A Turbulent Instability of anti-de Sitter

Gary Horowitz
UC Santa Barbara

O. Dias, J. Santos, G.H., 1109.1825
P. Bizon, A. Rostworowski, 1104.3702

Anti-de Sitter spacetime

This is the maximally symmetric solution to

$$S = \int d^4x \, \sqrt{-g} \, (R + 6)$$

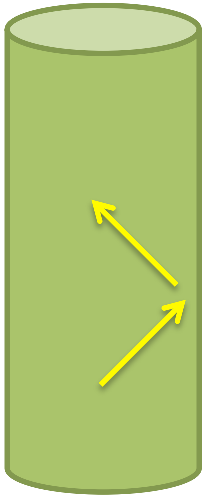
given by $ds^2 = -\left(r^2 + 1\right) dt^2 + \frac{dr^2}{r^2 + 1} + r^2 d\Omega^2$

These are global coordinates. The Poincare coordinates

$$ds^2 = r^2(-dt^2 + dx_i dx^i) + dr^2/r^2$$

do not cover the entire spacetime.

Conformally, AdS looks like the interior of a cylinder



With energy conserving boundary conditions, waves bounce off infinity and return in finite time.

The dual field theory lives on $S^2 \times R$.

At the linearized level, AdS appears just as stable as Minkowski space or de Sitter.

For Minkowski or dS, it has been shown that small but finite perturbations remain small.

This has never been shown for AdS.

WHY NOT?

It is just not true.

Claim: AdS is nonlinearly unstable

Generic small (but finite) perturbations of AdS become large and eventually form black holes.

The energy cascades from low frequency to high frequency modes in a manor reminiscient of the onset of turbulence.

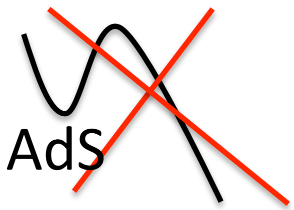
Doesn't this contradict the fact that AdS is supersymmetric?

Doesn't this contradict the fact that there is a positive energy theorem?

No

Positive Energy Theorem: If the matter satisfies a reasonable energy condition, then $E \geq 0$ for all nonsingular, asymptotically AdS initial data, and $E = 0$ if and only if the spacetime is AdS.

This ensures that AdS cannot decay.



It does not ensure that a small amount of energy added to AdS won't generically form a small black hole.

That is usually ruled out by arguing that waves disperse. This doesn't happen in AdS.

Example (Dafermos):

Consider $$S = \int R - (\nabla\phi)$$

This has a positive energy theorem and small nonlinear perturbations of Minkowski spacetime remain small.

Now consider $$S = \int R + (\nabla\phi)$$

No positive energy theorem, but Minkowski spacetime is still nonlinearly stable.

Why is AdS nonlinearly unstable?

AdS boundary conditions act like a confining box. Any finite excitation which is added to this box might be expected to eventually explore all configurations consistent with the conserved quantities – including small black holes.

One of the singularity theorems of GR shows that closed universes are generically singular (Hawking and Penrose, 1969). AdS is like a closed universe for the fields inside, so it should be generically singular.

Special solutions need not be singular

For each linearized gravitational mode, there is a corresponding nonlinear solution called a geon.

Geons are nonsingular and globally asymptotically AdS. There are an infinite number of them, but they are all special since they are

(1) Exactly periodic in time
(2) Invariant under a continuous symmetry

Intuitive picture

Geons are analogous to gravitational plane waves.

Colliding exact plane waves produce singularities.

Trying to superpose geons leads to many collisions which again produce singularities due to mutual focussing.

Perturbative construction of solutions

Expand: $$g = \bar{g} + \sum_i \epsilon^i h^{(i)}$$

At each order, have to solve:

$$\Delta_L h^{(i)}_{ab} = T^{(i)}_{ab}$$

where

$$2\Delta_L h^{(i)}_{ab} \equiv -\bar{\nabla}^2 h^{(i)}_{ab} - 2\bar{R}_a{}^c{}_b{}^d h^{(i)}_{cd} - \bar{\nabla}_a \bar{\nabla}_b h^{(i)} + 2\bar{\nabla}_{(a} \bar{\nabla}^c h^{(i)}_{b)c}.$$

Different types of perturbations

"Scalar type" perturbations: h_{ab} are constructed from spherical harmonics.

"Vector type" perturbations: h_{ab} are constructed from vector harmonics. (For S^2, these are ${}^*\nabla Y_{\ell m}$)

"Tensor type" perturbations only exist in higher dimensions.

At each order, can reduce the metric perturbation to two functions satisfying (Kodama, Ishibashi, 2003)

$$\Box_s \Phi^{(i)}_{\ell,m}(t,r) + V^{(i)}_{\ell}(r)\Phi^{(i)}_{\ell,m}(t,r) = \tilde{T}^{(i)}_{\ell,m}(t,r),$$

where $\Box_s$ is the wave operator associated with

$$ds^2 = -(r^2+1)dt^2 + \frac{dr^2}{r^2+1}$$

Boundary conditions

Regularity at the origin requires: $\Phi_{\ell,m} \sim \mathcal{O}(r^{\ell})$

Asymptotically:

$$\Phi_{\ell,m} \sim R_{\ell,m}(t) + \frac{S_{\ell,m}(t)}{r} + \mathcal{O}(r^{-2})$$

Surprisingly, to keep the metric fixed at infinity, we need to choose

$$S_{\ell,m}(t) = 0$$

First Order

The allowed frequencies are $\omega_\ell = 1 + \ell + 2p$

For p = 0, the solutions are

$$\Phi^{(1)}_{\ell,m}(t,r) = \frac{r^{\ell+1}}{(r^2+1)^{\frac{\ell+1}{2}}} a_{\ell,m} \cos(\omega_\ell \, t)$$

General structure

If the source has harmonic time dependence cos ωt, then the solution will have the same harmonic time dependence, EXCEPT when ω agrees with one of the normal mode frequencies.

Then we get a resonance and the solution grows linearly in time:

$$\Phi(t,r) = \cos(\omega t) R(r) \\ + t \sin(\omega t) L(r).$$

Example 1

Start with a single $\ell = 2, m = 2$ mode.

At second order – no resonances

At third order – one resonant term

but one can set the growing mode to zero by changing the frequency slightly

$$\omega_2 = 3 - \frac{14703}{17920}\epsilon^2$$

Continue in this way to construct geon.

The symmetry of the exact solution is not the same as the linearized solution.

Start with $\cos(\omega t - m\varphi)$ which is invariant under

$$\partial/\partial t + (\omega/m)\partial/\partial\phi$$

Higher order corrections have sources which are powers of $\cos(\omega t - m\varphi)$.

So you keep a symmetry of the above form, but since ω changes, so does the Killing field.

Example 2

Start with a linear combination of $\ell = 2, m = 2$ and $\ell = 4, m = 4$ mode.

At second order – no resonances

At third order – 4 resonant terms

growing mode in two can be removed by adjusting the frequencies of two original modes

growing mode of one is just absent

Last growing mode cannot be removed. This corresponds to $\ell = 6, m = 6$ with ω = 7.

Get a growing mode with higher frequency than we started with.

Energy is transferred to higher frequency modes.

Expect this to continue. When $\ell = 6, m = 6$ mode grows, it will source even higher frequency modes with growing amplitude.

Spherical scalar field collapse in AdS

(Bizon and Rostworowski, 2011)

Recall the situation when $\Lambda = 0$ (Choptuik, Christodoulou):

For any initial scalar field profile $\phi = \alpha f(r)$,

large α → large black hole

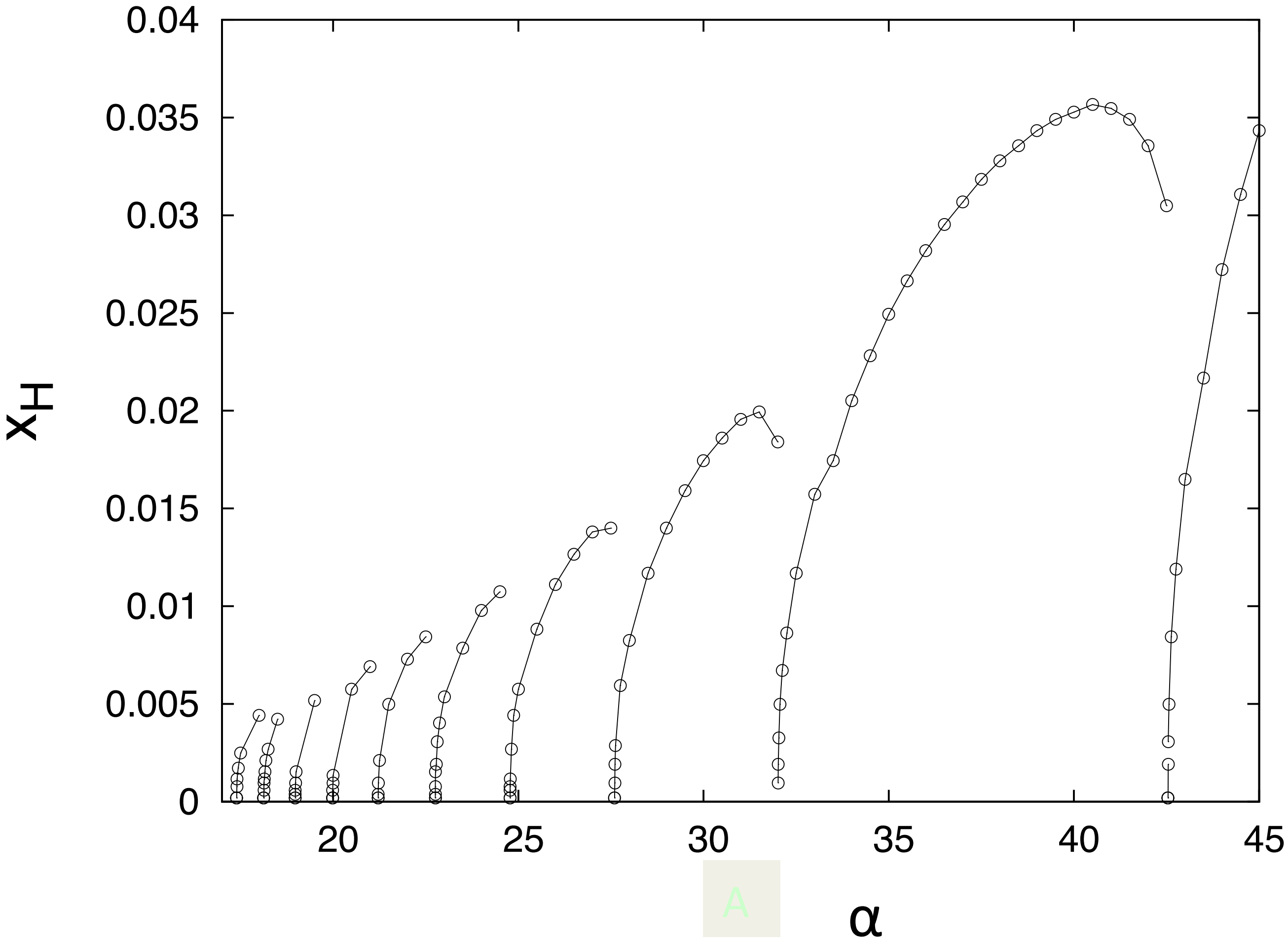
small α → waves scatter and go off to ∞

For a critical value α_*, the collapse forms a "zero mass black hole" i.e. a naked singularity. Near α_*:

$$M_{BH} \sim (\alpha - \alpha_*)^\gamma \text{ with } \gamma = .37$$

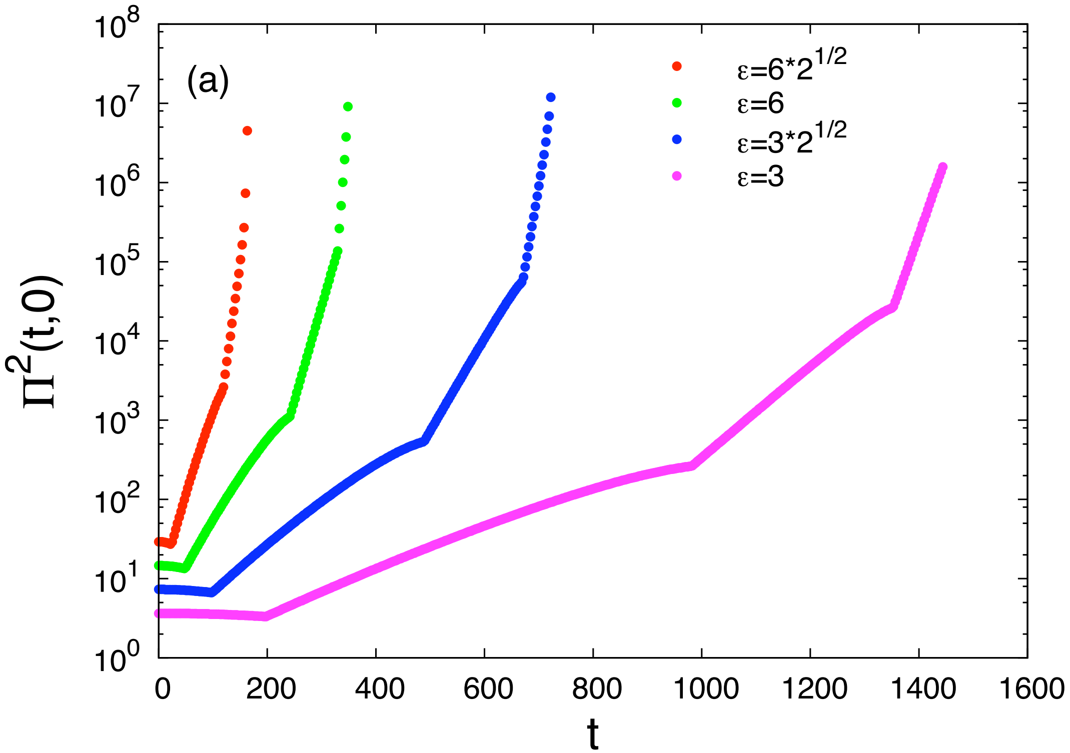
Repeating this in AdS one finds

Radius of initial black hole



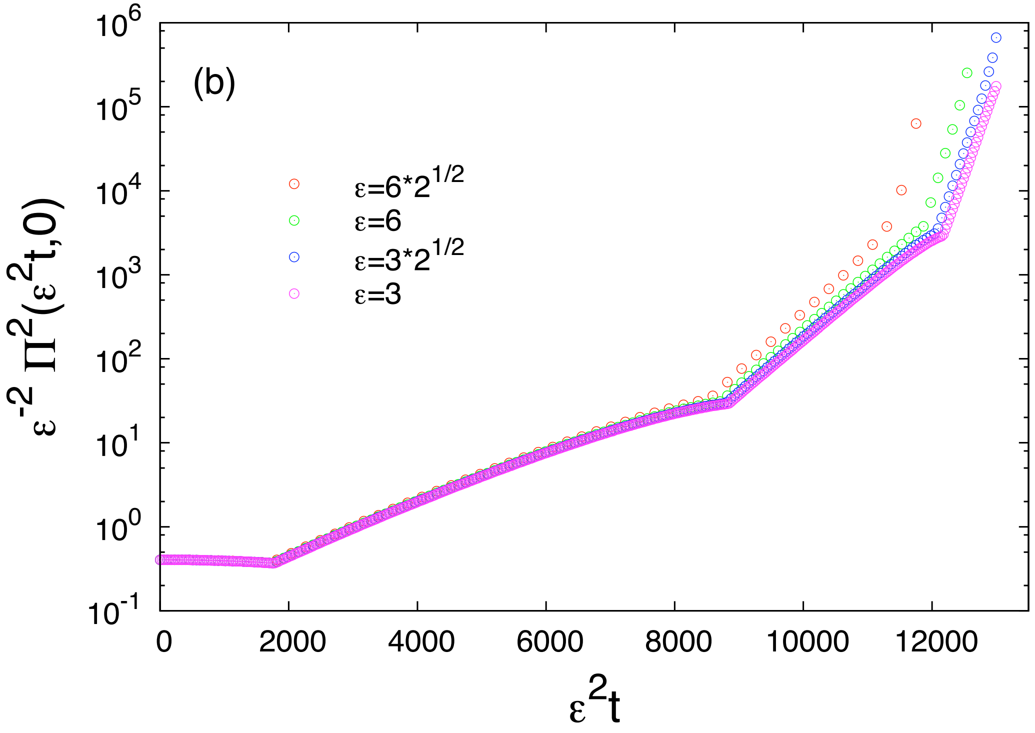
(Bizon and Rostworowski, 2011)

The Ricci scalar at the origin oscillates with period about two. Starting with small amplitude initial data, the maximum Ricci scalar behaves as follows:



(Bizon and Rostworowski, 2011)

If you rescale time and Ricci scalar by the amplitude, these curves all agree:



(Bizon and Rostworowski, 2011)

Conclusion of scalar field evolution in AdS:

No matter how small you make the initial amplitude, the curvature at the origin grows and you eventually form a small black hole.

What happens in string theory?

Consider IIB on $AdS_5 \times S^5$ with AdS radius L.

Two energy scales:

Planck energy E_P and string energy $E_s < E_P$

If initial energy is large $E > N^2/L$, form a 5D black hole.

If initial energy is $E_{corr} < E < N^2/L$, you form a 10D black hole. E_{corr} is energy of BH of size string scale.

If $E_s < E < E_{corr}$ you form an excited string.

If $E < E_s$, the cascade stops at frequencies $\omega = E$, and get a gas of particles in AdS.

Implications for Dual Field Theory

The fact that you evolve to the maximum entropy state can be viewed as thermalization.

All theories with a gravity dual will show this energy cascade like the onset of turbulence.

Puzzle: In 2+1 dimensions, classical turbulence has an inverse energy cascade due to a extra conserved quantity. Our results indicate that in a strongly coupled quantum theory, there is a standard energy cascade.

Interpretation of Geons

Linearized graviton is a spin 2 excitation.

Nonlinear geon is like a bose condensate of these excitations.

These high energy states do not thermalize!

These are different from the states discussed in (Freivogel, McGreevy and Suh, 1109.6013).

Boundary stress tensor

Contains alternating positive and negative energy regions around the equator.

Invariant under $\partial/\partial t + (\omega/m)\partial/\partial\phi$
which is timelike near the poles but spacelike near the equator.

Can (probably) add a small black hole inside the geon

Only constraint: Killing field of geon must be null on the horizon, so the black hole must rotate with $\Omega_H = \omega/m$.

Kerr-AdS is not the unique stationary black hole in AdS

Get a black hole with only one Killing field.

Most black holes have at least two symmetries:

1) Symmetries are needed to find exact solutions
2) There are theorems which show that stationary black holes must be axisymmetric (Hawking, 1972; Hollands, Ishibashi, Wald, 2006; Isenberg, Moncrief, 2006)

This is not true in AdS.

Superradiance

If a wave $e^{-i\omega t+im\phi}$ scatters off a rotating black hole with $\omega < m\Omega_H$, it can return with larger amplitude.

In AdS, the outgoing wave is reflected off infinity and the process repeats. Get a superradiant instability.

What is the endpoint?

If you perturb the black hole with a single mode, expect the exterior to settle down and look like a black hole inside a geon.

If you perturb with a superposition of modes, superradiance cause them to grow, but then the turbulent instability will cause higher frequency modes to be created.

What happens at late time?

Two possibilities:

- If the black hole absorbs the higher frequency modes faster than they can be created, might stabilize with gravitational waves sloshing around outside the black hole.

- The black hole exterior might continue to evolve toward higher and higher frequency.

Conclusions

(1) Anti-de Sitter spacetime is nonlinearly unstable: generic small perturbations become large and (probably) form black holes.

(2) For each linearized gravity mode, there is an exact, nonsingular geon.

(3) Dual field theory shows generic turbulent cascade to maximum entropy state – but there are special states (geons) that do not thermalize.

Open Questions

(1) Understand why the energy cascade in 2+1 quantum theory is different from the classical theory.

(2) Prove a singularity theorem for anti-de Sitter.

(3) Understand the late time behavior of superradiantly unstable black hole.

(4) Understand the space of CFT states that do not thermalize.