|k|) & (|l| = \sqrt{3}|k| \text{ and } |\mathbf{k}| \neq 0)\end{cases}\end{cases}$$



Distribution of ϕ_k in the two-dimensional wavenumber space. The region in which $\phi_k = 0$ is not colored. The two lines that bound the zero-valued region are $l = \pm\sqrt{3}k$

  - conserved when the nonlinearity is sufficiently weak (i.e. when $\gamma \gg 1$)
  - if conserved, energy should accumulate in a **wedge-shaped region W** in 2D wavenumber space:

$$W = \{(k, l)\ |\sqrt{3}|k| < |l|\}$$

(corresponding to the blank region in upper-right figure.)

4. Numerical experiments

- **Experimental setup**
  - Governing equation:
    - Long-wave limit of QG eq. + hyper viscosity

$$-\frac{\partial\psi}{\partial t}+\gamma\frac{\partial}{\partial x}\nabla^2\psi+\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}=(-1)^p\nu_p\nabla^{2p}\psi$$

(Freely decaying turbulence) $p = 10$, $\nu_p = 5\times10^{-35}$

Domain	2π×2π, double periodic
Discretization	Space : Fourier spectral method (K_T=85) Time : 4th order Runge-Kutta ($\Delta t = 2\times10^{-7}$)
Parameter	γ = 0, 0.25, 1, 5, 20
Initial condition	random Ψ field

- **Results**
  - 2D energy spectrum (average of 41 ensemble members) and their angular distribution

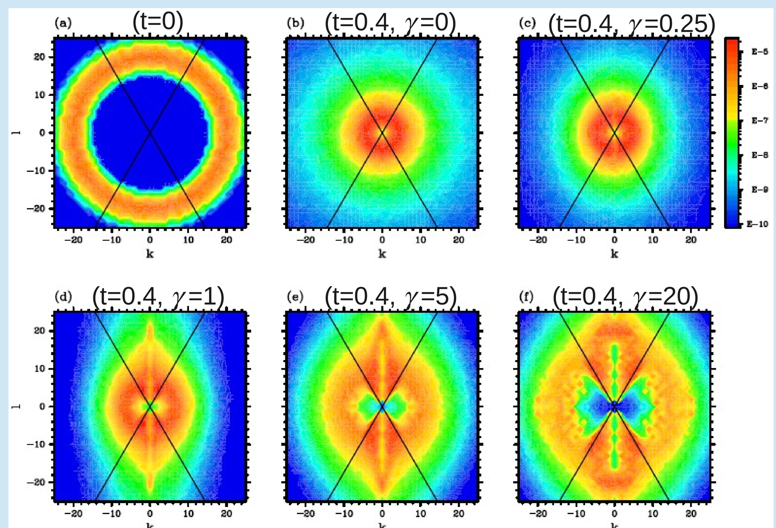


FIG. 3. (a) Two-dimensional energy spectrum at $t = 0$ within a rectangular domain of $|k|$, $|l| \le 25$ averaged over 41 ensemble members. (b)–(f) Same as (a) except that at $t = 0.4$ for $\gamma = 0$, 0.25, 1, 5, and 20, respectively. E$-x$ indicates 10^{-x}. In each figure, the lines of $l = \pm\sqrt{3}k$ are drawn for reference.

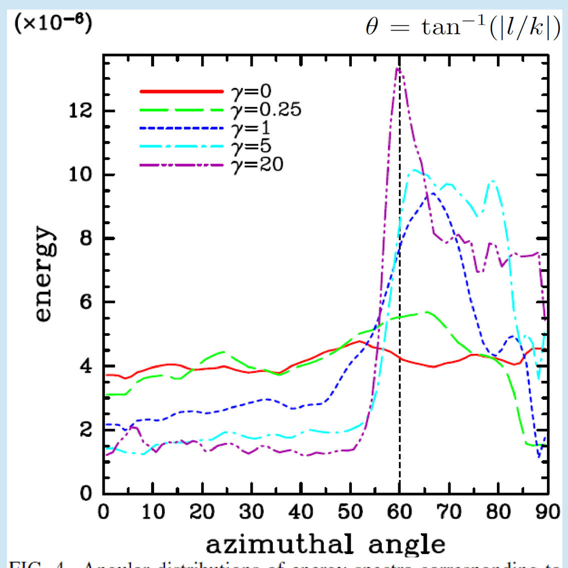
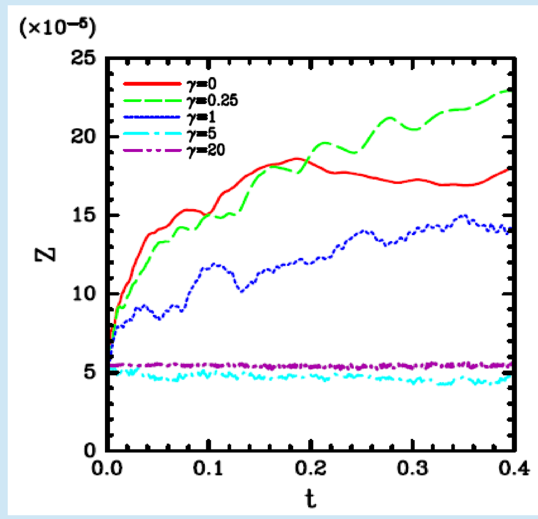


FIG. 4. Angular distributions of energy spectra corresponding to the final ($t = 0.4$) states shown as Figs. 3(b)–3(f).

- Clear contrast across the boundary (θ = 60°)

- γ=0 : Isotropic
- γ=0.25 : Slightly elongated along the l-axis
- γ=1, 5 : Accumulation in the wedge-shaped region W
- γ=20 : Concentration around θ = 60°
  (θ = tan |l/k|) (Connaughton et al., 2011)

  - Time-evolution of zonostrophy
    - Zonostrophy is well-conserved for larger value of γ(=5, 20).



Time-evolutions of zonostrophy(Z) for five γs.

5. Summary

- In the present study, we studied 2D turbulence governed by the long-wave limit of quasi-geostrophic equation and revealed:
  (1) The asymptotic form of zonostrophy.

(by numerical experiments)

  (2) The relationship between the parameter γ and anisotropy of 2D energy spectrum.
    - $\gamma > 1$: Accumulation in the wedge-shaped region W.
    - $\gamma \gg 1$: Concentration around θ = 60° .

＊ Accumulation in wedge-shaped region W can be explained by conservation of zonostrophy.

6. Discussion

- **1st baroclinic Rossby waves in the mid-latitude ocean**
  - wave length : several hundreds – thousand (km)
  - characteristic velocity scale : U ~ 1 (cm/s)
  - observed in the mid-latitude ocean (L_D ~ several tens (km))
  - **their wavenumber vector typically deviate from the zonal by 50°- 80°** (Glazman and Weichman, 2005)

  - For example, for 25°N latitude, βL_D^2 = 5.8 (cm/s) and we get:

$$\gamma = \frac{\beta L_D^2}{U} \sim 5$$

which satisfies the condition for anisotropy (γ > 1).

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