String Thermodynamics From Scratch

- Infrared and Ultraviolet in String Theory vs Effective Field Theory.
  - Vacuum Energy Density in String Theory is normalizable. Implications.
- Equilibrium String Thermodynamics in the Canonical Ensemble.
  - Implications of Thermal Duality Transformations. NO Hagedorn Transition.
  - Holographic (2d) Growth at High T vs Field Theoretic Growth at T < String scale.
- Type I and Type II Unoriented Open and Closed String Ensembles.
- Type I and Type I': The Long String Phase Transition is First Order!


Key elements in our work were motivated from J. Atick and E. Witten (1988).


Infrared & Ultraviolet in String Theory vs Field Theory

- Recall nature of UV/IR regulator dependence in the YM-Higgs theory:

\[ Z = \frac{1}{\text{Vol(gauge)}} \int dA d\Phi \ e^{-S_{\text{YM}}[A]-S[\Phi, \lambda]} = \int d\lambda d\Phi \ e^{-S_{\text{YM}}[\lambda]-S[\Phi, \lambda]} \]

\[ \langle O_1 \ldots O_n \rangle = \frac{1}{Z[A, \Phi]} \frac{1}{\text{Vol(gauge)}} \int dA d\Phi \ O_1[A, \Phi] \ldots O_n[A, \Phi] \ e^{-S_{\text{YM}}[A]-S[\Phi, \lambda]} . \]

- Only normalized correlators free of UV regulator ambiguity. Cancellation of IR divergences leaves renormalization scheme (MS) dependent finite corrections.

- 2d Quantum Gravity coupled to D Scalars = Embedded Worldsurfaces. Apparently: D-dim non-renormalizable EFT, in inv powers of string scale.
  In fact: String Theory is exactly renormalizable, a single Wilsonian ren’n’.

\[ S = \int d^2\xi \sqrt{g} \ [ \lambda R(g) + \mu + m^2 g^{\mu\nu} \partial_\mu X \partial_\nu X ] = 4\kappa \lambda (2 \cdot 2h - b - c) + \mu A + m^2 S_p[X, g] . \]

\[ Z = \frac{1}{\text{Vol(Diff)}} \int d^2X e^{-\frac{1}{2\kappa} R(g) - \kappa \lambda (2h - b - c) - \mu A + m^2 S_p[X, g]} = \int d^2X e^{-\frac{1}{2\kappa} R(g) - \kappa \lambda (2h - b - c) - \mu A + m^2 S_p[X, g]}. \]
2d Gauge Invariance and the Lack of Infrared Ambiguity

Critical String Theory: All renormalized 2d masses vanish (Weyl Inv).

\[ W_{\text{str}}[\mathbf{g}, \mathbf{X}] = \frac{1}{\text{Vol}(\text{Diff}_+ \times \text{Weyl})} \int \mathcal{D} \mathbf{X} \ e^{-S_{\text{eff}}[\mathbf{g}, \mathbf{X}, \mu, S, \lambda, \lambda', \lambda'']} \mathcal{D} \mu \mathcal{D} \lambda \mathcal{D} \lambda' \mathcal{D} \lambda''.
\]

One-loop effective potential = sum over connected vacuum graphs = an ordinary integral over worldsheet moduli. Finite, normalized, result.

\[ \mathbf{X}(\mathbf{a}, \mathbf{X}_n, \lambda_n; \mathbf{C}_n) \Rightarrow \prod_n \int \mathcal{D} \mu \mathcal{D} \lambda \mathcal{D} \lambda' \ e^{-\beta \mathbf{X}} = \left[ \det (\mathbf{\mu} \cdot \pi^{-1} \mathbf{A}) \right]^{-1/2} \Rightarrow \lim_{\beta \to \infty} \frac{d}{ds} \sum_{n=1}^{N} \mu_n \lambda_n, \]

A unique gauge invariant prescription exists for the regularization of such eigenvalue sums in 2d quantum gravity. Sommerfeld-Watson Transforms.

Correlation functions also unambiguously normalized in terms of string mass scale, dimensionless string coupling: a single Wilsonian renorm\'n.


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Free Energy of Closed Bosonic String Ensemble


Helmholtz Free Energy: \( F = -T \ W(T) = V \times \text{Effective Potential}. \)

"Heat Bath" = background spacetime geometry, fluxes and fields.

=> Canonical, rather than Microcanonical Ensemble. Comments.

\[ W_{\text{nc}} = 1^{2g} (4\pi \alpha')^{2g} \cdot \int \frac{d^{2g} \mathbf{X}}{2\pi} \tau_1^{2g-2+2/\mathbf{X}} \cdot e^{\frac{1}{\beta} \mathbf{X}} \cdot \prod_{n=1}^{N} \left[ 1 + e^{-2\beta n} \right]^{2g-2} \cdot \sum_{a, b, c, n=1}^{2g} \mathbf{e}^{a n} \mathbf{e}^{b n} \mathbf{e}^{[2a \mathbf{X} + b \mathbf{X} + c \mathbf{X}] / \mathbf{F}}. \]

* Spatial Volume: \( V = 1^{2g}. \) Thermodynamic Limit: \( \alpha' \to 0, L \to \infty. \)

* Momentum Modes \( \leftrightarrow n \in \mathbb{Z}. \) Winding Modes \( \leftrightarrow w \in \mathbb{Z}. \) Orientations \( \leftrightarrow \pm n, \pm w. \)

* 2d Field theory on a strip. Reparametrizable invariant sum on strips of all shape and size.

Thermal Duality = Euclidean timelike T-duality Transformation. In closed bosonic string theory, \( W(T) \) is thermal self-dual. Strong Holography.

* \( W(T) \) invariant under: \( \beta \to \frac{\beta'}{\beta}, \ n \leftrightarrow w, \ \beta' = 4 \pi \alpha' \implies F(T) = \lim_{T_T = \infty} T_T^{3/4} \frac{T_T^3}{T} = T_T^{3/4} \rho. \)

Tachyonic thermal spectrum, starting from \( T=0. \)

\[ a' \langle \text{mass}^2 \rangle = -1 + \frac{n^2 T_T^3 + w^2 T_T^3}{4 T_T^3}, \ n, w \in \mathbb{Z}. \) Relevant \( \leftrightarrow \) Irrelevant: \( T_{\pm n} = \frac{T_T^3}{4}, T_{\pm n} = 4 T_T^3. \)
Heterotic Ensemble: Absence of a Hagedorn Transition


- Thermal duality exchanges E8xE8 and Spin(32)/Z2 theories.
- Euclidean timelike Wilson line background <=> quantization of finite temp Yang-Mills gauge theory in modified axial gauge.

\[
W_{\text{str}}(\beta) = \int [d\pi] \beta^{N/2} \frac{d^4\tau}{2\pi^2} \tau^\gamma |\eta(\tau)|^{\alpha} \cdot \mathcal{Z}_{\text{str}}(\tau, \beta).
\]

- Modular Invariance and (1,0) Superconformal constraints, tachyon free spectrum.
- Interpolates between \( W_{\text{max}}(T) \) and \( W_{\text{spec}2/12}(\beta = 0) \). Let \( p' = \Gamma_{\alpha, \tau} \beta A_{\alpha} = (1^4, 0^+). \)
- \( \Gamma^{02, 0} : (p'; p_1, p_\alpha) = (p' + W \hat{\beta} A ; p_1 - p' A - W \hat{\beta} A^2, p_\alpha - p' A - W \hat{\beta} A^2). \)
- Thermal effects can break su(3) thru thermal mode dependent phases in the string path integral, idea due to Atick Witten, NPB310 (1988) 291.

\[
F(T) = \rho(T) V. \text{ limit } \tau \rightarrow T \text{ where } \rho(T)_{\text{max}} = \frac{T^4}{T^4} \rho^{(0)}_{\text{spec}2/12} = \frac{T^4}{T^4} \rho^{(0)}_{\text{spec}2/12} \cdot \text{ high temperature degrees of freedom grow as fast as in a 2d field theory.}
\]

Interpol'n implements therm duality, enabling tachyon free equilib'm ensemble.

Kosterlitz-Thouless Duality Phase Transition


- Thermodynamic Potentials:

\[
F(T) = -T W(T) + V \rho. P = \left\{ \frac{\partial F}{\partial V}, U = T \left( \frac{\partial W}{\partial T} \right), S = \left( \frac{\partial F}{\partial T} \right), C_v = T \left( \frac{\partial S}{\partial T} \right) \right\}. G = U - TS + PV.
\]

- Finite and normalizable functions of \((T, V)\). Thermodynamic Limit : \(a' \rightarrow 0, L \rightarrow \infty, \)
- Cosm Const \(\Rightarrow\) Ideal Fluid, negative pressure. \(\rho = \rho \Rightarrow\) Gibbs free energy vanishes.
- Thermodynamic potentials appear analytic in temperature in the vicinity of \(T_c\).

- Duality transition is in the Kosterlitz-Thouless Universality Class, (J. Kosterlitz and D. Thouless, J Phys C6 (1973) 1181.)

Denote: \([d\tau] = L^4 (4\pi a')^{10} \frac{d^4\tau}{2\pi^2} \tau^{10} |\eta(\tau)|^{10} \mathcal{Z}_{\text{str}}(\tau, \beta). \quad \gamma(\tau; \beta) = \frac{2\pi n}{\beta^1} + \frac{w^2 \beta^2}{4\pi a'} \mathcal{W}_{\text{str}}(\beta). \quad W_{\text{str}} = \int [d\tau] e^{\gamma(\tau_1, \ldots, \tau_N; \gamma, \ldots, \gamma')} \cdot F(\beta) = \frac{1}{\beta} W_{\text{str}}, U = -W_{\text{str}}, S = W_{\text{str}} - \beta W_{\text{str}} \gamma, C_v = \beta^2 W_{\text{str}} \gamma, \ldots
\]

- Infinite hierarchy of potentials, analytic in temperature in the vicinity of \(T_c\).
Type I and Type II Open and Closed String Theories


- Closed oriented string sector does not break supersymmetry at 1loop order.
- Type IIA and IIB strings have no Yang-Mills fields in the absence of a Ramond-Ramond sector \( \Rightarrow \) Type II thermal ensemble is unstable.

Consider Type IIB ensemble at infinitesimal temperature above \( T = T_{\text{crit}} \).

- Tachyonic winding modes dominate \( W \). The \( w = 1 \) mode is most relevant operator.
- AL (QFT) split: \( F = F_{\text{inv}} + F_{\text{inv}} \). String theory: 2d gauge inv picks unique UV regulator.
- \( g = \) bosonic vacuum degeneracy. Worldsheet RG flow is in the direction of lower \( g \).

- Thermal Phases in the Path Integral: \((-1)^F \) allowed, \((-1)^w \) disallowed by Modular Invariance.

- Mass Spectrum: \( \rho'(\text{mass})^2 = 1 + \frac{wT^2}{2T^2}, \quad n, w \in \mathbb{Z}_+ \).

*ONLY* the pure winding modes give non-vanishing tachyonic contributions to the Free Energy.

- Infrared stable fixed point is the noncompact supersymmetric vacuum.

Type I Unoriented Open and Closed String Theory


- Thermal Duality maps: Type IIA to Type IIB.
- \( \beta A_4 \) with D8branes to type IB with D9branes.
- Dbranes are half BPS states, so generic type IIA and IIB are type I' and IB.
- Closed oriented sector \( (\text{oneloop} = \text{torus}) \) does not break susy.

Consider Type IB with 32 D9branes, worldvolume Wilson line: \( \int dX^A A_A \), \( \beta A_4 = (1^+, 0^+) \).

- Breaks gauge group to \( SO(16) \times SO(16) \). Nonsupersymmetric. Note thermal phase.
- Oneloop vacuum energy receives contrib'ns from Annulus, Mobius strip, Klein Bottle.
- Cancellation of Ramond-Ramond scalar tadpole \( \Rightarrow \) absence of dilaton tadpole. \( F_{\text{inv}} = 0 \).

Thermal Dual Type I' with 16 D8branes on either orientifold plane. Momentum \( \leftrightarrow \) Windings.

\[
F(\beta) = \frac{1}{\beta} \mathcal{L}'(4\pi^4 \mathcal{A}^2) \int \frac{d^8 \mathbf{t}}{8!} t^{8s} e^{\gamma/12} \cdot \eta(\mathbf{it}) \cdot \sum_{k} 2^4 N^2 (Z_{m}^n - e^{c_{m}^{n}} Z_{m})
+ \left( Z_{m}^n - e^{c_{m}^{n}} Z_{m} \right) - 2^2 N (Z_{m}^n - e^{c_{m}^{n}} Z_{m}) \times e^{2 s \gamma / 2} e^{2 \gamma / 2}.
\]

Nonsupersymmetric Type I vacuum with susy broken by thermal effects, but without generation of a dilaton tadpole. Comment: Higher loops?
Holography in Open and Closed String Theory

S. Chaudhuri, hep-th/0409301.

Thermal Duality maps Type I’ with D8branes to type IB with D9branes.
Recall: Factorization limit dominated by the asymptotics of highest mass open string modes. This gives high temperature growth, above string scale. Short cylinder limit dominated by lowest open string modes (gauge).

\[ \lim_{\beta \to 0} F(\beta) = \frac{1}{\beta} \sum_{n \geq 0} \left[ \beta^{-4} (2a' \pi n^2) \right] \rho_{\text{high}} \]
\[ \lim_{\beta \to \infty} F(\beta) = \beta^{-10} \rho_{\text{low}}. \]

Strong Holography, at high temperatures above the string scale: growth in degrees of freedom only as fast as that in a 2d field theory. As expected, the growth below the string scale is that of a 9+1-d quantum field theory.

String Theory has many fewer degrees of freedom at high temperatures!

Note: Unlike \( F \), remaining thermodynamic potentials are non-vanishing.

Order Parameter for Long String Phase Transition


Recall expectation value of closed timelike loop = order parameter for thermal deconfinement transition in nonabelian gauge theory.

\[ W_c(R, \beta) = \lim_{\tau \to \infty} \int_0^\beta \, \frac{d \alpha}{\pi} \sum_{x \in \mathbb{Z}} \left[ \left( \frac{\Theta_n}{\eta} \right)^4 - \left( \frac{\Theta_{n+1}}{\eta} \right)^4 - e^{\frac{i \pi}{16} \alpha} \left( \frac{\Theta_n}{\eta} \right)^4 - \left( \frac{\Theta_{n+1}}{\eta} \right)^4 \right]. \]

Low energy limit of the Macroscopic Loop Amplitude in String Theory => Pair Correlation Function of Polyakov-Wilson Loops.

Macroscopic Loop Amplitude given by the reparametrization invariant sum over worldsheets with boundaries mapped to fixed curves in embedding target space.

\[ V(R, \beta) = - \left[ R \left( 1 + \frac{16 \pi^2 a' \beta}{R \beta^2} \right)^{\frac{1}{12}} \right], \quad \beta > \beta_c. \quad V(R, \beta) = - \left[ R \left( 1 + \frac{\beta^2}{R^2} \right)^{\frac{1}{12}} \right], \quad \beta < \beta_c. \quad \beta_c = 1/2 \sqrt{2} \pi a'^{\frac{1}{2}}. \]

Note: Long String Phase Transition occurs at temperature threshold for the production of the first thermal winding mode! (First Order).

Conclusions and Open Questions

  - Couplings in worldvolume gauge-gravity theory on braneworld.
- String Thermodynamics in Canonical Ensemble for all six supersymmetric string theories: type I and I’, type IIA and IIB, E8xE8 and Spin(32)/Z2.
  - No Hagedorn transition. Kosterlitz-Thouless and Long String Phase Transition.
  - Strong Holographic Growth in High T (T > String scale) Degrees of Freedom.
- Instabilities, using RG Flow analysis: type I unoriented open and closed string ensemble is stable. Nonsusy, with neither tachyons, nor tadpoles.
- Order Parameter for a Deconfining Phase Transition in the Worldvolume Gauge Theory: Long String Phase Transition (First Order).
- Microcanonical Ensemble of Type I unoriented open & closed strings?
- Stat Mech of Cosmic String Gas. Intercommutation vs Splitting?

(pedagogical review in hep-th/0409031)