Matrix Models in the Quantum Hall Effect

Outline

- Introduction: the Laughlin wave function
- Jain's idea & the Gauss law
- Maxwell-Chern-Simons matrix theory
  - two regimes:
    - $g=0$ "matrix QHE"
    - $g=\infty$ real QHE
- A conjecture

Work with M. Riccardi, I. Rodriguez (Florence)
Landau levels: one-body states

\[ H = \frac{1}{2m} (\vec{p}^2 - e\vec{A})^2, \quad A_i = \frac{B}{2} \varepsilon_{ij} x_j \]

\[ z = x_1 + i x_2, \quad \omega = \frac{1}{2} \left( \frac{2}{\partial \partial x_1} - i \frac{2}{\partial \partial x_2} \right) \]

\[ l = \sqrt{\frac{2\hbar c}{eB}} \text{ magnetic length } l \to 1 \]

\[ H = \omega (a^+ a + \frac{1}{2}) \]

\[ J = \vec{P} = b^+ b - a^+ a \]

\[ \begin{align*}
  a &= \frac{\pi}{2} + \delta \\
  a^+ &= \frac{\pi}{2} - \delta \\
  b &= \frac{\pi}{2} + \delta \\
  b^+ &= \frac{\pi}{2} - \delta
\end{align*} \]

[\[ a, a^+ = 1, \quad b, b^+ = 1, \quad [a, b] = [a, b^+] = 0 \]

- orbits have quantized radii \[ \pi r_n^2 B = n \phi_0, \quad \phi_0 = \frac{\hbar c}{e} \]
- degeneracy \[ D_A = \frac{BA}{\phi_0^2} = \frac{\phi}{\phi_0} = \# \text{ fluxes unit flux} \]
- filling fraction \[ \gamma = \frac{N}{D_A} \]
- Lowest Landau level: \[ \omega = \frac{eB}{mc} \gg kT \]

\[ 0 = a \psi_o = (\frac{\pi}{2} + \delta) \psi_o(\pi, \overline{\delta}), \quad \psi_0 = e^{-\frac{1}{2} \overline{|z|^2}} \varphi(\overline{z}) \text{ analytic} \]

- projection to LLL: \[ \begin{align*}
  a &= \frac{\pi}{2} + i \overline{p} = 0 \\
  a^+ &= \frac{\pi}{2} - i \overline{p} = 0
\end{align*} \]
Laughlin's quantum incompressible fluid

Electrons form a droplet of liquid without sound waves

\[
\begin{align*}
\text{Incompressible} & \equiv \text{density waves have a gap} \\
\text{Fluid} & \equiv \rho(\mathbf{r}) = \rho_0 = \text{const.}
\end{align*}
\]

\[z = x + iy\]

\[\begin{align*}
A &= \text{area of the droplet} \\
N &= \# \text{ of electrons} \\
\mathcal{D}_A &= \frac{BA}{\hbar c} = \# \text{ of degenerate Landau orbitals} \\
\rho &= \frac{N}{A} = \text{electron density} \\
\nu &= \frac{N}{\mathcal{D}_A} = \frac{N}{BA/E_0} = \text{filling fraction} = 1, \frac{1}{3}, \frac{1}{5}, \ldots \\
&= \text{density for quantum-mech. problem}
\end{align*}\]
Laughlin's trial wave function 

\[ \psi_{g.s.}(z_1, \ldots, z_N) = \prod_{i=1}^{N} \frac{1}{(z_i - z_j)} e^{-\sum |z_i|^2 / 2 \ell^2} \]

- \( \nu = 1 \) obvious gap for filled Landau level:
  \[ \text{gap} = \omega_c = \frac{eB}{mc} \gg k_B T \]

- \( \nu = \frac{1}{3} \)
  highly non-trivial gap due to repulsive electron-electron interaction:
  \[ \text{gap} = O\left( \frac{e^2}{\ell} \right) \]
  \[ \ell = \sqrt{\frac{2\hbar c}{eB}} \] (magnetic length)

quasi-hole excitation \( \approx \) vortex

\[ \psi_{q-h}(\eta; z_1, \ldots, z_N) = \prod_{i=1}^{N} (\eta - z_i) \prod_{i<j} (z_i - z_j) e^{2k+1} - \sum |z_i|^2 / 2 \ell^2 \]

- \( \nu = \frac{1}{2k+1} \)
  it has fractional charge 
  \[ Q = \frac{\nu}{2k+1} \]
  and fractional statistics 
  \[ \frac{\Theta}{\pi} = \frac{1}{2k+1} \]
\[ \Psi_{2q,n} (\eta_1, \eta_2; z_1, \ldots, z_n) = (\eta_1 - \eta_2)^\frac{1}{2k+1} \prod (\eta_1 - z_i) \prod (\eta_2 - z_i) \Psi_{\text{g.s.}} \]

\[ \Psi_{2q,n} (\eta_1, \eta_2 \to e^{i\pi/(\eta_1 - \eta_2)}) = e^{i\pi/(2k+1)} \Psi_{2q,n} (\eta_1, \eta_2) \]

Fractional statistics \( \frac{\Theta}{\pi} = \frac{1}{2k+1} = \frac{1}{3}, \frac{1}{5} \)

- Fractional statistics is a long-range correlation of vortices, independent of distance, i.e. "topological"

- Long-distance physics of incompressible fluid is universal, e.g. independent of type of repulsive interaction

- Laughlin's wave function is a "good representative" of the universality class

- \( \rightarrow \) low-energy effective Field theory
  - \( \rightarrow \) conformal field theory of massless edge excitations

- Lot of nice work
- Experimental confirmations
Non-relativistic effective field theories

- CFT description is very nice, also practical
- CFT almost completely determined by symmetries

BUT:
- cannot describe how the gapful ground states of incompressible fluids are formed
- cannot prove Laughlin's theory

→ need a non-relativistic theory
- dynamical gap is non-perturbative

→ try effective interactions & theories

- Jain's idea: the role of fluxes

\[
\frac{1}{v} = \frac{\text{# 1-p states}}{\text{# electrons}} = \frac{\text{# Fluxes}}{N} = \frac{1}{m} + 2k = \frac{B}{2\pi \rho_0}
\]

Removing \(2k\) fluxes per electron would give

\[
\frac{1}{v} \rightarrow \frac{1}{v^*} = \frac{1}{m}
\]

integer Hall effect
obvious non-interacting
theory with gap \(\frac{B^*}{m}\)

\[
B \rightarrow B^* = B - \Delta B, \quad \Delta B = 2k \frac{2\pi \rho_0}{m}
\]

eq. of motion of

U(1) Chern-Simons gauge theory
at plateaux $R_{xx} = \sigma_{xx} = 0 \rightarrow \text{gap}$

Laughlin's series $\nu = \frac{1}{2k+1} = 1/3, 1/5, 1/7 \quad k = 0, 1, 2, ...$

Jain's hierarchy $\nu = \frac{m}{2km + 1}$, $m = 1, 2, 3$

Ex: $k = 0$ integer QHE

$k = 1$, $\nu = \frac{m}{2m + 1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13} \rightarrow (\frac{1}{2})^-$

$\nu = \frac{m}{2m - 1} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \rightarrow (\frac{1}{2})^+$ "charge conjugate"
Add Chern-Simons effective interaction

\[ S_{NR} = S_{\text{Landau, levels}} + \frac{1}{\pi \hbar} \int d^3x \, A \cdot dA + J A \]

* each electron is given a magnetic charge \(2k\)
  \(\rightarrow\) Jain's "composite Fermion"

* effective integer Hall effect \(\rightarrow\) gap
  \(\rightarrow\) mean field theory + fluctuations
  \(\rightarrow\) very nice results
  \(\rightarrow\) simple
  \(\rightarrow\) difficult to improve

(Fradkin, Lopez; Halperin, Lee, Read; Shankar, ...)

• Another idea from strings (Susskind '01)

NR electrons \(\rightarrow\) DO branes

\[ \mathcal{L}_a(t), \alpha = 1, \ldots, N \]
\[ X_{\alpha}(t) \]

\(N\times N\) matrices

permutation symmetry \(\Pi_{\alpha \beta} : \alpha \leftrightarrow \beta\)

\(U(N)\) gauge symmetry \(X \rightarrow U X U^+\)

\(U(N)\) gauge theory in 0+1 dimensions of two Hermitian matrices \(\bar{X} = (X_1, X_2)\)

\(\rightarrow\) eigenvalues \(\bar{X}_\alpha\) \(\alpha\) coordinates \(\bar{X}_\alpha\)

+ additional "angular" variables \(V, W\)

\(X_1 = V \Lambda_1 V^+\), \(X_2 = W \Lambda_2 W^+\)
Gauss law
\[ G = i \sum_{i=1}^{2} [X_i, \Pi_i] = \Pi_N B \theta \]
const. background

→ gauge invariant states should satisfy it;
for \( \theta = \frac{1}{2\pi \rho_0} \), it amounts to Jain's relation
flux ↔ density
\[ B \theta = k \in \mathbb{Z}, \quad B = k \ 2\pi \rho_0 \]

• Two ways to satisfy Gauss law

I. matrix angular variables \( V, W \) are constrained
and induce a two-body repulsion among
eigenvalues of
Calogero type
\[ V = \sum_{\alpha \neq \beta} \frac{(B \theta)^2}{(x_\alpha - x_\beta)^2} \]

II. upon projection to lowest Landau level
\[ \Pi_1 = \frac{-B}{2} X_2, \quad \Pi_2 = \frac{B}{2} X_1 \quad \rightarrow \quad [X_1, X_2] = i \theta \]
noncommutative fields
\[ \rho_0 = \frac{N}{\mathcal{A}} = \frac{1}{2\pi \theta} \]
Maxwell-Chern-Simons Matrix Theory

\[ S = \int dt \text{Tr} \left[ \frac{m}{\hbar} (D_t X_i)^2 + \frac{\hbar}{2} \varepsilon_{ij} X^i D_t X^j + g [X_1, X_2]^2 \right] + \text{Tr} \left[ B \Theta A_0 - i \Psi^+ D_t \Psi \right] \]

- \[ D_t X_i = \dot{X}_i - i [A_0, X_i] \]
  \[ \Psi \text{ auxiliary vector} \]
  \[ \Psi(t) = \Psi_0 = \text{const.} \]

- U(N) gauge invariance \[ X_i \rightarrow U X_i U^+ \]
  \[ S \rightarrow S - i B \Theta \int dt \text{Tr} \left[ U^+ U \right] \]
  \[ B \Theta = \kappa \in \mathbb{Z} \]
  (Nair, Polykovvakos)

- Dimensional reduction from 2+1 dim. to 0+1: D0 branes

- Hamiltonian + Gauss law

\[ H = \text{Tr} \left[ \frac{\hbar}{m} A^+ A - g [X_1, X_2]^2 \right] , \]

\[ A = \frac{\hbar}{4} (X_1 + i X_2) + \frac{i}{2} (\Pi_1 + i \Pi_2) \]

\[ G = i [X_1, \Pi_1] + i [X_2, \Pi_2] - B \Theta + \Psi \Psi^+ \approx 0 \]

\[ \text{Tr} G = 0 \rightarrow \| \Psi \|^2 = B \Theta N = \kappa N \]

- Two solutions of \[ G \neq 0 \] are realized
  for \[ g = 0 \] and \[ g = \infty \], respectively

- Parameters: \[ \frac{B}{m}, \frac{g}{m} ; \quad B \Theta = \kappa \leftrightarrow \frac{1}{\nu} = \frac{B}{2\pi \rho_0} \] fixed
\[ g = \infty \text{ Limit: back to electrons} \]

\[ H = \frac{B}{4m} \text{Tr} \left[ (p_1 + \frac{B}{2} x_1)^2 + (p_2 - \frac{B}{2} x_1)^2 \right] - g \text{Tr} \left[ x_1, x_2 \right]^2 \]

\[ g = \infty \rightarrow [x_1, x_2] = 0 \quad \text{Normal Matrices} \]
\[ x_1 = U x_1 U^+ \quad x_2 = U x_2 U^+ \]

- \( U \in U(N) \) gauge d.o.f. \( \rightarrow U = 1 \)
- \( x^i = \text{diag} (x^i_\alpha) \quad \alpha = 1, \ldots, N \) eigenvalues & coordinates
- \( \Pi^i = U (p^i + \Gamma^i) U^+ \)
  
  diagonal, conjugate to \( x^i \)
- Gauss law: \( i [x_1, \Pi^1] + i [x_2, \Pi^2] = k \quad \pi^2 \quad \rightarrow \ Pi \)

\[ (\Gamma^i)_{\alpha \beta} = i \frac{x_\alpha - x_\beta}{2} \left( x_\alpha^2 - x_\beta^2 \right) \quad \alpha \neq \beta \quad k = 8 \pi \]

\( \rightarrow \text{induced interaction } \sqrt{2} \text{L}^2 \quad 2d \text{ Calogero} \)

\[ H = \frac{B}{4m} \text{Tr} \left[ \frac{(p_1 + \frac{B}{2} x_1)^2 + (p_2 - \frac{B}{2} x_2)^2 + \Gamma_1^2 + \Gamma_2^2}{\text{diagonal}} \right] \sum_{\alpha \neq \beta} \frac{k^2}{|x_\alpha - x_\beta|^2} \]

\( \rightarrow \text{back to original problem with } \sqrt{2} x_1 \rightarrow \sqrt{2} x_1^2 \)

\( \rightarrow \text{gap is dynamical & nonperturbative} \)
Physical states for $g=0$: Matrix $LL$

$$H = \frac{B}{m} \text{Tr} [\mathcal{A}^+ \mathcal{A}]$$

$$\mathcal{A} = \frac{\delta}{4} (X_1 + iX_2) + \frac{i}{2} (\pi_1 + i\pi_2)$$

$$G = i (X_1, \pi_1) + i (X_2, \pi_2) + 4\psi^+ - k = 0, \quad \theta = k$$

$$\text{Tr} G = 0 \quad \Rightarrow \quad \psi^+ \psi = Nk$$

- Landau levels of $N^2$ "particles" with coordinates $\vec{x}_a$
- Physical states $G_{\alpha\beta} \Psi(X_1, X_2, \psi) = 0$

$\rightarrow U(N)$ singlets with $Nk$ components $\psi$.

- **Claim**: allowed $g=\text{const. ground states}$ are (matrix extensions of) Laughlin and Jain states; they are all gapful.

$U(N)$ gauge symmetry $\rightarrow$ "kinematic" fractional QHE

- Start by filling the lowest Landau level

$$Q_{\alpha\beta} \Psi(X_1, X_2, \psi) = 0, \quad \Psi = e^{-\frac{1}{2} \text{Tr}(X^T X)}$$

analytic of $X = X_1 + iX_2$
• Solution of Gauss law (Hellermann, Von RamsdorK '01)

\[ \Phi(X, \psi) = \left[ \varepsilon_i \cdots \varepsilon_n \psi_i \cdots (X^2 \psi) \cdots (X^n \psi) \right]^k \]

represent it like a tree with different branches

• recover Laughlin wave function by diagonalizing

\[ \chi = N \Lambda V^{-1} \quad \Lambda = \text{diag} \left( \varepsilon_1, \ldots, \varepsilon_n \right) , \quad \Phi = V \Phi \]

\[ \Phi = \left[ \det N \prod_{\alpha \neq \beta} (\varepsilon_\alpha - \varepsilon_\beta) \prod_\alpha \phi_\alpha \right]^k \propto \prod_{\alpha < \beta} (\varepsilon_\alpha - \varepsilon_\beta)^k \]

• semiclassical limit of incompressible fluid

with \( p_0 = \frac{1}{2 \pi \Theta} \quad \frac{1}{V} = \frac{B}{2 \pi p_0} = B \Theta + 1 = k + 1 \) (Susskind)

N.C. Chern-Simons

• states with higher density (higher \( \nu \)) in LLL are not physical \([X_1, X_2] = i \Theta\)

→ "kinematic" repulsion

• quasi-particles have gap \( \omega = \frac{B}{m} \)
Jain's ground states $\frac{1}{v} = k + \frac{1}{m}$

start to fill higher Landau levels to achieve higher densities

- $\Pi$ LL: $(\alpha_{\alpha \beta})^2 \Psi = e^{-\frac{1}{2} \text{Tr}(X^+ X)} (2 \alpha_{\alpha \beta})^2 \Psi = 0$

$\Psi(x, x^+, \psi)$ at most linear in $X_{\alpha \beta}^+, \Phi_{\alpha \beta}^+$

$\Psi = \left[ \begin{array}{c} \psi_1 \\ \vdots \\ \psi_{2k-1} \\ \psi_{2k} \\ \vdots \\ \psi_{2k+1} \\ \vdots \\ \psi_N \end{array} \right] \cdot \left[ \begin{array}{c} X_{\alpha \beta}^+ \\ \vdots \\ \Phi_{\alpha \beta}^+ \end{array} \right] \left[ \begin{array}{c} \frac{N}{2} \text{matrices } X_{\alpha \beta}^+ \end{array} \right]$

→ upon diagonalization, it is a Slater determinant of $\Pi$ LL filling as hypothesized by Jain

$\frac{1}{v} = 2k + \frac{1}{2}$, $E_0 = \frac{B}{m} \cdot \frac{N}{2}$, gap $= \frac{B}{m}$

- Analysis extends to filling $\Pi$ LL and higher

- Full Maxwell-Chern-Simons theory is "weakly" non commutative

$G = \frac{B}{4} [x, x^+] + [x^+, \alpha] + [\alpha^+, x] - 3\theta + 4\psi^+ = 0$

it can vanish on higher LL w. $\alpha, \alpha^+ \neq 0$

this happens for $q \to \infty$ that forces $[x, x^+] \to 0$
"generalized" Jain hierarchy

\[ \phi \rightarrow \phi_{P_1} \phi_{P_2} \ldots \]

For \( \nu = 1 \)

\[ \phi_{1} = \phi \rightarrow \Pi(2 \times 3 - \phi_B) \text{ Filled 1st LL} \]

For \( \nu = 2 \)

\[ \phi_{2} = \phi \rightarrow \text{Filled 1st & 2nd} \]

For \( \nu = 3 \)

\[ \phi_{3} = \phi \rightarrow \text{quadratic in } X_{ij} \text{ \ first 3 LL filled} \]

- Jain fillings \( \frac{1}{\nu} = k + \frac{1}{m} \) are \( \phi_{k-1} \phi_{m} \)

- any product of \( k \) blocks \( \phi_{P_1} \ldots \phi_{P_k} \) is possible: it has increasingly higher energy \( E_0 \) and higher density

\[ \frac{1}{\nu} = 1 + \sum_{i=1}^{k} \frac{1}{P_i} < k + \frac{1}{m} \]

- higher densities are far from semiclassical limit of incompressible fluids \( \frac{1}{\nu} = k+1 = B_0 + 1 \)

- \( p \neq \text{ const?} \)

- additional d.o.f. in the fluid?
Phase diagram

\[ H = \text{Tr} \left[ \frac{B}{m} A^+ A - g [x_1, x_2]^2 \right] \]

**LLL:** \( B \gg m \), \( A \approx 0 \)

Chern-Simons Matrix Model

\[ S = \text{Tr} \left[ \frac{B}{2m} \sum \epsilon_{ij} \dot{x}^i \dot{x}^j + B \Theta A_0 \right] \]

(Susskind, Polychronakos)

\[ G \approx 0 \quad [x_1, x_2] = i \Theta \]

\[ \rho_0 = \frac{1}{2 \pi \Theta} \quad \text{incompressible fluid} \]

\[ \frac{1}{\nu} = \frac{B}{2 \pi \rho_0} = B \Theta = k \quad \text{Laughlin filling} \]

“Kinematic” repulsion

Complete reduction to eigenvalues

\[ G = i [x_1, \pi_1] + i [x_2, \pi_2] - B \Theta \]

(\( \pi_i \))

\[ \left( \pi_i \right)_{\alpha \beta} = \frac{i B \Theta (x_i^\alpha - x_i^\beta)}{\left| x_i^\alpha - x_i^\beta \right|^2} \]

Induced interaction

\[ H = \text{Tr} \left( \pi_1^2 + \pi_2^2 \right) + \ldots \]

\[ = \sum_{\alpha \neq \beta} \frac{(B \Theta)^2}{\left| x_\alpha - x_\beta \right|^2} + \ldots \]

Gap is dynamical

\[ \ldots \]

Interaction
Conjecture on Maxwell-Chern-Simons MM

\[ H = \text{Tr} \left[ \frac{B}{m} A^+ A - g [x_1, x_2]^2 \right] \]

\[ \frac{B}{m} \]

\[ g = 0 \text{ Matrix QHE} \]
- all expected states with \( p = \text{const} \) & gap
- \( N^2 \) d.o.f.

\[ g = \infty \text{ physical QHE} \]
- \( [x_1, x_2] = 0 \to \) eigenvalues
- Calogero interaction \( \propto \) Coulomb inter.

As \( g: 0 \to \infty \), kinematic repulsion is replaced by Calogero interaction; matrix angular d.o.f. projected out

Conjecture

As \( g: 0 \to \infty \), gapful \( p = \text{const.} \) ground states prepared at \( g = 0 \) remain gapful for all \( g \) values and have smooth \( g = \infty \) limit
Conjecture: no phase change for $0 < g < \infty$

For densities that admit gapful $p$-cost ground states near $g \sim 0$

- Problem: find method to analyze interaction $g \text{tr}([X_1, X_2]^2)$

- Maxwell-Chern-Simons matrix theory could provide another effective non-relativistic theory of fractional QHE

- It generalizes the Chern-Simons matrix theory (Susskind, Polychronakos, ...) that was too much constrained (in particular, $g$ interaction is meaningless)