The Haldane Phase as a Symmetry-Protected Topological Phase and Quantum Entanglement

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Haldane “Conjecture”

Heisenberg antiferromagnetic chain

\[ \mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1} \]

S = 1/2, 3/2, 5/2, ...

“massless” = gapless, power-law decay of spin correlations

S = 1, 2, 3, ...

“massive” = non-zero gap, exponential decay of spin correlations
AKLT model/state

\[ H = J \sum_j \left[ \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} (\vec{S}_j \cdot \vec{S}_{j+1})^2 \right] \]

Exact groundstate: (Affleck–Kennedy–Lieb–Tasaki 1987)

- \( S=1 \)
- \( S=1/2 \)
- Symmetrization (=projection to \( S=1 \))
- Singlet pair of two \( S=1/2 \)'s - "valence bonds"
- Non-zero gap, exponential decay of correlations (supporting the Haldane conjecture)
Order in AKLT state?

Groundstate of the AKLT model: **UNIQUE** *(for periodic boundary condition)*

Correlation function of any local operator decays exponentially

There is no local order parameter; no symmetry is broken spontaneously

No order, that's it?
\[ \mathcal{H} = J \sum_j \left( \vec{S}_j \cdot \vec{S}_{j+1} + D (S^z_j)^2 \right) \]

Why quantum phase transition?

Because there is some kind of order (topological order) in the "Haldane phase"
Edge states

Consider a chain with open boundary condition

“free” $S=1/2$ appears at each end, interacting with each other. Effective coupling:

$$J_{\text{eff}} \sim e^{-L/\xi}$$

$2\times2=4$ groundstates below the Haldane gap (nearly degenerate)

Kennedy (1990)
Hidden (string) order

In $S^z$ basis, + and - alternate, with 0's in between. No long-range order w.r.t. local observables, but a hidden (topological) order measurable by the "string order parameter"

$$O_{\text{str}}^\alpha \equiv \lim_{|j-k| \to \infty} \langle S_j^\alpha e^{i\pi \sum_{l=j}^{k-1} S_l^\alpha} S_k^\alpha \rangle$$

Den Nijs & Rommelse (1989)
Hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

\[ \mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1} \]

Kennedy & Tasaki (1992)

\[ \mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1} \]

non-local unitary transformation

\[ U = \prod_{j<k} e^{i \pi S_j^z S_k^x} \]

[simple expression by M.O. (1992)]

\[ \tilde{\mathcal{H}} = U \mathcal{H} U^{-1} \]

[well-defined only for open b.c.]

\[ \tilde{\mathcal{H}} = J \sum_j \left[ S_j^x e^{i \pi S_j^z} S_{j+1}^x + S_j^y e^{i \pi (S_j^z + S_{j+1}^z)} S_{j+1}^y + S_j^z e^{i \pi S_j^z} S_{j+1}^z \right] \]

Global discrete symmetry

($\pi$-rotation about $x$, $y$, $z$ axes = $\mathbb{Z}_2 \times \mathbb{Z}_2$)
Spontaneous breaking of hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

4-fold groundstate degeneracy for $\tilde{\mathcal{H}}$

$U H U^{-1} = \tilde{\mathcal{H}}$

4-fold groundstate degeneracy for $\mathcal{H}$ only with the open b.c.! = edge states

Ferromagnetic order for $\tilde{\mathcal{H}}$

$U \left( S_j^z e^{i\pi \sum_{l=j}^{k-1} S_i^z S_k^z} \right) U^{-1} = S_j^z S_k^z$

String order for $\mathcal{H}$
When does it work?

This was not discussed (as far as I know) in 1990s

The Kennedy-Tasaki transformation is nonlocal -- if the transformed Hamiltonian $\tilde{\mathcal{H}}$ is nonlocal, the argument does not work.

Because the transformation is self-dual, for $\tilde{\mathcal{H}}$ to be local, the original Hamiltonian must have global $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry ($\pi$-rotation about $x, y, z$ axes)

This means that $S=1$ Haldane phase is a topological phase protected by global $D_2$ symmetry

Pollmann, Berg, Turner, M.O. 2009
Other symmetries?

AKLT model: edge state with $S=1/2$

Does the edge state survive in more general models?

Consider perturbations to AKLT model

Generic perturbations will lift the edge degeneracy!

However, if the perturbation respect time reversal, it should keep the “Kramers degeneracy” of $S=1/2$ edge state

i.e. time reversal symmetry protects Haldane phase

cf.) edge state of topological insulator
Yet another symmetry

\[ \mathcal{H} = J \sum_j \vec{\hat{S}}_j \cdot \vec{\hat{S}}_{j+1} + B_x \sum_j S^x_j + D \sum_j (S^z_j)^2 \]

Gu and Wen, 2009

\( \mathcal{D}_2 \) symmetry (\( \pi \)-rotation about x,y,z axes): lost

string order does not work as an order parameter

Time reversal: lost

edge state does not characterize the Haldane phase

Nevertheless, Haldane phase is still distinct from other phases by QPTs

Protected by inversion symmetry!

Protected by inversion symmetry!
I: space inversion (parity)

valence bond: 

\[ |(j, j + 1)\rangle \equiv \frac{1}{\sqrt{2}} (|\uparrow\rangle_j |\downarrow\rangle_{j+1} - |\downarrow\rangle_j |\uparrow\rangle_{j+1}) \]

\[ I|(j, j + 1)\rangle = |(-j, -j - 1)\rangle = -|(-j - 1, -j)\rangle \]

\[ I|(-\frac{1}{2}, \frac{1}{2})\rangle = -|(-\frac{1}{2}, \frac{1}{2})\rangle \]

this vb makes AKLT state P-odd!

\[ I|(-\frac{5}{2}, -\frac{3}{2})\rangle|(-\frac{3}{2}, \frac{5}{2})\rangle = |(-\frac{5}{2}, -\frac{3}{2})\rangle|(-\frac{3}{2}, \frac{5}{2})\rangle \]

each vb pair is P-even
S=1 AKLT state is “P-odd”.
Now consider any perturbation, keeping P-invariance. The adiabatically connected state remains P-odd.

On the other hand, a trivial groundstate is P-even.
Any adiabatic evolution of the trivial state is also P-even as long as P-invariance is kept.

There must be a phase transition between the two groundstates (robustness of Haldane phase)

\[ |\Psi\rangle = \bigotimes_j |\psi\rangle_j \]
Symmetry Protection

S=1 Haldane phase is “protected” by ANY one of

(I) Spontaneous breaking of hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, robust in the presence of $D_2(=\mathbb{Z}_2 \times \mathbb{Z}_2)$ symmetry [\(\pi\)-rotation about x,y, and z axes]

(II) Kramers degeneracy of edge spins, robust in the presence of time-reversal

(III) Space Inversion symmetry about a bond center

(Gu-Wen/Pollmann-Berg-Turner-M.O.)
What about $S>1$?

The concept of hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry can be generalized to any integer $S$

$$U = \prod_{j<k} e^{i\pi S_j^z S_k^x}$$

[M.O. (1992)]

The hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is unbroken in $S=2,4,6,8,...$ AKLT state while broken in $S=1,3,5,7,...$ AKLT state!

“even-odd effect”
What does it mean?

The hidden $Z_2 \times Z_2$ symmetry is unbroken in the (uniform) $S=2$ AKLT state.

Q (1992): Is it indistinguishable from a trivial state, or are we just unaware of appropriate hidden order/symmetry?
What does it mean?

The hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is unbroken in the (uniform) $S=2$ AKLT state.

Q (1992): Is it indistinguishable from a trivial state, or are we just unaware of appropriate hidden order/symmetry?

A (2009): It is essentially indistinguishable from a trivial state!
Edge state for $S=2$

**S=2 AKLT state**: each end has $S=1$ (3-fold deg.)
The degeneracy will be lifted by perturbations, and generically no degeneracy remains! (no Kramers degeneracy)

If we keep the SU(2) symmetry, the presence of the $S=1$ edge state makes the system distinct from trivial states? In general, the answer is NO.
Kramers vs. non-Kramers

\( S_b = 0 \) vs. 1

- singlet
- triplet

\( S_b \) can change by level crossing at the edge (w/o bulk transition)

cf.) Todo et al. (2001)

\( S_b = 0 \) vs. 1/2

- singlet
- doublet

Kramers theorem requires all the edge levels be doubly degenerate!
The degeneracy can be only removed by bulk phase transition.
Intrinsic parity for $S>1$ chains

I: lattice inversion

The intrinsic parity is even, because you flip two valence bonds.

In general, intrinsic link parity is even (odd), if the number of valence bonds is even (odd)!
S=2 Haldane state

None of the 3 symmetries protects the “Haldane phase” as a distinct topological phase!

The S=2 “Haldane state” could be adiabatically connected to a trivial state

Is this really the case?

Yes! There exists a 1-parameter family of Matrix Product State (and corresponding Hamiltonian) interpolating S=2 AKLT state and large-D state

Pollmann, Berg, Turner, M.O. 2009
S=2 phase diagram

\[ \mathcal{H} = J \sum_{j} [S_{j}^{x} S_{j+1}^{x} + S_{j}^{y} S_{j+1}^{y} + \Delta S_{j}^{z} S_{j+1}^{z}] + D \sum_{j} (S_{j}^{z})^2 \]

conjecture by M.O. 1992

(figure taken from Tonegawa et al. arXiv:1011.6568)
S=2 phase diagram

DMRG result
Schollwock et al.
1995~1996
(figures from arXiv:1011.6568)

conjecture
M.O. 1992
S=2 phase diagram

Tonegawa et al. arXiv:1011.6568
exact diagonalization + “level spectroscopy”
(finite size scaling using CFT+perturbation)

Haldane state
connected to
Large-D
(trivial?)
phase?

“ID phase”
topological)?
Level spectroscopy

Okamoto and Nomura 1992

e.g. \[ H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \alpha \vec{S}_j \cdot \vec{S}_{j+2} \] \( S=1/2 \)

KT transition between fluid (TL liquid) phase and spontaneously dimerized phase

crossing of excited states \( \Leftrightarrow \) phase transition

size dependence of crossing point is weak, despite KT!

\[ \alpha_c \sim 0.2411 \]

crossing point is \( \alpha=0.25 \) already for 4 spins!
Haldane phase (odd $S$)

Topological phase protected by (any one of) 3 symmetries

Different mechanism for each symmetry?

Is there a “universal feature” of the Haldane phase?

Entanglement!

$|\Psi\rangle \neq |\psi_a\rangle_A |\psi_b\rangle_B$
Entanglement Spectrum

\[ |\Psi\rangle = \sum_{a=1,N_A}^{N_A} \sum_{b=1,N_B}^{N_B} \Psi_{ab} |\psi_a\rangle_A |\psi_b\rangle_B \]

\[ \Psi_{ab} \text{ } N_A \times N_B \text{ matrix} \]

Schmidt decomposition

\[ |\Psi\rangle = \sum_j \Lambda_j |\phi_j\rangle_A |\phi_j'\rangle_B \]

\[ \rho_A = \sum_j \Lambda_j^2 |\phi_j\rangle_A A A \langle \phi_j| \]

\{\Lambda_j\} \text{ Entanglement Spectrum}

\[ S_E = - \sum_j \Lambda_j^2 \log \Lambda_j^2 \]

Entanglement spectrum contains more information than entanglement entropy!
Entanglement Spectrum

\[ |\Psi\rangle = \sum_{\mu} \Lambda_{\mu} |\phi_{\mu}^{A}\rangle_A |\phi_{\mu}^{B}\rangle_B \]

The entire entanglement spectrum has exact double degeneracy in the Haldane phase!

\[ \Lambda_1 = \Lambda_2, \Lambda_3 = \Lambda_4, \Lambda_5 = \Lambda_6, \ldots \]

This degeneracy is protected by any one of the three symmetries.

Minimal entanglement entropy \( \log(2) \) when

\[ \Lambda_1 = \Lambda_2 = 1/\sqrt{2}, \Lambda_{\alpha} = 0 (\alpha \geq 3) \]
“Odd parity” state

\[|\Psi\rangle \sim \sum_{\alpha} \lambda^{(\alpha)} (|\alpha, 1\rangle_A |\alpha, 2\rangle_B - |\alpha, 2\rangle_A |\alpha, 1\rangle_B)\]

Exact two-fold degeneracy in the entire entanglement spectrum
Time evolution

We introduced adiabatic weakening of one bond

\[ J_0(t) \vec{S}_0 \cdot \vec{S}_1 \rightarrow J_0(t) \rightarrow 0 \]

Inversion symmetry of the Hamiltonian is kept (although translation symmetry is lost)

The degeneracy of the spectrum survives in the Haldane phase!
Degeneracy is universal

TABLE I. The different symmetries which can stabilize the Haldane phase. For each class of symmetries, the table shows whether string order, edge states, or the degeneracy of the entanglement spectrum are necessarily present. The symmetry under $\pi$ rotations about a pair of orthogonal axes is represented by the dihedral group $D_2$.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>String order</th>
<th>Edge states</th>
<th>Degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_2 (=Z_2 \times Z_2)$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Time reversal</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Inversion</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Summary

- Odd # of valence bonds
- [inversion symmetry]
- Kramers degeneracy of edge spins
- Odd intrinsic link parity
- [D₂ symmetry]
- Hidden Z₂xZ₂ symmetry breaking (dNR string order)
- [time-reversal invariance]
- Exact double degeneracy of entire entanglement spectrum