Rattleback Reversals

A Prototype of Chiral Dynamics

Keith Moffatt

Trinity College, Cambridge

in collaboration with Tadashi Tokieda
I lectured here at KITP on Euler’s Disk and its Finite Time Singularity in 2000

Euler’s Disk led to the Rising Egg Problem – a table-top example of dissipative (slow, secular) instability (with Y. Shimomura and Michal Branicki)

The Rising Egg led to the Rattleback – chiral dynamics, and a table-top ‘model’ of geomagnetic reversals

Toys provide surprising phenomena, dynamical insights Once these are understood, …, and fun!
Wolfgang Pauli, Neils Bohr, and the tippe-top, ~ 1955
A distinguished precedent for the investigation of spinning objects!
**Rattleback**: otherwise known as the *celt* or the *wobblestone*

A rigid body that exhibits the property of *spin asymmetry*: it can spin quite smoothly on a table in one sense, but when spun in the opposite sense, a *pitching instability* develops which extracts energy from the spin to such an extent that this spin actually reverses in sign.


Walker used the word *celt* to describe such an object
Slow-time evolution equations:

\[
\begin{align*}
\frac{dA}{dt} &= (4N_0 p_1) \cdot A, \\
\frac{dB}{dt} &= (0 N_0 p_2) \cdot B, \\
\frac{dN}{dt} &= 0 4A^2 + B^2 \cdot 0 p_3 \cdot N,
\end{align*}
\]

(\(p_1, p_2, p_3\) are friction parameters)

Let \(N(t) = \text{spin}\)

\(A(t) = \text{amplitude of pitching instability}\)

\(B(t) = \text{amplitude of rolling instability}\)
Figure 1. Solution of equations (7a,b, 8) with $\circ = 4$ and initial conditions $A(0) = B(0) = 0.01$, $N(0) = 0.5$. Blue shows $N(w)$, red $A(w)$ (pitching), and green $B(w)$ (rolling). Rapid reversals of $N$ (from positive to negative) are induced by the pitching instability, and slow reversals (from negative to positive) by the rolling instability.
Figure 2. Solution of equations (9a,b, 10) with $o = 4$, dissipation parameters $p_1 = 0.04$, $p_2 = 0.08$, $p_3 = 0.1$; initial conditions and colour code as in figure 1. The pitching instability still induces a spin reversal, but the subsequent rolling instability is not sufficiently strong to induce a second reversal against the effects of dissipation.
Figure 3. Same as figure 2, except that $N(0) = 0.5$. Here, a single weak reversal is induced at a late stage ($t \approx 19$) by the rolling instability, and the pitching instability is not excited.
An example with reduced friction, showing four reversals, as observed with a carefully constructed Cambridge rattleback

\( (p_1 = 0.01, \ p_2 = 0.02, \ p_3 = 0.025) \)
On the word \textit{celt}

According to the \textit{Oxford English Dictionary}:

\begin{quote}
“Though always confused with ‘Celt’ (pronounced \textit{kelt}) as in Celtic peoples, it is a separate word, pronounced \textit{selt}: an implement with chisel-shaped edge, of bronze or stone, ..., found among the remains of prehistoric man.”
\end{quote}

\textit{Stylo ferreo, et plumbi lamina, vel \textit{celte} sculpantur in silice.}

\textit{Job xix 24, Vulgate}

\begin{quote}
\textit{Oh that my words were now written!  Oh that they were printed in a book!  That they were graven with an iron pen and lead, in the rock for ever.}
\end{quote}

\textit{Job xix 23,24 Authorized Version}

\begin{quote}
(An appropriate sentiment for those submitting papers to scientific Journals!)
\end{quote}
Pitching and Rolling

Consider a uniform solid ellipsoid, mass $M$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad a > b > c$$

Take $z = -c$, $\theta = 0$

$$ \mathbf{g} = (0, 0, -g) $$

Use $M, c, \sqrt{\frac{g}{c}}$ as units of mass, length, time

[equivalently, $M = c = g = 1$]

Then $P$ is $(0, 0, -1)$

Principal moments of inertia at $P$ are

$$\kappa = \frac{a^2 + b^2}{5}, \quad \beta = \frac{a^2 + 2b^2}{5}, \quad \gamma = \frac{a^2 + 5b^2}{5}$$

Near $P$, contact surface is locally

$$z = -1 + \frac{x^2}{2a^2} + \frac{y^2}{2b^2}$$

If disturbed from rest, the oscillation modes:

$$x + a \omega_1^2 = 0 \quad \omega_1^2 = \frac{5(a^2 - 1)}{a^2 + 6} \quad \text{Pitching}$$

$$y + a_2 \omega_2^2 = 0 \quad \omega_2^2 = \frac{5(6b^2 - 1)}{3b^2 + 6} \quad \text{Rolling}$$

Chiral distortion

Now imagine mass redistribution inside ellipsoid so that principal axes of inertia at $P$ are rotated through a small angle about vertical (axis of eigenvector of $\mathbf{g}$).

Near $P$, surface is locally

$$z = -1 + \frac{x^2}{2a^2} + \frac{y^2}{2b^2} + \frac{\eta^2}{28}$$

$\eta = \text{chirality parameter}$
Chiral instability

The body can spin with angular velocity \((0, 0, n)\)

Effect of chirality is to destabilize pitching or rolling, or both [H. Bondi, 1956, Proc Roy. Soc. A]

We suppose
- \(|n| << 1\)
- \(|N| << 1\), (small deformation)
- \(a^2 >> b^2 >> 1\), \(\varepsilon - 1 = O(\varepsilon)\) long rattleback

Then Bondi’s equations for the oscillatory modes can be arranged, at leading order in \(a^2 >> 1\) in the form

\[
\begin{align*}
\ddot{x} + \omega_x^2 x + N\dot{y} & = (\text{garbage}), \\
\dot{y} + \omega_y^2 y + N\dot{x} & = (\text{garbage})
\end{align*}
\]

when \(N = 6(1 + \frac{n^2}{2})\)

Garbage terms (linear w x + y, and small) more \(\omega_x, \omega_y\), slightly along real axis

Only the terms \(N\dot{y}\) and \(N\dot{x}\) are (jointly) responsible for instability.

So in fact, it is sufficient to consider the simplified system (dispose of the garbage!)

\[
\begin{align*}
\ddot{x} + \omega_x^2 x + N\dot{y} & = 0 \\
\dot{y} + \omega_y^2 y + N\dot{x} & = 0
\end{align*}
\]

So we retain:
- a small change in pitching mode due to chirality
- a small change in rolling mode due to Coriolis

[cp: equation for \(n\omega\)-dynamo

\[
\begin{align*}
\frac{\partial A}{\partial t} & = \alpha B + \eta D^2 A \\
\frac{\partial B}{\partial t} & = (D \Delta A_y).D U + \eta D^2 B
\end{align*}
\]

let \(x(t), y(t) \sim e^{i\omega t}\)

\[
\begin{vmatrix}
-\omega^2 + \omega_x^2 & -\chi \omega_y^2 \\
\chi \omega_y^2 & -\omega^2 + \omega_y^2
\end{vmatrix} = 0
\]

\[
(\omega^2 - \omega_x^2)(\omega^2 - \omega_y^2) + iN\chi \omega^3 = 0
\]

for \(|N\chi| << 1\), perturbed roots are

\[
\omega_1 - i\tau_1, \quad \omega_2 - i\tau_2
\]

where \(\tau_1 = \frac{N\chi \omega_x^2}{2(\omega_x^2 - \omega_y^2)}\), \(\tau_2 = \frac{-N\chi \omega_y^2}{2(\omega_x^2 - \omega_y^2)}\)
The amplitudes $A_0(t), B_0(t)$ of pitching and rolling evidently satisfy

$$\frac{dA_0}{dt} = \nu_1 A_0, \quad \frac{dB_0}{dt} = \nu_2 B_0.$$

Take $N > 0$; then these become

$$\frac{dA_0}{dt} = \lambda N A_0, \quad \frac{dB_0}{dt} = -N B_0,$$

where now $N(t)$ varies on the slow time-scale.

Hence $A_0 B_0^\lambda = \text{cst.}$
If we neglect dissipative effects (i.e., assume no-slip condition, neglect air friction, etc.) then energy (kinetic + potential) is also conserved

\[
E_{\text{kinetic}} = \frac{1}{2} \beta (x^2 + \omega^2 z^2) = \frac{1}{2} \beta \omega_0^2 A_0^2
\]

\[
E_{\text{potential}} = \frac{1}{2} \omega_0 (y^2 + \omega_0^2 y^2) = \frac{1}{2} \omega_0 \omega_0^2 B_0^2
\]

\[
E_{\text{total}} = \frac{1}{2} Y n^2 = \frac{7}{6} Y N^2 / 25(1+\gamma^2)
\]

Rescale \( A_0 \) and \( B_0 \):

\[
A = 5(1 + \frac{1}{5}) \sqrt{\frac{Y}{B}} A_0, \quad B = 5(1 + \frac{1}{5}) \sqrt{\frac{Y}{B}} B_0
\]

Then \( E = E_r + E_i \cdot E = \text{cst.} \) because

\[
A^2 + B^2 + N^2 = \text{cst.}
\]

So we have:

\[
\frac{dA}{dt} = \lambda NA
\]

\[
\frac{dB}{dt} = -NB
\]

\[
\frac{dN}{dt} = -\lambda A^2 + B^2
\]

with integral,

\[
A^2 + B^2 + N^2 = E = \text{cst.}
\]

\[
\Delta R = \frac{1}{\lambda} \Delta t
\]