Koopmanism's World Wide Quest to Rule

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Appendix A

Koopman operator

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2013-03-01 PC I have created this chapter in order to prepare a possible appendix for ChaosBook.org; it is already formatted so it can be simply moved from here to the ChaosBook.

So far we have mostly focused on computation of eigenvalues of evolution operators. Here we shall discuss the role of their eigenfunctions. This is easiest to explain for systems with stable equilibria and periodic orbits, for which the dynamics is described by Koopman operators. We shall show how here how the *nonlinear* dynamics of transient states on the way to a stable solution is captured by the eigenfunctions of the *linear* Koopman operator.

A.1 Koopmania

The *Koopman operator* action on an observable $a(x)$ (a bounded and smooth state space function that associates a scalar to state x) is to replace it by its downstream value time t later, $a(x) \to a(x(t))$, evaluated at the trajectory point $x(t)$:

$$\begin{aligned} \left[\mathcal{K}^t a\right](x) &= a(f^t(x)) = \int_{\mathcal{M}} dy\, \mathcal{K}^t(x,y)\, a(y) \\ \mathcal{K}^t(x,y) &= \delta\big(y - f^t(x)\big)\,. \end{aligned} \tag{A.1}$$

Given an initial density of representative points $\rho(x)$, the state space average of $a(x)$ evolves as

$$\begin{aligned} \langle a \rangle_\rho(t) &= \frac{1}{|\rho_{\mathcal{M}}|}\int_{\mathcal{M}} dx\, a(f^t(x))\, \rho(x) = \frac{1}{|\rho_{\mathcal{M}}|}\int_{\mathcal{M}} dx\, \left[\mathcal{K}^t a\right](x)\, \rho(x) \\ &= \frac{1}{|\rho_{\mathcal{M}}|}\int_{\mathcal{M}} dx\, dy\, a(y)\, \delta\big(y - f^t(x)\big)\, \rho(x)\,. \end{aligned}$$

The 'propagator' $\delta(y - f^t(x))$ can be interpreted as belonging to the Perron-Frobenius operator, so the two operators are adjoint to each other, [1]

$$\int_{\mathcal{M}} dx \, \left[\mathcal{K}^t a\right](x)\, \rho(x) = \int_{\mathcal{M}} dy \, a(y) \, \left[\mathcal{L}^t \rho\right](y) \,. \tag{A.2}$$

The Koopman and Perron-Frobenius operators describe the dynamics in complementary ways. Koopman advances the trajectory by time t, Perron-Frobenius depends on the trajectory point time t in the past. Perron-Frobenius propagates a conserved quantity (a density of initial conditions) forward in time. The growth (or decay) of the density depends on the compression (or expansion) of a volume occupied by a set of trajectories. The dynamics of an observable depends on the other hand on one single trajectory. [2]

[3] The family of Koopman operators $\left\{\mathcal{K}^t\right\}_{t\in\mathbb{R}_+}$ forms a semigroup parameterized by time, $\mathcal{K}^t\mathcal{K}^{t'} = \mathcal{K}^{t+t'}$, $\mathcal{K}^0 = \mathbf{1}$ with the generator of infinitesimal time translations defined by

$$\mathcal{A}^\dagger \;=\; \lim_{t\to 0^+} \frac{1}{t}\left(\mathcal{K}^t - \mathbf{1}\right) \,.$$

If the flow is finite-dimensional and invertible, $\mathcal{A}^\dagger$ is a generator of a group. The explicit form of $\mathcal{A}^\dagger$ follows from expanding dynamical evolution up to first order:

$$\mathcal{A}^\dagger a(x) \;=\; \lim_{t\to 0^+} \frac{1}{t}\left(a(f^t(x)) - a(x)\right) \;=\; v_i(x)\partial_i a(x) \,. \tag{A.3}$$

This is by definition the time derivative, so the time-evolution equation for $a(x)$ is

$$\left(\frac{d}{dt} - \mathcal{A}^\dagger\right) a(x) \;=\; 0 \,. \tag{A.4}$$

[4] We formally write the solution to (A.4) as

$$a(x(t)) = e^{t\mathcal{A}^\dagger} a(x_0) = \mathcal{K}^t a(x_0) \,,$$

so the finite time Koopman operator (A.1) can be recovered by exponentiating the time-evolution generator $\mathcal{A}^\dagger$. The generator $\mathcal{A}^\dagger$ looks very much like the generator of translations. For example, for a constant velocity field dynamical evolution is nothing but a translation by time $\times$ velocity:

$$e^{tv\frac{\partial}{\partial x}} a(x) = a(x + tv) \,. \tag{A.5}$$

[1] Predrag: this is unclear: "(analogous to the shift from the Heisenberg to the Schrödinger picture in quantum mechanics)", so I dropped it

[2] Predrag: pretty confusing... Improve

[3] Predrag: insert this somewhere: " The Koopman operator $\mathcal{K}^t$ is the appropriate evolution operator to represent the dynamics of attractors and their stable manifolds. "

[4] Predrag: (A.4) looks different from my generator for Perron-Frobenius operator, chapter measure.tex; replaced (B.2) here. The '-' sign bugs me - it is not material derivative? Recheck - follow the clear discussion of Lasota-Mackey [24] Sect. 7.6

The Koopman / Perron-Frobenius operators are non-normal, non-self-adjoint operators, so their left and right eigenvectors differ. The right eigenvectors of a Perron-Frobenius operator are the left eigenvectors of the Koopman, and vice versa. That is, [5]

$$\mathcal{A}\phi_\alpha(x) = s_\alpha \phi_\alpha(x)\,, \quad \mathcal{A}^\dagger \psi_\alpha(x) = s_\alpha^* \psi_\alpha(x)\,, \qquad \alpha = 0, 1, 2, \cdots$$

The left and right eigenfunctions satisfy the bi-orhogonality condition with respect to L^2 norm, [6]

$$\int_\mathcal{M} dx\, \phi_\alpha^* \psi_\beta \;= \delta_{\alpha\beta}\,. \tag{A.6}$$

While one might think of a Koopman operator as an 'inverse' of the Perron-Frobenius operator, the notion of *adjoint* is the right one, especially in settings where flow is not time-reversible, as is the case for dissipative PDEs (infinite dimensional flows contracting forward in time) and for stochastic flows.

Given the left and right eigenfunctions, we can express the evolution of an observable as

$$a(x(t)) = \left[\mathcal{K}^t a\right](x_0) = \sum_\alpha c_\alpha e^{s_\alpha t} \psi_\alpha(x_0) \tag{A.7}$$

where

$$c_\alpha = \int_\mathcal{M} dx\, a(x)\, \phi_\alpha^*(x)\ .$$

This expansion suggests an alternative description of nonlinear dynamics, which is the (linear) evolution of observables in an infinite-dimensional space. In principle, this allows the study of full nonlinear dynamics using linear operator-theoretical tools.

Example A.1. ***Spectrum of a 1D linear system:*** *Consider a 1D system with a single equilibrium* [7]

$$\dot{x} = \lambda x, \tag{A.8}$$

If the observable $a(x)$ is a smooth, real-analytical function, the Koopman operator spectrum can be identified from its Taylor expansion,

$$\left\{ \begin{array}{rcl} s_k &=& k\lambda \\ \phi_k &=& \delta^{(k)}(x) \\ \psi_k &=& x^k \end{array} \right. \qquad \textit{when} \quad \lambda < 0 \qquad \textit{(attractor)} \tag{A.9}$$

and

$$\left\{ \begin{array}{rcl} s_k &=& -(k+1)\lambda \\ \phi_k &=& x^k \\ \psi_k &=& \delta^{(k)}(x) \end{array} \right. \qquad \textit{when} \quad \lambda > 0 \qquad \textit{(repeller)} \tag{A.10}$$

for $k = 0, 1, \cdots$. Here the superscript $^{(k)}$ refers to the kth derivative. We observe the duality between the right/left eigenfunctions and the repelling/attracting points. When $\lambda < 0$, any neighborhood of representative points shrinks to a point and asymptotically the density becomes a singular function. On the other hand, any smooth observable

[5] Predrag: harmonize ϕ_α, ψ_α with ChaosBook

[6] Predrag: in general complex, so ϕ_α^*, s^*, right?

[7] Predrag: we do this for discrete time in chapter converg.tex, and we also do it for Fokker-Planck operator. Changed $\alpha \to k$ to harmonize with those examples. Might change all of them back, to harmonize with Perron-Frobenius operator spectrum notation...

has the asymptotic limit $a(0)$. Koopman operator $\mathcal{K}^t$ is thus the appropriate evolution operator to represent the dynamics in stable manifolds, since the observable dynamics goes along with the flow.

Example A.2. *Spectrum of a 1D nonlinear system:* *As an example of how the effects of nonlinearity are captured by expansion into eigenfunctions of the Koopman operator, consider the stable nonlinear system:*

$$\dot{x} \quad = \lambda x - x^3, \qquad \lambda < 0 \tag{A.11}$$

where the only equilibrium point is the attracting fixed point $x_q = 0$. The difference between (A.11) and the linear system in (A.8), is the presence of a cubic nonlinear term. However, the nonlinear coordinate transformation

$$y = g(x) = \frac{x}{\sqrt{x^2 - \lambda}} \tag{A.12}$$

transforms (A.11) into a linear system $\dot{y} = \lambda y$, whose spectrum is already determined by (A.9). The Koopman spectrum in terms of the coordinate x is thus

$$\begin{cases} s_k & = \quad k\lambda \\ \phi_k(x) & = \quad \delta^{(k)}\left(x - g^{-1}(y)\right)|_{y=0} \\ \psi_k(x) & = \quad \left(x/\sqrt{x^2-\lambda}\right)^k \end{cases} \tag{A.13}$$

where $k = 0, 1, \cdots$ and the derivative of δ is with respect to y. Comparing to (A.9), the Koopman eigenvalues are not modified by the cubic nonlinear term in (A.11), but the term $\sqrt{x^2 - \lambda}$ appears in the Koopman eigenfunctions.

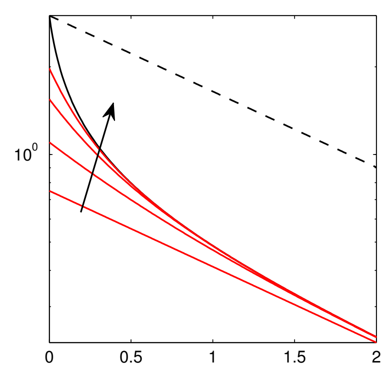

Figure A.1: (black line) The trajectory $x(t)$ of (A.11) plotted on logarithmic scale as a function of time, for $\lambda = -0.6$. (red/gray lines) Reconstructions of the trajectory based on the expansion (A.14) – including up to the ϕ_1, ϕ_3, ϕ_5 or ϕ_7 left eigenfunction of $\mathcal{K}^t$. (dashed line) The trajectory of the linearized system, with x^3 neglected in (A.11).

Consider the expansion (A.7) of a position $x(t)$ at time t considered as an observable, $a(x(t)) = x(t)$,

$$x(t) = \left(\frac{-\lambda}{{x_0}^2 - \lambda}\right)^{1/2} x_0\, e^{\lambda t} + \frac{1}{\sqrt{-\lambda}} \left(\frac{x_0}{{x_0}^2 - \lambda}\right)^{3/2} e^{3\lambda t} + \cdots \tag{A.14}$$

In figure A.1, the trajectory $x(t)$ (black line) obtained by integrating (A.11) starts out by a rapid decay to the stable manifold of the stable fixed point, followed by an exponential decay along the manifold to $x_q = 0$. In a purely linear analysis, the state evolves as $x_{lim}(t) = x_0 e^{\lambda t}$ (dashed black line in the figure). A linear analysis provides the exponential decay rate, but fails to describe the curved trajectory in its initial stages. In the *figure the first non-zero expansion terms and the superposition of gradually increasing number of modes are shown with red lines. Whereas the Koopman eigenvalues provide the asymptotic decay rate, the Koopman eigenfunctions provide the direction as well as an amplitude. Including higher order terms in the expansion, eventually the full state trajectory can be recovered by a number of Koopman eigenfunctions, and thus the transient nonlinear dynamics preceding the infinitesimal linear region can be captured.*

A.2 Koopman eigenvalues for a limit cycle

[8] The $[(d-1)\times(d-1)]$-dimensional monodromy matrix $\mathbf{M}_{ij} = \partial_j P_i(\hat{x}_a)$ of dimension governs the dynamics of the small perturbation $\delta\hat{x}$ within a Poincaré section.

[9] Even though the monodromy matrix $\mathbf{M}(\hat{x})$ depends upon $\hat{x}$ (the 'starting' point of the periodic orbit), its eigenvalues do not, so we may write for its eigenvectors $\mathbf{e}^{(j)}$ (sometimes referred to as 'covariant Lyapunov vectors,' or, for periodic orbits, as 'Floquet vectors')

$$\mathbf{M}(x)\,\mathbf{e}^{(j)}(x) = \Lambda_j\,\mathbf{e}^{(j)}(x)\,, \qquad \Lambda_j = e^{\lambda^{(j)}T}\,. \tag{A.15}$$

where Floquet exponents $\lambda^{(j)} = \mu^{(j)} \pm i\omega^{(j)}$ are independent of x. We order the Floquet multipliers as [10]

$$|\Lambda_1| \geq |\Lambda_2| \geq \cdots \geq |\Lambda_{d-1}|\,. \tag{A.16}$$

The limit cycle is stable if $|\Lambda_1| < 1$.

The two most important characteristics of the limit cycle are thus the fundamental frequency and the leading Lyapunov exponent, defined by

$$\omega = \frac{2\pi}{T}, \qquad \mu = \frac{1}{T}\ln|\Lambda_1|, \tag{A.17}$$

respectively.

Here we follow the derivations presented in refs. [25, 21], [11] except that the analysis is restricted to the simpler case of a stable limit cycle. The trace of the Koopman operator is,

$$\mathrm{tr}\,\mathcal{K}^t = \int_{\mathcal{M}} \mathcal{K}^t(x,x)dx.$$

where $\mathcal{K}^t$ is the kernel. Inspired [12] by this definition, we define the trace of Koopman operator as

$$\mathrm{tr}\,\mathcal{K}^t = \int_{\mathcal{M}} \delta(x - f^t(x))dx. \tag{A.18}$$

[8] Shervin: To my understanding, much of this section is repetition of Chapter 17 and 18. I guess one could (perhaps at later stage of this experiment) start directly from the more general expression of the spectral determinant in Chapter 18, and then derive (A.27) in the specific case of a single stable limit cycle? I leave this section intact in this revision.

[9] Predrag: clipped from Chapter invariants.tex

[10] Predrag: I number complex pairs consecutively, so not $<$ but $\leq$. [2013-02-27 Shervin] OK, agree.

[11] Predrag: Ref. [21] follows ref. [26], not sure it is any better than the original. Should recheck... [2013-02-27 Shervin] OK.

[12] Shervin: "Integral operators are in L^2 space as well as compact, that is they can in many aspects be treated as finite rank matrices. The kernel of $\mathcal{K}^t$ is singular and the operator is certainly not in L^2. However,

From (A.18), one observes that the trace $\mathcal{K}^t$ receives a contribution whenever the trajectory returns to the starting point after r repeats of the limit cycle period T. [13]

[14] To proceed, we decompose the propagator f_t into two parts, the $(d–1)$-dimensional 2CB
Poincaré map P and a 1-dimensional return-time function τ. The Poincaré map captures only the transverse part of the periodic dynamics, since the flow component tangent to the trajectory, which is not in the span of the Poincaré surface, has not been taken into account. [15] Assuming the longitudinal state component has a certain mean velocity v as it traverses the limit cycle, one may transform this component to a time coordinate system using the relation vdt. Thus the full dynamics is described by the Poincaré map P and by the first return function $\tau(\hat{x})$ that provides the (non-constant) time interval between successive points $\hat{x}$ on Poincaré surface, e.g. $t_{k+1} = t_k + \tau(\hat{x}_k)$. [16] Applying τ recursively, we may write $((k+1)$th time as a function first point and initial time,

$$t_{k+1} = t_1 + \sum_{j=0}^{k-1} \tau(P^j \hat{x}_1). \tag{A.19}$$

Now, factor the kernel of $\mathcal{K}^t$ (A.18) into two parts

$$\mathrm{tr}\,\mathcal{K}^t = \int_{\mathcal{P}(\hat{\S})=0} d\hat{x} \int_0^{\tau(\hat{x})} dt\, \delta\big(\hat{x} - P^k \hat{x}\big)\, \delta\left(t - \sum_{j=0}^{k-1} \tau(P^j \hat{x})\right), \tag{A.20}$$

where P^k and τ are defined above and in (A.19), respectively. We treat the two Dirac delta functions separately, starting with P^k. First recall that the Dirac delta function applied to a scalar-valued function $g(x)$, is

$$\int \delta(g(x))dx = \int \delta(x)|g'(0)^{-1}|dx = \sum_j \frac{1}{|g'(x_j)|},$$

if the dynamical system $\mathbf{f}$ is real-analytic, it has an analytic continuation to a complex extension of the state space, where the singularity can be “removed” and the problem reduced to the standard theory of integral operators [27].” This footnote is based on Chapter "Why does it work?" Do you agree with my interpretation? [2013-03-10 PC] Tentatively yes. But I am always unsure about these things...

[13]Predrag: To Shervin: this trace does not ‘count’. See ChaosBook.org/chapter/count.pdf “Topological zeta function for flows” for the continuous time counting trace.

[14]Predrag: yet another name for the time-forward map: ‘propagator’

[15]Shervin: We have determined the stability of a limit cycle in terms of the eigenvalues of the linearized Poincaré map. One can make use of that analysis to write the trace as a function of the Jacobian matrix $\mathbf{M}$. However, this is only part of the dynamics, since the flow component tangent to the limit cycle, which is not in the span of the Poincaré surface, has not be taken into account. The longitudinal component does not affect the stability, since exactly after one period T the trajectory returns to its initial position, e.g. there is no decay or growth of perturbations along the tangential direction, which would yield a Jacobian matrix that equals one. Nevertheless longitudinal direction is necessary for the description of the full dynamics, and hence it contributes to the trace of $\mathcal{K}^t$.

The longitudinal component does not affect the stability, since exactly after one period T the trajectory returns to its initial position, e.g. there is no decay or growth of perturbations along the tangential direction, which would yield a Jacobian matrix that equals one. Although, this longitudinal component does not affect the stability it is necessary for the description of the full dynamics, which means that it is significant for determining the Koopman eigenvalues.

[16]Predrag: refer to ChaosBook.org/chapter/maps.pdf “Poincaré sections.”

where x_j are the roots of $g(x)$. This property may be generalized to $d-1$ dimensions and applied to the Dirac-delta in (A.20),

$$\int_{\mathcal{P}(\overline{u})=0} d\hat{x}\, \delta(\hat{x} - P^k(\hat{x})) = \frac{1}{|\det(\mathbf{I} - \mathbf{M}^r)|}, \tag{A.21}$$

where $\mathbf{I}$ denotes the identity matrix. The second part of the trace can be written as [25]

$$\int_0^{\tau(\hat{x})} \delta(t - \sum_{j=0}^{k-1} \tau(P^j \hat{x}))dt = T \sum_{r=1}^{\infty} \delta(t - rT). \tag{A.22}$$

Inserting the identities (A.21) and (A.22) in (A.20), we get the trace formula for a single limit cycle of period T,

$$\mathrm{tr}\,\mathcal{K}^t = T \sum_{r=1}^{\infty} \frac{\delta(t - rT)}{|\det(\mathbf{I} - \mathbf{M}^r)|}\,, \tag{A.23}$$

which was first derived in ref. [26], here given in the special case of a single limit cycle. The trace formula is a sum whose terms are nonzero only for integers of the cycle period. The rth nonzero term describes how much after the rth return to the Poincaré section a small neighborhood volume (i.e. a tube) of the stable limit cycle has retracted. This relation thus connects the trace of $\mathcal{K}^t$ to the dynamics in the local stable manifold of the limit cycle.

The Koopman eigenvalues are the poles of the Laplace transform of trace of $\mathcal{K}^t$

$$\int_0^{\infty} e^{-st} \mathrm{tr}\,\mathcal{K}^t dt = \mathrm{tr}\,\frac{1}{s - \mathcal{A}}\,,$$

i.e., the poles of the resolvent of $\mathcal{A}$. By inserting (A.23) in the left-hand side of above equation one obtains,

$$\mathrm{tr}\,\frac{1}{s - \mathcal{A}} = \frac{\partial}{\partial s} \ln(\det(s - \mathcal{A})),$$

where $\det(s - \mathcal{A})$ is the spectral determinant,

$$\det(s - \mathcal{A}) = \exp\left[-\sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{-sTr}}{|\det(\mathbf{I} - \mathbf{M}^r)|}\right].$$

Now, since the determinant does not depend on the basis which $\mathbf{M}$ is described in, we may write it in terms of the eigenvalues of $\mathbf{M}$,

$$\frac{1}{|\det(\mathbf{I} - \mathbf{M}^r)|} = \prod_{k=1}^{d-1} \frac{1}{1 - \Lambda_k^r}, \tag{A.24}$$

where we have assumed that $|\Lambda_k| < 1$ for all k.

Denominators can be expanded in Taylor series such as

$$(1 - x)^{-1}(1 - y)^{-1} = 1 + x + y + x^2 + xy + y^2 + \dots,$$

when $|x| < 1, |y| < 1$. Each term in the product (A.24) may thus be written as an infinite sum. Define a *multi-index* as an array of d non-negative integers $j_k = 0, 1, 2, \dots$:

$$\boldsymbol{j} = [j_1, j_2, \dots, j_d] \in \mathbb{N}^d \,,$$

Consider next the product of $d-1$ Floquet multipliers

$$\boldsymbol{\Lambda} = \Lambda_1 \Lambda_2 \cdots \Lambda_{d-1} = e^{T(\mu^{(1)} + \mu^{(2)} + \cdots + \mu^{(d-1)})} \,,$$

(the imaginary parts of complex pairs cancel in the exponent), and define [17]

$$\boldsymbol{\mu} = [\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(d-1)}] \in \mathbb{R}^d \,.$$

Λ can now be raised to $\boldsymbol{j}$th power as

$$\boldsymbol{\Lambda}^{\boldsymbol{j}} = e^{T\boldsymbol{\mu}\cdot\boldsymbol{j}} = \Lambda_1^{j_1} \Lambda_2^{j_2} \cdots \Lambda_{d-1}^{j_{d-1}} \,. \tag{A.25}$$

Using multi-index notation (A.25) we may write (A.24) as [18]

$$\frac{1}{|\det(\mathbf{I} - \mathbf{M}^r)|} = \sum_{\boldsymbol{j}} \Lambda^{r\boldsymbol{j}} \,,$$

and consequently the spectral determinant as

$$\det(s - \mathcal{A}) = \exp\left[-\sum_{r=1}^{\infty} \frac{1}{r} (e^{-sT} \sum_{\boldsymbol{j}} \Lambda^{\boldsymbol{j}})^r \right] .$$

Applying the identity $\sum x^r/r = -\ln(1-x)$, we obtain the final form of the spectral determinant for a stable limit cycle

$$\det(s - \mathcal{A}) = \prod_{\boldsymbol{j}}^{\infty} \left(1 - e^{-sT} \Lambda^{\boldsymbol{j}}\right) \,. \tag{A.26}$$

The zeros of $\det(s - \mathcal{A}) = 0$ are given by the zeros of individual terms in the product:

$$e^{-T(s - \boldsymbol{\mu}\cdot\boldsymbol{j})} = 1 \,.$$

Taking the logarithm of both sides, we obtain

$$s_{\boldsymbol{j},m} = \boldsymbol{\mu}\cdot\boldsymbol{j} + 2\pi i m/T = \boldsymbol{\mu}\cdot\boldsymbol{j} + im\omega \tag{A.27}$$

with $m = 0, \pm 1, \pm 2, \dots$. For our particular choice of analytic observables the spectrum of $\mathcal{K}^t$ is reduced to its minimal components, namely any integer multiple of the

[17] Predrag: ponder- complex pairs contribute here with multiplicity 2

[18] Predrag: Note that I have edited multi-index notation in half of your formulas - please recheck. [2013-02-27 Shervin] It makes the equations cleaner. I will incorporate this into the paper. I went through the new formulas, and they seem OK. [2013-03-10 PC] I have started using them in chapter 1. But they are more complicated, it might be easier to start with the stable limit cycle first...

stability eigenvalues. Thus, for any stable limit cycle, the Koopman eigenvalues form a lattice on the lower half of the complex plane. The marginal eigenvalues on the horizontal imaginary axis corresponding to $j = 0$ correspond to the non-decaying time-averaged mean ($m = 0$) and periodic dynamics ($m \neq 0$) on the limit cycle. [19] The remaining eigenvalues $j \neq 0$ are decaying and describe the transient behavior of flow in the local stable manifold of the limit cycle.

Example A.3. ***Spectrum of a stable limit cycle:*** *Consider the three dimensional system* [20]

$$\begin{cases} \dot{x} &= \mu x - y - xz \\ \dot{y} &= \mu y - x - yz \\ \dot{z} &= -z + x^2 + y^2 \,,. \end{cases} \tag{A.28}$$

for $\mu \gtrsim 0$. The system has an unstable fixed point

$$x_q = (x, y, z) = 0,$$

and an attracting limit cycle

$$x_a = (\sqrt{\mu}\cos t, \sqrt{\mu}\sin t, \mu).$$

In figure A.2, a typical trajectory starting near x_q is shown. The trajectory grows exponentially with the exponent $\lambda_q > 0$ and after a transient time, approaches the stable limit cycle exponentially fast with the exponent $\lambda_a < 0$.

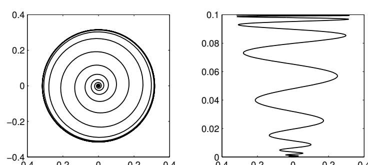

Figure A.2: State trajectory starting close to $x_q = 0$ and with $\mu = 1/10$ for the system (A.28) in x, y-plane (left) and the (x, z)-plane (right).

The set of discrete Koopman/Perron-Frobenius eigenvalues is simply the union of the eigenvalues associated with the fixed point and the limit cycle. One may thus treat the two critical elements separately using the formulas derived from the trace of the operators. Here we only consider the spectrum pertaining to stable limit cycle. By considering the Poincaré section given by the plane $y = 0$ and its associated monodromy matrix, one arrives at [21] [22]

$$\Lambda = -2\mu, \qquad \omega = 2\pi \,.$$

According to formula (A.27) the Koopman/Perron-Frobenius eigenvalues $\{j, m\} = \{j_1, 0, \dots, 0, m\}$ corresponding to this leading Floquet exponent are,

$$s_{j,m} = j\Lambda + im\omega = -2\mu j + mi2\pi$$

[19] Predrag: I like 'periodic dynamics' - it is Fourier decomposition of the initial density, which can rotate but cannot change, along the cycle in components m. Do I understand you right? [2013-02-27 Shervin] Yes, but I guess I refer to the Fourier decomposition of the observable at the initial point on the trajectory.

[20] Predrag: will have to rename μ, conflicts with ChaosBook real part of λ

[21] Predrag: here μ is dimensionally a Floquet exponent, but Λ is reserved for *multipliers*. I would like to $\mu \to \lambda$, $\Lambda \to \mu$

[22] Shervin: Should we include the spectrum of the unstable fixed point? The eigenfunctions are Dirac delta functions, also due to the interaction with the limit cycle, one will have degenerate eigenvalues, Jordan blocks etc.**Predrag 2013-04-15:** best not to complicate?

for $j = 0, 1, 2, \dots$ and $m = 0, \pm 1, \pm 2, \dots$. The expansion of the state observable into the leading complex Koopman eigenfunctions ($j = 0, 1$ and $m = 0, 1$) associated with (A.28) is

$$x(t) = v_{0,0} + v_{0,1}\, e^{it} + v_{1,0}\, e^{-2\mu t} + c.c + \dots$$

with,

$$v_{0,0} = (0, 0, \mu), \tag{A.29}$$

$$v_{0,1} = \frac{\sqrt{\mu}}{2}(1, 0, 0) + \frac{i\sqrt{\mu}}{2}(0, 1, 0), \tag{A.30}$$

$$v_{1,0} = \frac{c\sqrt{\mu}}{2}(0, 0, \frac{r^2 - \mu}{r^2}), \tag{A.31}$$

where c is some constant and $r^2 = x^2 + y^2$.

The first two modes resolve the attractor dynamics; $v_{0,0}$ represents the average asymptotic value, and $v_{0,1}$ the periodic asymptotic solution with unit frequency on the attractor. These two Koopman modes correspond to the three first (real) empirical Karhunen-Loève or proper orthogonal decomposition modes. A robust low-order representation of the flow should in addition to the limit cycle also, at least in some sense, capture the dynamics of the corresponding attracting inertial manifold, that connects the unstable fixed point with the limit cycle. This is the role of the transient mode $v_{1,0}$; the function $(r^2 - \mu)/r^2$ is singular near the fixed point and zero at the limit cycle and points in the direction z, i.e. from x_q to x_a.

Commentary

Remark A.1. Koopman operators. The "Heisenberg picture" in dynamical systems theory has been introduced by Koopman and Von Neumann [28, 29], see also ref. [24]. Inspired by the contemporary advances in quantum mechanics, Koopman [28] observed in 1931 that $\mathcal{K}^t$ is unitary on $L^2(\mu)$ Hilbert spaces. The Koopman operator is the classical analogue of the quantum evolution operator $\exp\left(i\hat{H}t/\hbar\right)$ – the kernel of $\mathcal{L}^t(y, x)$ introduced in (**??**) (see also sect. **??**) is the analogue of the Green function discussed here in chapter **??**. The relation between the spectrum of the Koopman operator and classical ergodicity was formalized by von Neumann [29]. We shall not use Hilbert spaces here and the operators that we shall study *will not* be unitary. [23] For a discussion of the relation between the Perron-Frobenius operators and the Koopman operators for finite dimensional deterministic invertible flows, infinite dimensional contracting flows, and stochastic flows, see Lasota-Mackey [24] and Gaspard [21].

⇓PRIVATE

In the theory of semigroups (Pazy 1983), the linear operator $\mathcal{A}$ is the infinitesimal generator of the operator $\mathcal{K}^t$.

⇑PRIVATE

A.3 Flotsam

The vector-valued observable $a(x) : \mathcal{M} \to \mathbb{R}^m$ is defined as

$a(x) = [g_1(x), g_2(x), ..., g_m(x)]$,

where each component $g_j(x)$ is a scalar function over $\mathcal{M}$.

[23] Predrag: recheck the unitarity claims...

Using a Dirac-delta function, the Koopman operator for any bounded observable function $a(x)$ can be written as

$$\mathcal{K}^t a(x) = \int_{\mathcal{M}} \delta\left(y - f^t(x)\right) a(y) dy \,,$$

where x belongs to the manifold $\mathcal{M} \subset \mathbb{R}^n$. This form of the Koopman operator is similar to the form of integral operators, for which one may define...

Then can define [24]

$$\boldsymbol{j}! = j_1! j_2! \cdots j_d! \,, \qquad |\boldsymbol{j}| = j_1 + j_2 + \cdots + j_d \,.$$

$$\boldsymbol{\partial} = [\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d}] \,, \qquad \boldsymbol{\partial}^{\boldsymbol{j}} = \frac{\partial^{j_1}}{\partial x_1^{j_1}} \frac{\partial^{j_2}}{\partial x_2^{j_2}} \cdots \frac{\partial^{j_d}}{\partial x_d^{j_d}} \,.$$

A.4 Exercises

Exercise A.1. ***Perron-Frobenius operator is the adjoint of the Koopman operator.*** *Check (A.2) - it might be wrong as it stands. Pay attention to presence/absence of a Jacobian.*

Exercise A.2. ***Nonlinear system mapped into a linear one.*** *(A.11) and the linear system in (A.8), is the presence of a cubic nonlinear term. Show the nonlinear coordinate transformation (A.12)*

$$y = g(x) = \frac{x}{\sqrt{x^2 - \lambda}}$$

transforms (A.11) into a linear system $\dot{y} = \lambda y$.

Exercise A.3. ***Stability of a limit cycle.*** *Show that the system (A.28) has an attracting limit cycle $x_a = (\sqrt{\mu}\cos t, \sqrt{\mu}\sin t, \mu)$.*

Solution A.1 - Perron-Frobenius operator is the adjoint of the Koopman operator.

Solution A.2 - Nonlinear system mapped into a linear one.

$$\begin{aligned}\dot{y} &= \frac{dg}{dx}\dot{x} = \left(-\frac{1}{2}\frac{2x^2}{(x^2-\lambda)^{3/2}} + \frac{1}{(x^2-\lambda)^{1/2}}\right) x(\lambda - x^2) \\ &= \frac{\lambda\, x}{(x^2-\lambda)^{1/2}} = \lambda\, y \,. \end{aligned} \tag{A.32}$$

Solution A.3 - Stability of a limit cycle. *Go to polar coordinates* [25]

$$\left(\begin{array}{c}\dot{r}\\ \dot{\theta}\end{array}\right) = \left(\begin{array}{cc}\cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r}\end{array}\right)\left(\begin{array}{c}\dot{x}\\ \dot{y}\end{array}\right) . \tag{A.33}$$

[24] Predrag: is it useful to rewrite the sums as $\sum_{\boldsymbol{j}} = \sum_{k=0}^{\infty}\sum_{|\boldsymbol{j}|=k}$? Probably no. Use 'weight ordering' as a generalization of the 'stability ordering'. Define 'amplitude ordering' in QM.

[25] Predrag: 2013-04-18 have not succeeded in making this work yet...

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