Properties of monopole operators in 3d gauge theories

Silviu S. Pufu
Princeton University

Based on:

- arXiv:1303.6125
- arXiv:1309.1160 (with Ethan Dyer and Mark Mezei)
- work in progress with Ethan Dyer, Mark Mezei, and Subir Sachdev

KITP, January 31, 2014
Consider (compact) $U(1)$ gauge theory in 3 dimensions:

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}^2 + \text{matter w/ integer } U(1) \text{ charges.}$$

IR dynamics:

- No matter $\implies$ confinement [Polyakov].
- Lots of matter $\implies$ interacting CFT [Appelquist, ...]. Maxwell term is irrelevant.
- Can be studied in $1/N_f$ expansion.

Few matter fields: the theory may confine.
- Analog of $\chi_{SB}$ if we have few fermions [Pisarski; Vafa, Witten; ...]

**When** should we expect an interacting CFT? What is the operator spectrum of this CFT?
Monopole operators

- A $U(1)$ gauge theory in 3d has a global $U(1)_{\text{top}}$ symmetry with conserved current $j^\text{top}_\mu = \frac{1}{4\pi} \epsilon_{\mu\nu\rho} F^{\nu\rho}$. Monopole operators are local operators with non-zero $U(1)_{\text{top}}$ charge.

- Monopole operators are “disorder operators” that insert a monopole singularity in the gauge field.

  - A monopole of charge $q \in \mathbb{Z}/2$ centered at the origin:
    \[
    \int_{S^2} F = 4\pi q.
    \]

  - Monopole operator $\mathcal{M}_q(0)$ satisfies OPE
    \[
    \mathcal{M}_q(0) \int_{S^2} F \sim \mathcal{M}_q(0) 4\pi q + \cdots.
    \]

  - Lots of operators with given $q$.  

Monopole operators

- A $U(1)$ gauge theory in 3d has a global $U(1)_{\text{top}}$ symmetry with conserved current $j^{\text{top}}_\mu = \frac{1}{4\pi} \epsilon_{\mu \nu \rho} F^{\nu \rho}$. Monopole operators are local operators with non-zero $U(1)_{\text{top}}$ charge.

- Monopole operators are “disorder operators” that insert a monopole singularity in the gauge field

  - A monopole of charge $q \in \mathbb{Z}/2$ centered at the origin:

    $$\int_{S^2} F = 4\pi q.$$ 

  - Monopole operator $\mathcal{M}_q(0)$ satisfies OPE

    $$\mathcal{M}_q(0) \int_{S^2} F \sim \mathcal{M}_q(0) 4\pi q + \cdots.$$ 

- Lots of operators with given $q$. 
In CFT: use state-operator correspondence and study the theory on $S^2 \times \mathbb{R}$.

Let $\mathcal{M}_q(0)$ correspond to the ground state in the sector of monopole flux $\int_{S^2} F = 4\pi q$ [Borokhov, Kapustin, Wu ’02].
Roles played by monopole operators

- **Mechanism for confinement**: monopole proliferation in theories where $U(1)_{\text{top}}$ is explicitly broken microscopically [Polyakov].

  - Key insight: failure of the Bianchi identity implies Wilson loop area law.

  - If monopole operators are irrelevant $\Rightarrow$ confinement (e.g. [Hermele, Senthil, Fisher, Lee, Nagaosa, Wen]).

- **Order parameters** for continuous phase transitions between two ordered phases (i.e. which evade the Landau-Ginsburg-Wilson paradigm) [Sachdev, Read '89].

  - A discrete $\mathbb{Z}_k$ subgroup of $U(1)_{\text{top}} \times SO(2)_{\text{rot}}$ is a symmetry of the lattice Hamiltonian.

  - Relevant monopole ops that transform under $\mathbb{Z}_k$: order parameters.

  - Relevant monopole ops invariant under $\mathbb{Z}_k$: can cause confinement.
Roles played by monopole operators

- **Mechanism for confinement**: monopole proliferation in theories where $U(1)_{\text{top}}$ is explicitly broken microscopically \cite{Polyakov}.
  - Key insight: failure of the Bianchi identity implies Wilson loop area law.
  - If monopole operators are irrelevant $\implies$ confinement (e.g. \cite{Hermele, Senthil, Fisher, Lee, Nagaosa, Wen}).

- **Order parameters** for continuous phase transitions between two ordered phases (i.e. which evade the Landau-Ginsburg-Wilson paradigm) \cite{Sachdev, Read '89}.
  - A discrete $\mathbb{Z}_k$ subgroup of $U(1)_{\text{top}} \times SO(2)_{\text{rot}}$ is a symmetry of the lattice Hamiltonian.
  - Relevant monopole ops that transform under $\mathbb{Z}_k$: order parameters.
  - Relevant monopole ops invariant under $\mathbb{Z}_k$: can cause confinement.
Monopole operators as order parameters

Square lattice antiferromagnet with $SU(N)$ spins at each site.

**J-Q model:**

$$H = J \sum_{\langle ij \rangle} S^\alpha_\beta(i) S^{\beta_\alpha}(j) + Q \times \text{(four spins)}$$

$$S = \int d^2 r \, d\tau \left[ |(\partial_\mu - iA_\mu)z_\alpha|^2 + s|z_\alpha|^2 + u(|z_\alpha|^2)^2 + \frac{1}{4e^2} F_{\mu\nu}^2 \right]$$

- At the critical point $e = \infty$ and $s$ is tuned to zero $\implies$ same universality class as the $\mathbb{CP}^{N-1}$ model [Motrunich, Vishwanath ’04; Senthil, Balents, Fisher, Sachdev, Vishwanath ’04].
- Néel order: $\langle z_\alpha \rangle \neq 0$; VBS order: $\langle M_{1/2} \rangle \neq 0$. [Sachdev, Read]
Spin liquids

In other models, obtain compact QED with some number of fermions $\Rightarrow$ “spin liquids” [Wen, ...].

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu \nu}^2 + \sum_{a=1}^{N_f} \psi_\alpha^\dagger (i \not\! \partial + \mathcal{A}) \psi_\alpha.$$ 

Monopole operators that transform under lattice symmetries $\Rightarrow$ VBS order parameters.

Monopoles invariant under lattice symmetries can cause confinement. Non-trivial CFT exists only if they are irrelevant [Heremele, Senthil, Fisher, Lee, Nagaosa, Wen ’04].
Properties of monopole operators in CFT

- Task: determine quantum numbers under conformal group (scaling dimension, Lorentz spin) and under the flavor symmetry.

- Method: $1/N_f$ expansion. Good because gauge field fluctuations are suppressed.

- In fermionic theory, for instance:

  $$ S = \int d^3x \sqrt{g} \sum_{\alpha=1}^{N_f} \psi_\alpha^\dagger (i\mathcal{D} + \mathcal{A}) \psi_\alpha. $$

- Integrate out $\psi$:

  $$ Z = \int DA \exp \left[ N_f \text{ tr } \log (i\mathcal{D} + \mathcal{A}) \right] $$

- Saddle point approximation $A = \mathcal{A} + a$ for some saddle $\mathcal{A}$.

- At large $N_f$, gauge field fluctuations are suppressed by $1/\sqrt{N_f}$.  

Silviu Pufu (Princeton University)
Properties of monopole operators in CFT

- Task: determine quantum numbers under conformal group (scaling dimension, Lorentz spin) and under the flavor symmetry.
- Method: $1/N_f$ expansion. Good because gauge field fluctuations are suppressed.
- In fermionic theory, for instance:

$$S = \int d^3x \sqrt{g} \sum_{\alpha=1}^{N_f} \psi_\alpha^\dagger (i\not{\partial} + \not{A}) \psi_\alpha.$$  

Integrate out $\psi$:

$$Z = \int DA \exp \left[ N_f \text{tr} \log (i\not{\partial} + \not{A}) \right]$$

- Saddle point approximation $A = \mathcal{A} + a$ for some saddle $\mathcal{A}$.
- At large $N_f$, gauge field fluctuations are suppressed by $1/\sqrt{N_f}$. 

Silviu Pufu (Princeton University)
A brief history of the $1/N_f$ expansion

**The leading order epoch** (treat gauge field as background):
- [Murthy, Sachdev ’90] : bosonic theory, scaling dim’s to order $O(N_f)$.
- [Borokhov, Kapustin, Wu ’02] : fermionic theory, scaling dim’s to order $O(N_f)$, flavor charges of $M_{1/2}$.
- [Metlitski, Hermele, Senthil, Fisher ’08] : bosonic theory, scaling dim’s to order $O(N_f)$.
Assumption: rotationally-invariant saddle.

**The subleading order epoch** (fluctuations of the gauge field):
- [SSP ’13] : fermionic theory, scaling dim of $M_{1/2}$ to order $O(N_f^0)$.
- [Dyer, Mezei, SSP ’13] : fermionic theory, scaling dim’s to order $O(N_f^0)$ for all $M_q$; flavor symmetry charges; generalization to QCD.
- [Dyer, Mezei, SSP, Sachdev, in progress] : bosonic theory, scaling dim’s to order $O(N_f^0)$. 
A brief history of the $1/N_f$ expansion

The leading order epoch (treat gauge field as background):
- [Murthy, Sachdev ’90]: bosonic theory, scaling dim’s to order $O(N_f)$.
- [Borokhov, Kapustin, Wu ’02]: fermionic theory, scaling dim’s to order $O(N_f)$, flavor charges of $\mathcal{M}_{1/2}$.
- [Metlitski, Hermele, Senthil, Fisher ’08]: bosonic theory, scaling dim’s to order $O(N_f)$.

Assumption: rotationally-invariant saddle.

The subleading order epoch (fluctuations of the gauge field):
- [SSP ’13]: fermionic theory, scaling dim of $\mathcal{M}_{1/2}$ to order $O(N_f^0)$.
- [Dyer, Mezei, SSP ’13]: fermionic theory, scaling dim’s to order $O(N_f^0)$ for all $\mathcal{M}_q$; flavor symmetry charges; generalization to QCD.
- [Dyer, Mezei, SSP, Sachdev, in progress]: bosonic theory, scaling dim’s to order $O(N_f^0)$. 

Scaling dimensions in fermionic theory

- The IR scaling dimension of $M_q$ are [Borokhov, Kapustin, Wu ’02; SSP ’13; Dyer, Mezei, SSP ’13]:

<table>
<thead>
<tr>
<th>$q$</th>
<th>scaling dimension [$M_q$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>$0.265N_f - 0.0383 + O(1/N_f)$</td>
</tr>
<tr>
<td>1</td>
<td>$0.673N_f - 0.194 + O(1/N_f)$</td>
</tr>
<tr>
<td>3/2</td>
<td>$1.186N_f - 0.422 + O(1/N_f)$</td>
</tr>
<tr>
<td>2</td>
<td>$1.786N_f - 0.706 + O(1/N_f)$</td>
</tr>
<tr>
<td>5/2</td>
<td>$2.462N_f - 1.04 + O(1/N_f)$</td>
</tr>
</tbody>
</table>

- $M_{1/2}$ is irrelevant, provided:

$$[M_{1/2}] > 3 \implies N_f \geq 12.$$  

- So we expect the theory does not confine whenever $N_f \geq 12$.
- $F$-theorem $\implies$ confinement impossible for $N_f \geq 12$. Also [Grover].
Scaling dimensions in fermionic theory

- The IR scaling dimension of $\mathcal{M}_q$ are [Borokhov, Kapustin, Wu ’02; SSP ’13; Dyer, Mezei, SSP ’13] :

<table>
<thead>
<tr>
<th>$q$</th>
<th>scaling dimension $[\mathcal{M}_q]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$1/2$</td>
<td>$0.265N_f - 0.0383 + O(1/N_f)$</td>
</tr>
<tr>
<td>1</td>
<td>$0.673N_f - 0.194 + O(1/N_f)$</td>
</tr>
<tr>
<td>$3/2$</td>
<td>$1.186N_f - 0.422 + O(1/N_f)$</td>
</tr>
<tr>
<td>2</td>
<td>$1.786N_f - 0.706 + O(1/N_f)$</td>
</tr>
<tr>
<td>$5/2$</td>
<td>$2.462N_f - 1.04 + O(1/N_f)$</td>
</tr>
</tbody>
</table>

- $\mathcal{M}_{3/2}$ is irrelevant, provided:

$$\left[\mathcal{M}_{3/2}\right] > 3 \implies N_f \geq 4.$$ 

- So we expect the theory does not confine whenever $N_f \geq 4$ if $\mathcal{M}_{1/2}$ and $\mathcal{M}_1$ transform non-trivially under lattice symmetries.
Flavor quantum numbers in **fermionic** theory

- $N_f$ fermions in $U(1)$ gauge theory have $SU(N_f)$ flavor symmetry.
- How do monopole operators transform in representations of $SU(N_f)$?
- **Step 1:** Ground state of $N_f$ fermions on $S^2 \times \mathbb{R}$ in the presence of uniform magnetic flux $F = q \sin \theta d\theta \wedge d\phi$.
- **Step 2:** Take into account the effect of having a dynamical gauge field.
Step 1: Fermions on $S^2$ in uniform magnetic flux

- The Dirac equation

$$(i\slashed{D} + A)\psi^\alpha = 0$$

has solutions with energy $\pm \sqrt{(j + 1/2)^2 - q^2}$ transforming in the spin-$j$ irrep of $SU(2)_{\text{rot}}$.

- $2|q|N_f$ zero-energy modes with $j = |q| - 1/2$ (for each flavor there are $2j + 1$ zero-energy modes).

  - Creation and annihilation operators $c_{jm}^{\alpha \dagger}$ and $c_j^\alpha$, respectively.

- Ground state: non-zero energy modes are not excited.

- Ground state Hilbert space $\mathcal{G}$ has dimension $2^{2|q|N_f}$. It is spanned by $|\Omega\rangle$, $c_{jm}^{\alpha \dagger}|\Omega\rangle$, $c_{jm_1}^{\alpha_1 \dagger}c_{jm_2}^{\alpha_2 \dagger}|\Omega\rangle$, etc.
Step 1: Fermions on $S^2$ in uniform magnetic flux

- The Dirac equation

\[(i\mathcal{D} + A)\psi^\alpha = 0\]

has solutions with energy $\pm \sqrt{(j + 1/2)^2 - q^2}$ transforming in the spin-$j$ irrep of $SU(2)_{\text{rot}}$.

- $2|q|N_f$ zero-energy modes with $j = |q| - 1/2$ (for each flavor there are $2j + 1$ zero-energy modes).

  - Creation and annihilation operators $c_{jm}^{\alpha\dagger}$ and $c_{jm}^\alpha$, respectively.

- Ground state: non-zero energy modes are not excited.

- Ground state Hilbert space $\mathcal{G}$ has dimension $2^{2|q|N_f}$. It is spanned by $|\Omega\rangle$, $c_{jm}^{\alpha\dagger}|\Omega\rangle$, $c_{jm_1}^{\alpha_1\dagger}c_{jm_2}^{\alpha_2\dagger}|\Omega\rangle$, etc.
Step 2: Dynamical gauge field

- At IR fixed point, ignore Maxwell field. The action is

\[ S = \int d^3x \sum_{\alpha = 1}^{N_f} \psi_\alpha^\dagger (i\mathcal{D} + \mathcal{A} + \mathcal{a})\psi_\alpha. \]

- The path integral over \( a_\mu \) imposes \( j^\mu = \sum_{\alpha = 1}^{N_f} \psi_\alpha^\dagger \gamma^\mu \psi_\alpha = 0. \)

- The space of physical ground states \( \mathcal{G}_{\text{phys}} \) consists of those states of \( \mathcal{G} \) that satisfy

\[ j^\mu(x)|\chi\rangle = 0. \]
The current is:

$$j^\mu(x) = \sum_{m,m'} \left( c_{jm}^{\alpha\dagger} c_{jm'}^\alpha - \frac{N_f}{2} \delta_{mm'} \right) S^\dagger_{q,jm}(\theta, \phi) \gamma^\mu S_{q,jm'}(\theta, \phi) + \cdots.$$ 

Integrate $j^\tau(x)$ against $Y_{00}(\theta, \phi) \implies$ total charge constraint:

$$\sum_{m,\alpha} c_{jm}^{\alpha\dagger} c_{jm}^\alpha |\chi\rangle = |qN_f|\chi\rangle.$$ 

Integrate $j^\tau(x)$ against $Y_{1m}(\theta, \phi) \implies$ total spin vanishes.

Integrate $j^\tau(x)$ against $Y_{\ell m}(\theta, \phi), \ell > 1 \implies$ more complicated constraints.

All $\ell \geq 1$ constraints: invariance under the $SU(2|q|)$ that rotates the $m$ index of $c_{jm}^{\alpha\dagger}$. 

Silviu Pufu (Princeton University)
Package all $c^{\alpha \dagger}_{jm}$ into a column vector of length $2|q|N_f$ and let $SU(2|q|N_f)$ act on it.

Clearly, $SU(2|q|N_f) \supset SU(N_f) \times SU(2|q|)$.

Group theory question: find $SU(2|q|)$ singlets under the decomposition of the rank-$|q|N_f$ anti-symmetric tensor irrep of $SU(2|q|N_f)$.

Answer: only one $SU(2|q|)$ singlet transforming in the $SU(N_f)$ irrep with Young diagram
Fun with group theory

- Package all $c_{jm}^{\alpha \dagger}$ into a column vector of length $2|q|N_f$ and let $SU(2|q|N_f)$ act on it.

- Clearly, $SU(2|q|N_f) \supset SU(N_f) \times SU(2|q|)$.

- Group theory question: find $SU(2|q|)$ singlets under the decomposition of the rank-$|q|N_f$ anti-symmetric tensor irrep of $SU(2|q|N_f)$.

- Answer: only one $SU(2|q|)$ singlet transforming in the $SU(N_f)$ irrep with Young diagram

```
\begin{array}{c}
N_f/2 \\
2|q| \\
\end{array}
```
Quantum numbers of monopole operators in fermionic theory

- $M_q$ transforms under $SU(N_f)$ as [Dyer, Mezei, SSP ’13] :

\[
N_f / 2 \left\{ \begin{array}{c}
2 |q| \\
\end{array} \right. 
\]

- These operators are Lorentz scalars b/c $SU(2)_{\text{rot}} \subset SU(2|q|)$.

- Scaling dimensions

<table>
<thead>
<tr>
<th>$q$</th>
<th>scaling dimension $[M_q]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>$0.265N_f - 0.0383 + O(1/N_f)$</td>
</tr>
<tr>
<td>1</td>
<td>$0.673N_f - 0.194 + O(1/N_f)$</td>
</tr>
<tr>
<td>3/2</td>
<td>$1.186N_f - 0.422 + O(1/N_f)$</td>
</tr>
<tr>
<td>2</td>
<td>$1.786N_f - 0.706 + O(1/N_f)$</td>
</tr>
<tr>
<td>5/2</td>
<td>$2.462N_f - 1.04 + O(1/N_f)$</td>
</tr>
</tbody>
</table>
Generalization to $U(N_c)$ QCD w/ $N_f$ fundamentals

- Monopole operator with lowest scaling dimension:
  \[ \Delta = 0.265 N_f - 0.0383 - (N_c - 1)0.516 + O(N_c^2/N_f). \]

- Predictions:
  - $U(1)$ deconfines when $N_f \geq 12$,
  - $U(2)$ when $N_f \geq 14$,
  - $U(3)$ when $N_f \geq 16$, etc.

- More complicated monopole operators are parameterized by \( \{q_a \in \mathbb{Z}/2\}, a = 1, 2, \ldots, N_c \). They transform as
  \[
  N_f/2 \left\{ 2 \sum_{a=1}^{N_c} |q_a| \right\}
  \]

- Not all \( \{q_a\} \) yield independent operators!! [Dyer, Mezei, SSP ’13]
Open questions

- Can the properties of monopole operators I described be verified through lattice QED/QCD or through the conformal bootstrap?

- Do monopole operators play any role in non-supersymmetric (bosonization) dualities?
  
  \[ U(N)_k + \text{fundamental boson: } \Delta \propto k \text{ if } k \gg N. \]
  
  \[ U(N)_k + \text{fundamental fermion: } \Delta \propto k^{3/2} \text{ if } k \gg N. \]
Open questions

- Can the properties of monopole operators I described be verified through lattice QED/QCD or through the conformal bootstrap?

- Do monopole operators play any role in non-supersymmetric (bosonization) dualities?
  - $U(N)_k$ + fundamental boson: $\Delta \propto k$ if $k \gg N$.
  - $U(N)_k$ + fundamental fermion: $\Delta \propto k^{3/2}$ if $k \gg N$. 

Silviu Pufu (Princeton University)
Tentative comments on the critical **bosonic** theory

- Assuming spherically symmetric flux through $S^2$ [Sachdev, Murthy ’90; Metlitski, Hermele, Senthil, Fisher ’08; Dyer, Mezei, SSP, Sachdev, in progress]:

<table>
<thead>
<tr>
<th>$q$</th>
<th>scaling dimension [$\mathcal{M}_q$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>$0.125 N_b + 0.0603 + O(1/N_b)$</td>
</tr>
<tr>
<td>1</td>
<td>$0.311 N_b - 0.233 + O(1/N_b)$</td>
</tr>
<tr>
<td>3/2</td>
<td>negative mode</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Lorentz scalars invariant under $SU(N)$ global symmetry.

- $\mathcal{M}_{1/2}$ is irrelevant, provided:

  $$[\mathcal{M}_{1/2}] > 3 \implies N_b \geq 24.$$ 

- So we expect the theory does not confine whenever $N_b \geq 24$. 