Parity-breaking hydrodynamics in 2+1 dimensions

Pavel Kovtun

University of Victoria

KITP, Santa Barbara, September 27, 2011

Comments

- Talk based on work *in progress* with

  Kristan Jensen
  Matthias Kaminsky
  René Meyer
  Adam Ritz
  Amos Yarom

- Some points will be very well known, some hopefully new
- I will probably miss many references – please point them out!
- Will talk about relatvistic systems, can take NR limit

Outline

1. Normal relativistic hydro
2. Hydro for systems without parity
3. Magnets and gravitomagnets
4. Fluctuations: a problem with classical hydro
5. A systematic way to treat fluctuations
6. A toy model with fluctuations
7. Conclusions

Outline

Why 2+1 dim relativistic hydro?

How to write down the hydro equations

- Non-relativistic hydro: conservation of energy, momentum, and particle number currents
- Normal relativistic hydro: conservation of the energy-momentum tensor, plus possibly other currents

$$\partial_\mu T^{\mu\nu} = 0\,, \qquad \partial_\mu J^\mu = 0\,.$$

- Open Landau-Lifshitz, vol.6

$$T^{\mu\nu} = P\eta^{\mu\nu} + (\epsilon{+}P)u^\mu u^\nu + \tau^{\mu\nu}\,,$$
$$J^\mu = nu^\mu + \nu^\mu\,.$$

- $\tau^{\mu\nu}$, ν^μ contain derivatives of u^μ, T, μ, describe dissipation

$$\tau_{\mu\nu} = -\eta\left(\partial_\mu u_\nu{+}\partial_\nu u_\mu{+}u_\mu u^\lambda\partial_\lambda u_\nu{+}u_\nu u^\lambda\partial_\lambda u_\mu\right) - (\zeta{-}\eta)(\eta_{\mu\nu}{+}u_\mu u_\nu)\,,$$
$$\nu^\mu = -\sigma T\left[\partial_\mu(\mu/T) + u_\mu u^\lambda\partial_\lambda(\mu/T)\right]$$

A more systematic way

Boost invariance is broken by a preferred frame; timelike vector u^μ.
Decompose $T_{\mu\nu}$ and J_μ with respect to u_μ:

$$T_{\mu\nu} = \mathcal{E} u_\mu u_\nu + \mathcal{P}\Delta_{\mu\nu} + (q_\mu u_\nu + q_\nu u_\mu) + t_{\mu\nu}\,,$$
$$J_\mu = \mathcal{N} u_\mu + j_\mu$$

- the projector is

$$\Delta_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu$$

- q_μ and j_μ are transverse, $t_{\mu\nu}$ is transverse, symm., traceless
- $\mathcal{E}$, $\mathcal{P}$, q_μ etc. are functions of local T, μ, u, and their derivatives

A more systematic way

Boost invariance is broken by a preferred frame; timelike vector u^μ.
Decompose $T_{\mu\nu}$ and J_μ with respect to u_μ:

$$
\begin{aligned}
T_{\mu\nu} &= \mathcal{E} u_\mu u_\nu + \mathcal{P} \Delta_{\mu\nu} + (q_\mu u_\nu + q_\nu u_\mu) + t_{\mu\nu}\,, \\
J_\mu &= \mathcal{N} u_\mu + j_\mu
\end{aligned}
$$

- the projector is

$$\Delta_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu$$

- q_μ and j_μ are transverse, $t_{\mu\nu}$ is transverse, symm., traceless
- $\mathcal{E}$, $\mathcal{P}$, q_μ etc. are functions of local T, μ, u, and their derivatives

Two steps to proceed further:

1. Out of equilibrium, can redefine the fields $T(x)$, $\mu(x)$, $u_\mu(x)$ to simplify the decomposition
2. Expand in powers of derivatives of $T(x)$, $\mu(x)$, $u_\mu(x)$

A more systematic way (2)

A more systematic way (2)

1. Need to choose $u_\mu(x)$, $T(x)$, $\mu(x)$
   - Choose $u_\mu =$ velocity of energy flow, or $q_\mu = 0$ (Landau frame)
   - Choose T so that $\mathcal{E} = \epsilon$ local thermodynamic energy density
   - Choose μ so that $\mathcal{N} = n$ local thermodynamic charge density

A more systematic way (2)

1. Need to choose $u_\mu(x)$, $T(x)$, $\mu(x)$
   - Choose $u_\mu =$ velocity of energy flow, or $q_\mu = 0$ (Landau frame)
   - Choose T so that $\mathcal{E} = \epsilon$ local thermodynamic energy density
   - Choose μ so that $\mathcal{N} = n$ local thermodynamic charge density
2. Derivative expansion
   - Expand non-equilibrium pressure

$$\mathcal{P} = P - \zeta \Delta_{\mu\nu} \partial^\mu u^\nu + O(\partial^2)$$

   - Expand non-equilibrium stress

$$t_{\mu\nu} = -\eta \Big[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \Delta_{\mu\nu}\Delta_{\alpha\beta} \Big] \partial^\alpha u^\beta + O(\partial^2)$$

   - Expand non-equilibrium current

$$j_\mu = -\sigma T \Delta_{\mu\nu} \partial^\nu \Big(\frac{\mu}{T}\Big) + \chi_{\rm T} \Delta_{\mu\nu} \partial^\nu T + O(\partial^2)$$

Comments:

- This gives the standard relativistic version of the Navier-Stokes equations, as described e.g. in Landau-Lifshitz, vol.6
- To solve the hydro equations, one needs to know $P(T, \mu)$, and three dissipative transport coefficients η, ζ, and σ.
- The equations allow for instantaneous propagation of dissipation, which is embarrassing in a relativistic theory. Can be cured by adding $O(\partial^2)$ terms
  Israel, Israel+Stewart, 1976
- A complete classification of $O(\partial^2)$ terms in relativistic hydro only appeared recently, with a lot of help from the AdS/CFT correspondence
  Baier+Romatschke+Son+Starinets+Stephanov, 2007
  Bhattacharyya+Hubeny+Minwalla+Rangamani, 2007
- The $O(\partial^2)$ terms by themselves are ill-defined because of the mode-mode coupling effects, just like in non-relativistic hydro
  DeSchepper+VanBeyeren+Ernst, 1974
  Ernst+Dorfman, 1975

Outline

How to write down the hydro equations

How to write down the hydro equations

Symmetry of the microscopic description descends to the symmetry of the macroscopic (hydrodynamic) description

How to write down the hydro equations

Symmetry of the microscopic description descends to the symmetry of the macroscopic (hydrodynamic) description

- If Parity is not a symmetry of the microscopic theory, the hydro equations will know about it.

How to write down the hydro equations

Symmetry of the microscopic description descends to the symmetry of the macroscopic (hydrodynamic) description

- If Parity is not a symmetry of the microscopic theory, the hydro equations will know about it.
- Specifically, the expansions of $\mathcal{P}$, $t_{\mu\nu}$, and j_μ will contain Parity-odd terms.

How to write down the hydro equations

Symmetry of the microscopic description descends to the symmetry of the macroscopic (hydrodynamic) description

- If Parity is not a symmetry of the microscopic theory, the hydro equations will know about it.
- Specifically, the expansions of $\mathcal{P}$, $t_{\mu\nu}$, and j_μ will contain Parity-odd terms.
- Let me now write down these terms.

How to write down the hydro equations (2)

Can project a general vector onto a direction orthogonal to u_μ by

$$\text{either}\ \ \Delta_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu\,,\ \ \text{or}\ \ \Sigma_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda} u^\lambda$$

How to write down the hydro equations (2)

Can project a general vector onto a direction orthogonal to u_μ by

$$\text{either} \quad \Delta_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu \,, \ \text{or} \quad \Sigma_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda} u^\lambda$$

$$\mathcal{P} = P - \zeta \, \Delta_{\mu\nu} \partial^\mu u^\nu - \tilde{\zeta} \, \Sigma_{\mu\nu} \partial^\mu u^\nu + O(\partial^2) \,,$$

How to write down the hydro equations (2)

Can project a general vector onto a direction orthogonal to u_μ by

$$\text{either} \quad \Delta_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu \,, \;\; \text{or} \quad \Sigma_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda} u^\lambda$$

$$\mathcal{P} = P - \zeta\, \Delta_{\mu\nu} \partial^\mu u^\nu - \tilde{\zeta}\, \Sigma_{\mu\nu} \partial^\mu u^\nu + O(\partial^2)\,,$$

$$t_{\mu\nu} = \; -\eta \Big[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \Delta_{\mu\nu}\Delta_{\alpha\beta} \Big] \partial^\alpha u^\beta$$

$$- \tilde{\eta} \Big[\Delta_{\mu\alpha}\Sigma_{\nu\beta} + \Delta_{\nu\alpha}\Sigma_{\mu\beta} + \Sigma_{\mu\alpha}\Delta_{\nu\beta} + \Sigma_{\nu\alpha}\Delta_{\mu\beta} \Big] \partial^\alpha u^\beta + O(\partial^2)\,,$$

How to write down the hydro equations (2)

Can project a general vector onto a direction orthogonal to u_μ by

$$\text{either} \quad \Delta_{\mu\nu} \equiv \eta_{\mu\nu} + u_\mu u_\nu \,, \ \text{or} \quad \Sigma_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda} u^\lambda$$

$$\mathcal{P} = P - \zeta\, \Delta_{\mu\nu}\partial^\mu u^\nu - \tilde{\zeta}\, \Sigma_{\mu\nu}\partial^\mu u^\nu + O(\partial^2)\,,$$

$$t_{\mu\nu} = -\eta\Big[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \Delta_{\mu\nu}\Delta_{\alpha\beta}\Big]\partial^\alpha u^\beta$$
$$- \tilde{\eta}\Big[\Delta_{\mu\alpha}\Sigma_{\nu\beta} + \Delta_{\nu\alpha}\Sigma_{\mu\beta} + \Sigma_{\mu\alpha}\Delta_{\nu\beta} + \Sigma_{\nu\alpha}\Delta_{\mu\beta}\Big]\partial^\alpha u^\beta + O(\partial^2)\,,$$

$$j_\mu = -\sigma T \Delta_{\mu\nu}\partial^\nu\Big(\frac{\mu}{T}\Big) + \chi_{\rm T}\Delta_{\mu\nu}\partial^\nu T - \tilde{\sigma} T \Sigma_{\mu\nu}\partial^\nu\Big(\frac{\mu}{T}\Big) + \tilde{\chi}_{\rm T}\Sigma_{\mu\nu}\partial^\nu T + O(\partial^2).$$

The red terms would be forbidden in a Parity-invariant system

Parity-breaking terms: pressure

Recall

$$\mathcal{P} = P - \zeta\,\Delta_{\mu\nu}\partial^\mu u^\nu - \tilde{\zeta}\,\Sigma_{\mu\nu}\partial^\mu u^\nu + O(\partial^2)\,,$$

Parity-breaking terms: pressure

Recall

$$\mathcal{P} = P - \zeta\,\Delta_{\mu\nu}\partial^\mu u^\nu - \tilde{\zeta}\,\Sigma_{\mu\nu}\partial^\mu u^\nu + O(\partial^2)\,,$$

ζ : Conventional bulk viscosity, $\mathcal{P} = P - \zeta(\partial_x v_x + \partial_y v_y) + \dots$
Contributes to off-equilibrium entropy production.

Parity-breaking terms: pressure

Recall

$$\mathcal{P} = P - \zeta\,\Delta_{\mu\nu}\partial^\mu u^\nu - \tilde{\zeta}\,\Sigma_{\mu\nu}\partial^\mu u^\nu + O(\partial^2)\,,$$

- ζ : Conventional bulk viscosity, $\mathcal{P} = P - \zeta(\partial_x v_x + \partial_y v_y) + \ldots$
  Contributes to off-equilibrium entropy production.
- $\tilde{\zeta}$: Allowed by symmetry, once Parity is broken,
  $\mathcal{P} = P - \zeta(\partial_x v_x + \partial_y v_y) - \tilde{\zeta}(\partial_x v_y - \partial_y v_x) + \ldots$
  Does not contribute to off-equilibrium entropy production
  Is related to the equilibrium response to vorticity

Parity-breaking terms: stress

Recall

$$t_{\mu\nu} = -\eta\Big[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \Delta_{\mu\nu}\Delta_{\alpha\beta}\Big]\partial^\alpha u^\beta$$

$$-\,\tilde\eta\Big[\Delta_{\mu\alpha}\Sigma_{\nu\beta} + \Delta_{\nu\alpha}\Sigma_{\mu\beta} + \Sigma_{\mu\alpha}\Delta_{\nu\beta} + \Sigma_{\nu\alpha}\Delta_{\mu\beta}\Big]\partial^\alpha u^\beta + O(\partial^2)\,.$$

Parity-breaking terms: stress

Recall

$$t_{\mu\nu} = -\eta\Big[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \Delta_{\mu\nu}\Delta_{\alpha\beta}\Big]\partial^\alpha u^\beta$$
$$- \tilde{\eta}\Big[\Delta_{\mu\alpha}\Sigma_{\nu\beta} + \Delta_{\nu\alpha}\Sigma_{\mu\beta} + \Sigma_{\mu\alpha}\Delta_{\nu\beta} + \Sigma_{\nu\alpha}\Delta_{\mu\beta}\Big]\partial^\alpha u^\beta + O(\partial^2)\,.$$

η : Conventional shear viscosity, $T_{xy} \sim \eta\,(\partial_x v_y + \partial_y v_x)$
Contributes to off-equilibrium entropy production.

$$\eta = \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\, G^{\mathrm{ret}}_{T_{xy}T_{xy}}(\omega, \boldsymbol{k}{=}0)$$

Parity-breaking terms: stress

Recall

$$t_{\mu\nu} = -\eta\Big[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \Delta_{\mu\nu}\Delta_{\alpha\beta}\Big]\partial^\alpha u^\beta$$

$$-\tilde{\eta}\Big[\Delta_{\mu\alpha}\Sigma_{\nu\beta} + \Delta_{\nu\alpha}\Sigma_{\mu\beta} + \Sigma_{\mu\alpha}\Delta_{\nu\beta} + \Sigma_{\nu\alpha}\Delta_{\mu\beta}\Big]\partial^\alpha u^\beta + O(\partial^2)\,.$$

η : Conventional shear viscosity, $T_{xy} \sim \eta\,(\partial_x v_y + \partial_y v_x)$
Contributes to off-equilibrium entropy production.

$$\eta = \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\,G^{\mathrm{ret}}_{T_{xy}T_{xy}}(\omega,\boldsymbol{k}{=}0)$$

$\tilde{\eta}$: Hall viscosity, $T_{xy} \sim \tilde{\eta}\,(\partial_x v_x - \partial_y v_y)$
Does not contribute to off-equilibrium entropy production

$$\tilde{\eta} = \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\,G^{\mathrm{ret}}_{T_{xy}T_{xx}}(\omega,\boldsymbol{k}{=}0)$$

See the talk by D.T.Son earlier at this program

Parity-breaking terms: current

Recall

$$j_\mu = -\sigma T \Delta_{\mu\nu} \partial^\nu \left(\frac{\mu}{T} \right) + \chi_{\rm T} \Delta_{\mu\nu} \partial^\nu T - \tilde{\sigma} T \Sigma_{\mu\nu} \partial^\nu \left(\frac{\mu}{T} \right) + \tilde{\chi}_{\rm T} \Sigma_{\mu\nu} \partial^\nu T + O(\partial^2).$$

Parity-breaking terms: current

Recall

$$j_\mu = -\sigma T \Delta_{\mu\nu} \partial^\nu \left(\frac{\mu}{T}\right) + \chi_{\rm T} \Delta_{\mu\nu} \partial^\nu T - \tilde{\sigma} T \Sigma_{\mu\nu} \partial^\nu \left(\frac{\mu}{T}\right) + \tilde{\chi}_{\rm T} \Sigma_{\mu\nu} \partial^\nu T + O(\partial^2).$$

- σ : Conventional charge conductivity, proportional to the charge diffusion constant. Contributes to off-equilibrium entropy production.

Parity-breaking terms: current

Recall

$$j_\mu = -\sigma T \Delta_{\mu\nu} \partial^\nu \left(\frac{\mu}{T}\right) + \chi_{\rm T} \Delta_{\mu\nu} \partial^\nu T - \tilde{\sigma} T \Sigma_{\mu\nu} \partial^\nu \left(\frac{\mu}{T}\right) + \tilde{\chi}_{\rm T} \Sigma_{\mu\nu} \partial^\nu T + O(\partial^2).$$

- σ : Conventional charge conductivity, proportional to the charge diffusion constant. Contributes to off-equilibrium entropy production.
- $\chi_{\rm T}$: Must be zero, in order to have positive entropy production, or in order to have $\lim_{\boldsymbol{k}\to 0} G^{\rm ret}_{nn}(\omega{=}0, \boldsymbol{k}) = (\partial\rho/\partial\mu)_T$.

Parity-breaking terms: current

Recall

$$j_\mu = -\sigma T \Delta_{\mu\nu}\partial^\nu\left(\frac{\mu}{T}\right) + \chi_{\rm T}\Delta_{\mu\nu}\partial^\nu T - \tilde{\sigma} T \Sigma_{\mu\nu}\partial^\nu\left(\frac{\mu}{T}\right) + \tilde{\chi}_{\rm T}\Sigma_{\mu\nu}\partial^\nu T + O(\partial^2).$$

- σ : Conventional charge conductivity, proportional to the charge diffusion constant. Contributes to off-equilibrium entropy production.
- $\chi_{\rm T}$: Must be zero, in order to have positive entropy production, or in order to have $\lim_{\boldsymbol{k}\to 0} G^{\rm ret}_{nn}(\omega{=}0, \boldsymbol{k}) = (\partial\rho/\partial\mu)_T$.
- $\tilde{\sigma}$: produces the Hall charge conductivity without magnetic field. Does not contribute to off-equilibrium entropy production.
- $\tilde{\chi}_{\rm T}$: does not have to vanish, is a *thermodynamic* parameter. Does not contribute to off-equilibrium entropy production.

We would like to:

- Write down the Kubo formulas for the new transport coefficients
- Give the new coefficients physical interpretation

Couple the system to infinitesimal external electromagnetic and gravitational fields, look at the hydrodynamic response

Outline

Add external E&M and gravity to hydro

- Conservation equations will change:
  external fields do work on the system
- Thermodynamics will change:
  For example, $P(T, \mu)$ becomes $P(T, \mu, B)$ in external B field
- Constitutive relations will change:
  For example, one must have $J_i = \sigma E_i + \ldots$ in external $\mathbf{E}$ field

How conservation equations change

Follow the standard GR prescription:

$$\partial_\mu T^{\mu\nu} = 0 \quad \rightarrow \quad \nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda$$
$$\partial_\mu J^{\mu} = 0 \quad \rightarrow \quad \nabla_\mu J^{\mu} = 0$$

See e.g. the lectures Herzog, arXiv:0904.1975 in the context of hydro

How conservation equations change

Follow the standard GR prescription:

$$\partial_\mu T^{\mu\nu} = 0 \quad \to \quad \nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda$$
$$\partial_\mu J^{\mu} = 0 \quad \to \quad \nabla_\mu J^{\mu} = 0$$

See e.g. the lectures Herzog, arXiv:0904.1975 in the context of hydro

Can equivalently rewrite as

$$\partial_\mu \mathcal{T}^{\mu\nu} = -\Gamma^{\nu}_{\mu\lambda} \mathcal{T}^{\mu\lambda} + F^{\nu\lambda} \mathcal{J}_\lambda \,,$$
$$\partial_\mu \mathcal{J}^{\mu} = 0 \,,$$

where $\mathcal{T}^{\mu\nu} \equiv \sqrt{-g}\, T^{\mu\nu}$, $\mathcal{J}^{\mu} \equiv \sqrt{-g}\, J^{\mu}$

How thermodynamics changes: E&M fields

How thermodynamics changes: E&M fields

- An arbitrary external electromagnetic field will drive the system out of equilibrium. However, time-independent external magnetic field B will allow the system to stay in equilibrium, hence $P = P(T, \mu, B)$.

How thermodynamics changes: E&M fields

- An arbitrary external electromagnetic field will drive the system out of equilibrium. However, time-independent external magnetic field B will allow the system to stay in equilibrium, hence $P = P(T, \mu, B)$.
- Without external fields, $T^{\mu\nu} = \mathrm{diag}(\epsilon, P, P)$ in equilibrium. This is *not true* once external B field is present. Instead,

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & P - mB & 0 \\ 0 & 0 & P - mB \end{pmatrix}, \quad m = \left(\frac{\partial P}{\partial B}\right)_{T,\mu}$$

The subtraction is due to the force by the magnetic field on the boundary currents; survives in the thermodynamic limit

Cooper+Halperin+Ruzin, 1996

How thermodynamics changes: gravity fields

How thermodynamics changes: gravity fields

- An arbitrary external gravity perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ will drive the system out of equilibrium. Is there a gravity analogue B^{G} of the magnetic field B that will allow the system to stay in equilibrium?

How thermodynamics changes: gravity fields

- An arbitrary external gravity perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ will drive the system out of equilibrium. Is there a gravity analogue $B^{\rm G}$ of the magnetic field B that will allow the system to stay in equilibrium?
- Recall that linearized gravity is quite similar to E&M:
  Einstein equations $\Rightarrow$ Maxwell equations
  Geodesic equation $\Rightarrow$ Lorentz force law
  If $F^{\rm G}_{\mu\nu} \equiv \partial_\mu h_{0\nu} - \partial_\nu h_{0\mu}$, with $h_{0x} = -\frac{1}{2}B^{\rm G}y$, $h_{0y} = \frac{1}{2}B^{\rm G}x$

$$\partial_\mu T^{\mu\nu} = -F^{{\rm G}\,\nu\lambda}T_{0\lambda} + F^{\nu\lambda}J_\lambda$$

How thermodynamics changes: gravity fields

- An arbitrary external gravity perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ will drive the system out of equilibrium. Is there a gravity analogue $B^{\rm G}$ of the magnetic field B that will allow the system to stay in equilibrium?
- Recall that linearized gravity is quite similar to E&M:
  Einstein equations $\Rightarrow$ Maxwell equations
  Geodesic equation $\Rightarrow$ Lorentz force law
  If $F^{\rm G}_{\mu\nu} \equiv \partial_\mu h_{0\nu} - \partial_\nu h_{0\mu}$, with $h_{0x} = -\frac{1}{2} B^{\rm G} y$, $h_{0y} = \frac{1}{2} B^{\rm G} x$

$$\partial_\mu T^{\mu\nu} = -F^{{\rm G}\,\nu\lambda} T_{0\lambda} + F^{\nu\lambda} J_\lambda$$

- $B^{\rm G}$ does no work on massive particles, just like B does no work on charged particles. Guess: $P = P(T, \mu, B, B^{\rm G})$

How thermodynamics changes: gravity fields (2)

- However, B^{G} is not a scalar.
- Can find a pseudoscalar which reduces to B^{G} in the fluid rest frame: this is vorticity $\Omega \equiv -\Sigma^{\mu\nu}\nabla_\mu u_\nu$.
- Hence the equilibrium pressure is

$$P = P(T, \mu, B, \Omega)\,,$$

where $\Omega \equiv -\Sigma^{\mu\nu}\nabla_\mu u_\nu$, $B \equiv -\frac{1}{2}\Sigma^{\mu\nu}F_{\mu\nu}$

- Similarly, there must be vortical subtractions to pressure,

$$T^{ij} = (P - mB - m_\Omega \Omega)\delta^{ij}\,, \qquad m_\Omega \equiv \left(\frac{\partial P}{\partial \Omega}\right)_{T,\mu,B}$$

Magnetic and gravitomagnetic subtractions

- In the equilibrium with space-dependent magnetization, there are bound currents, unrelated to transport: $J^i = \epsilon^{ij}\partial_j m$, in the rest frame of the fluid
  Cooper+Halperin+Ruzin, 1996
- Similarly, there must be bound momentum density, unrelated to transport: $T^{0i} = \epsilon^{ij}\partial_j m_\Omega$, in the rest frame of the fluid
- Covariantize:

$$J^\mu_{\rm bound} = \partial_\nu M^{\mu\nu}\,, \qquad M^{\mu\nu} = m\Sigma^{\mu\nu}$$

$$T^{\mu\nu}_{\rm bound} = u^{(\mu}\partial_\lambda M_\Omega^{\nu)\lambda}\,, \qquad M_\Omega^{\mu\nu} = m_\Omega\Sigma^{\mu\nu}$$

Magnetic and gravitomagnetic subtractions

- In the equilibrium with space-dependent magnetization, there are bound currents, unrelated to transport: $J^i = \epsilon^{ij}\partial_j m$, in the rest frame of the fluid
  Cooper+Halperin+Ruzin, 1996
- Similarly, there must be bound momentum density, unrelated to transport: $T^{0i} = \epsilon^{ij}\partial_j m_\Omega$, in the rest frame of the fluid
- Covariantize:

$$J^\mu_{\rm bound} = \partial_\nu M^{\mu\nu}\,, \qquad M^{\mu\nu} = m\Sigma^{\mu\nu}$$

$$T^{\mu\nu}_{\rm bound} = u^{(\mu}\partial_\lambda M_\Omega^{\nu)\lambda}\,, \qquad M_\Omega^{\mu\nu} = m_\Omega\Sigma^{\mu\nu}$$

Therefore, in static equilibrium:

$$T^{\mu\nu} = \begin{pmatrix} \epsilon - m_\Omega\Omega & 0 & 0 \\ 0 & P - mB - m_\Omega\Omega & 0 \\ 0 & 0 & P - mB - m_\Omega\Omega \end{pmatrix}, \quad J^\mu = (n - m\Omega, 0, 0)$$

Gravitomagnets

The vorticity Ω is the gravitational analogue of B

$$\text{Ferromagnets}: \; m = \left(\frac{\partial P}{\partial B}\right) \text{ is non-zero at } B = 0$$

$$\text{Gravito-ferromagnets}: \; m_\Omega = \left(\frac{\partial P}{\partial \Omega}\right) \text{ is non-zero at } \Omega = 0$$

- Having non-zero m_Ω requires parity-breaking
- Simple example: a gas of free massive fermions in 2+1 dim is both a ferromagnet and a gravito-ferromagnet.
- Less simple example: an electrically charged AdS_4 black hole with an axion profile is both a ferromagnet and a gravito-ferromagnet.

Parity-breaking black holes

Gauge/string correspondence is a duality:

Gravitational system in $d{+}2$ dim	$\Leftrightarrow$	QFT on the $d{+}1$ dim boundary
Black hole physics in $d{+}2$ dim	$\Leftrightarrow$	Thermal physics in $d{+}1$ dim QFT
Large-scale dynamics of bh	$\Leftrightarrow$	Hydrodynamics in QFT

For a review, see Hubeny+Rangamani, arXiv:1006.3675

Parity-breaking black holes

Gauge/string correspondence is a duality:

Gravitational system in $d+2$ dim	$\Leftrightarrow$	QFT on the $d+1$ dim boundary
Black hole physics in $d+2$ dim	$\Leftrightarrow$	Thermal physics in $d+1$ dim QFT
Large-scale dynamics of bh	$\Leftrightarrow$	Hydrodynamics in QFT

For a review, see Hubeny+Rangamani, arXiv:1006.3675

Need parity-breaking AdS black holes in 3+1 dimensions

Parity-breaking black holes

Gauge/string correspondence is a duality:

Gravitational system in $d{+}2$ dim $\Leftrightarrow$ QFT on the $d{+}1$ dim boundary
Black hole physics in $d{+}2$ dim $\Leftrightarrow$ Thermal physics in $d{+}1$ dim QFT
Large-scale dynamics of bh $\Leftrightarrow$ Hydrodynamics in QFT

For a review, see Hubeny+Rangamani, arXiv:1006.3675

Need parity-breaking AdS black holes in 3+1 dimensions

- Can break parity through the gravitaional "axion": add

$$\theta(r)\epsilon^{\lambda\rho\alpha\beta}R^{\mu}{}_{\nu\alpha\beta}R^{\nu}{}_{\mu\lambda\rho}$$

to the bulk action, can evaluate $\tilde{\eta}$.

Son+Saremi, 2011

Parity-breaking black holes

Gauge/string correspondence is a duality:

Gravitational system in $d+2$ dim $\Leftrightarrow$ QFT on the $d+1$ dim boundary
Black hole physics in $d+2$ dim $\Leftrightarrow$ Thermal physics in $d+1$ dim QFT
Large-scale dynamics of bh $\Leftrightarrow$ Hydrodynamics in QFT

For a review, see Hubeny+Rangamani, arXiv:1006.3675

Need parity-breaking AdS black holes in 3+1 dimensions

- Can break parity through the gravitaional "axion": add

$$\theta(r)\epsilon^{\lambda\rho\alpha\beta}R^{\mu}{}_{\nu\alpha\beta}R^{\nu}{}_{\mu\lambda\rho}$$

  to the bulk action, can evaluate $\tilde{\eta}$. Son+Saremi, 2011

- Or can break parity through the conventional axion: add

$$\theta(r)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

  to the bulk action, can evaluate $\tilde{\sigma}$.

Now I want to switch gears

- I would like to talk some more about hydro in general, regardless of parity breaking
- So far, hydro was presented as a *classical* theory, i.e. as a set of partial differental equations
- Hydro is more than just a classical theory: just like there are quantum fluctuations in the QFT vacuum, there are thermal fluctuations in the equilibrium state
- These fluctuations may significantly change what you thought was classical hydrodynamics

The rest of the talk will be about these fluctuations, and is not specifically related to parity breaking

Outline

1. Normal relativistic hydro
2. Hydro for systems without parity
3. Magnets and gravitomagnets
4. Fluctuations: a problem with classical hydro
5. A systematic way to treat fluctuations
6. A toy model with fluctuations
7. Conclusions

Let us start with viscosity

Viscosity measures rate of momentum transfer between layers of fluid

$$\eta = \rho v_{\mathrm{th}} \ell_{\mathrm{mfp}}$$

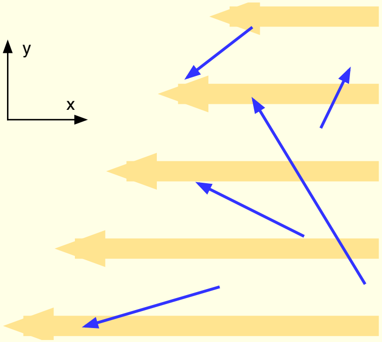
Maxwell, 1860

Let us start with viscosity

Viscosity measures rate of momentum transfer between layers of fluid

$$\eta = \rho v_{\mathrm{th}} \ell_{\mathrm{mfp}}$$

Maxwell, 1860

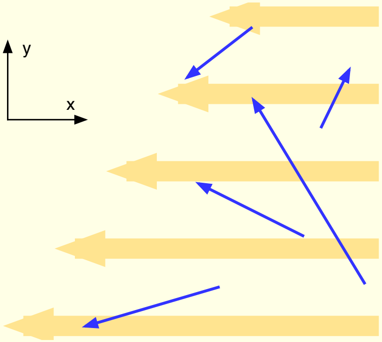


Let us start with viscosity

Viscosity measures rate of momentum transfer between layers of fluid

$$\eta = \rho v_{\rm th} \ell_{\rm mfp}$$

Maxwell, 1860



$$\ell_{\rm mfp} \sim \frac{1}{n\sigma} \sim \frac{T}{\lambda^2}$$

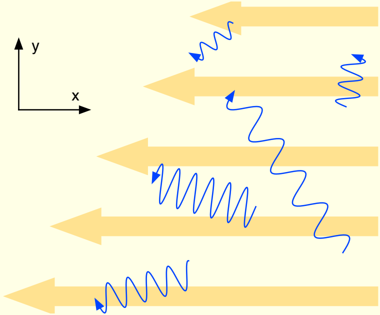
$$\eta_0 \sim \frac{N^2 T^3}{\lambda^2}$$

Let us start with viscosity (2)

Elementary excitations are not the only way to transfer momentum.
Momentum can also be transfered by collective excitations.

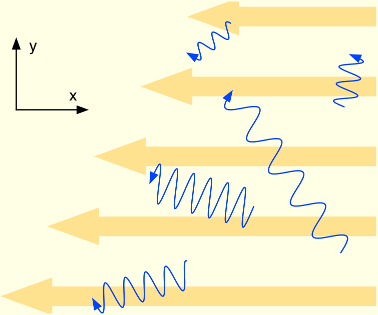
Let us start with viscosity (2)

Elementary excitations are not the only way to transfer momentum.
Momentum can also be transfered by collective excitations.



Let us start with viscosity (2)

Elementary excitations are not the only way to transfer momentum.
Momentum can also be transfered by collective excitations.

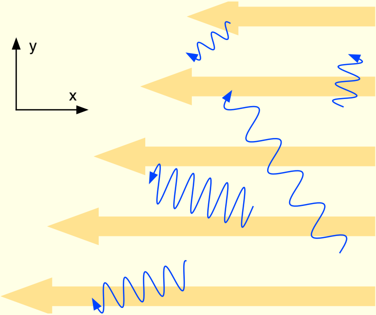


$$\ell_{\rm mfp} \sim \frac{1}{\frac{\eta}{\epsilon+P}\boldsymbol{k}^2}$$

$$\eta_1 \sim \int^{k_{\max}} d^3k \frac{T}{\frac{\eta_0}{\epsilon+P}\boldsymbol{k}^2} \sim \frac{k_{\max}T^2}{\eta_0/s}$$

Let us start with viscosity (2)

Elementary excitations are not the only way to transfer momentum. Momentum can also be transfered by collective excitations.



$$\ell_{\rm mfp} \sim \frac{1}{\frac{\eta}{\epsilon+P}\boldsymbol{k}^2}$$

$$\eta_1 \sim \int^{k_{\rm max}} d^3k \frac{T}{\frac{\eta_0}{\epsilon+P}\boldsymbol{k}^2} \sim \frac{k_{\rm max}T^2}{\eta_0/s}$$

- Total viscosity $\eta_{\rm total} = \eta_0 + \eta_1$ is bounded from below
- This integral IR finite in $d = 3+1$, but IR divergent in $d = 2+1$

Forster+Nelson+Stephen, 1977

Long-time tails

Start with $\boldsymbol{J} = -D\boldsymbol{\nabla} n + n\boldsymbol{v}$, take $\boldsymbol{k} = 0$. Schematically:

$$\begin{aligned}
\langle \boldsymbol{J}(t)\boldsymbol{J}(0)\rangle &\supset \int d^d x \, \langle n(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})n(0)\boldsymbol{v}(0)\rangle \\
&= \int d^d x \, \langle n(t,\boldsymbol{x})n(0)\rangle \langle \boldsymbol{v}(t,x)\boldsymbol{v}(0)\rangle \\
&\sim \int d^d k \, e^{-D\boldsymbol{k}^2 t} e^{-\gamma_\eta \boldsymbol{k}^2 t} \\
&\sim \left[\frac{1}{(D+\gamma_\eta)t}\right]^{d/2}
\end{aligned}$$

See e.g. Arnold+Yaffe, PRD 1997
(known since late 1960's)

Long-time tails

Start with $\boldsymbol{J} = -D\boldsymbol{\nabla} n + n\boldsymbol{v}$, take $\boldsymbol{k} = 0$. Schematically:

$$
\begin{aligned}
\langle \boldsymbol{J}(t)\boldsymbol{J}(0)\rangle &\supset \int d^d x \, \langle n(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})n(0)\boldsymbol{v}(0)\rangle \\
&= \int d^d x \, \langle n(t,\boldsymbol{x})n(0)\rangle\langle \boldsymbol{v}(t,x)\boldsymbol{v}(0)\rangle \\
&\sim \int d^d k \, e^{-D\boldsymbol{k}^2 t} e^{-\gamma_\eta \boldsymbol{k}^2 t} \\
&\sim \left[\frac{1}{(D+\gamma_\eta)t}\right]^{d/2}
\end{aligned}
$$

See e.g. Arnold+Yaffe, PRD 1997
(known since late 1960's)

When FT, the convective contribution to $S(\omega)$ is

$$
S(\omega) \sim \omega^{1/2}\,, \quad d = 3
$$
$$
S(\omega) \sim \ln(\omega)\,, \quad d = 2
$$

Correction to Kubo formulas

Recall Kubo formula for the diffusion constant:

$$D\chi T = \lim_{\omega\to 0} \frac{1}{2d} S_{ii}(\omega, \boldsymbol{k}{=}0)$$

Correction to Kubo formulas

Recall Kubo formula for the diffusion constant:

$$D\chi T = \lim_{\omega\to 0} \frac{1}{2d} S_{ii}(\omega, \boldsymbol{k}{=}0)$$

This was derived in linear response. With the non-linear temrs:

$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const}\ \omega^{1/2}\right) , \quad d = 3$$

$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const}\ \ln(\omega)\right) , \quad d = 2$$

Correction to Kubo formulas

Recall Kubo formula for the diffusion constant:

$$D\chi T = \lim_{\omega\to 0} \frac{1}{2d} S_{ii}(\omega, \boldsymbol{k}{=}0)$$

This was derived in linear response. With the non-linear temrs:

$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const}\ \omega^{1/2}\right)\ , \quad d = 3$$

$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const}\ \ \ln(\omega)\right)\ , \quad d = 2$$

Same applies to shear viscosity:

$$\eta^{\text{full}} = \lim_{\omega\to 0} \left(\eta + \text{const}\ \omega^{1/2}\right)\ , \quad d = 3$$

$$\eta^{\text{full}} = \lim_{\omega\to 0} \left(\eta + \text{const}\ \ \ln(\omega)\right)\ , \quad d = 2$$

Correction to Kubo formulas

Recall Kubo formula for the diffusion constant:

$$D\chi T = \lim_{\omega\to 0} \frac{1}{2d} S_{ii}(\omega, \boldsymbol{k}{=}0)$$

This was derived in linear response. With the non-linear temrs:

$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const}\ \omega^{1/2}\right) , \quad d = 3$$

$$D^{\text{full}} = \lim_{\omega\to 0} \left(D + \text{const}\ \ln(\omega)\right) , \quad d = 2$$

Same applies to shear viscosity:

$$\eta^{\text{full}} = \lim_{\omega\to 0} \left(\eta + \text{const}\ \omega^{1/2}\right) , \quad d = 3$$

$$\eta^{\text{full}} = \lim_{\omega\to 0} \left(\eta + \text{const}\ \ln(\omega)\right) , \quad d = 2$$

In 2+1 dimensional hydro, transport coefficients blow up

Problems with second-order hydro

Problems with second-order hydro

In linearized second order hydro:

$$G^R_{xy,xy}(\omega, \boldsymbol{k}) = P - i\omega\eta + \left(\eta\tau_\Pi - \frac{\kappa}{2}\right)\omega^2 - \frac{\kappa}{2}\boldsymbol{k}^2 + \ldots$$

Baier+Romatschke+Son+Starinets+Stephanov, 2007

Problems with second-order hydro

In linearized second order hydro:

$$G^R_{xy,xy}(\omega,\boldsymbol{k}) = P - i\omega\eta + \left(\eta\tau_\Pi - \frac{\kappa}{2}\right)\omega^2 - \frac{\kappa}{2}\boldsymbol{k}^2 + \dots$$

Baier+Romatschke+Son+Starinets+Stephanov, 2007

But this gets seriously modified by 1-loop hydro fluctuations,

$$G^R_{xy,xy}(\omega,\boldsymbol{k}{=}0) = P - i\omega\eta - \mathrm{const}\,|\omega|^{3/2}(1 + i\,\mathrm{sign}(\omega)) + \dots$$

Problems with second-order hydro

In linearized second order hydro:

$$G^R_{xy,xy}(\omega, \boldsymbol{k}) = P - i\omega\eta + \left(\eta\tau_\Pi - \frac{\kappa}{2}\right)\omega^2 - \frac{\kappa}{2}\boldsymbol{k}^2 + \dots$$

Baier+Romatschke+Son+Starinets+Stephanov, 2007

But this gets seriously modified by 1-loop hydro fluctuations,

$$G^R_{xy,xy}(\omega, \boldsymbol{k}{=}0) = P - i\omega\eta - \text{const}\,|\omega|^{3/2}(1 + i\,\text{sign}(\omega)) + \dots$$

Blindly apply Kubo formula

$$\eta\tau_\Pi - \frac{\kappa}{2} = \lim_{\omega\to 0}\frac{1}{2}\frac{\partial^2}{\partial\omega^2}\text{Re}\,G^R_{xy,xy}(\omega, \boldsymbol{k}{=}0) \to \infty$$

Problems with second-order hydro

In linearized second order hydro:

$$G^R_{xy,xy}(\omega,\boldsymbol{k}) = P - i\omega\eta + \left(\eta\tau_\Pi - \frac{\kappa}{2}\right)\omega^2 - \frac{\kappa}{2}\boldsymbol{k}^2 + \dots$$

Baier+Romatschke+Son+Starinets+Stephanov, 2007

But this gets seriously modified by 1-loop hydro fluctuations,

$$G^R_{xy,xy}(\omega,\boldsymbol{k}{=}0) = P - i\omega\eta - \mathrm{const}\,|\omega|^{3/2}(1 + i\,\mathrm{sign}(\omega)) + \dots$$

Blindly apply Kubo formula

$$\eta\tau_\Pi - \frac{\kappa}{2} = \lim_{\omega\to 0}\frac{1}{2}\frac{\partial^2}{\partial\omega^2}\mathrm{Re}\,G^R_{xy,xy}(\omega,\boldsymbol{k}{=}0) \to \infty$$

This means τ_Π does not exist in classical hydro

Comment

- Hydro fluctuations suggest a lower bound on viscosity and have implications for QGP physics Kovtun+Moore+Romatschke, 2011
- Current hydro simulations of QGP are blind to these effects because they simply solve the classical hydro equations and ignore the fluctuations
- Holographic fluids are blind to these effects because the fluctuation corrections are $1/N^{\#}$ suppressed. Transport coefficients come out finite in *classical* gravity. Long-time tails come from quantum corrections to classical gravity
  Kovtun+Yaffe, 2003
  Caron-Huot + Saremi, 2009
- This is an example where $\omega \to 0$ limit does not commute with large-N limit.

Outline

1. Normal relativistic hydro
2. Hydro for systems without parity
3. Magnets and gravitomagnets
4. Fluctuations: a problem with classical hydro
5. A systematic way to treat fluctuations
6. A toy model with fluctuations
7. Conclusions

Brownian particle

$$m\frac{d^2x}{dt^2} = -(6\pi\eta a)\frac{dx}{dt} + f(t)\,,$$

$(6\pi\eta a)$ = friction coefficient (Stokes law)
$f(t)$ = random force

Brownian particle

$$m\frac{d^2x}{dt^2} = -(6\pi\eta a)\frac{dx}{dt} + f(t)\,,$$

$(6\pi\eta a)$ = friction coefficient (Stokes law)
$f(t)$ = random force

Take $q \equiv \frac{dx}{dt}$, $\Rightarrow$ Langevin equation:

$$\boxed{\dot{q}(t) + \gamma q(t) = \xi(t)}$$

Brownian particle

$$m\frac{d^2x}{dt^2} = -(6\pi\eta a)\frac{dx}{dt} + f(t)\,,$$

$(6\pi\eta a) =$ friction coefficient (Stokes law)
$f(t) =$ random force

Take $q \equiv \frac{dx}{dt}$, $\Rightarrow$ Langevin equation:

$$\boxed{\dot{q}(t) + \gamma q(t) = \xi(t)}$$

Noise properties:

$$\langle \xi(t) \rangle = 0\,, \qquad \langle \xi(t)\xi(t') \rangle = 2T\delta(t-t')\,.$$

Correlation function of $q(t)$

Correlation function of $q(t)$

- Take the Langevin equation $\dot{q}(t) + \gamma q(t) = \xi(t)$
- Solve for $q(t)$ in terms of $\xi(t)$
- Find $\langle q(t)q(t')\rangle$ by averaging over $\xi(t)$
- When $\gamma t, \gamma t' \gg 1$, find

$$\langle q(t)q(t')\rangle = \frac{C}{2\gamma} e^{-\gamma|t-t'|}$$

- Fourier transform:

$$S(\omega) = \frac{C}{\omega^2 + \gamma^2}$$

Path integral for Brownian particle

Let us now represent the Brownian motion as Quantum Mechanics
(0+1 dimensional quantum field theory)

Path integral for Brownian particle

Path integral for Brownian particle

Step 1 Write Langevin equation as $EoM \equiv (\dot{q} + \frac{\partial F}{\partial q} - \xi) = 0$

Path integral for Brownian particle

Step 1 Write Langevin equation as $EoM \equiv (\dot{q} + \frac{\partial F}{\partial q} - \xi) = 0$

Step 2 Gaussian noise:

$$\langle ... \rangle = \int \mathcal{D}\xi \, e^{-W[\xi]} (...) \,, \quad \text{where} \quad W[\xi] = \frac{1}{2C} \int dt' \, \xi(t')^2 \,.$$

Path integral for Brownian particle

Step 1 Write Langevin equation as $EoM \equiv (\dot{q} + \frac{\partial F}{\partial q} - \xi) = 0$

Step 2 Gaussian noise:

$$\langle ... \rangle = \int \mathcal{D}\xi \, e^{-W[\xi]} (...) \,, \quad \text{where} \quad W[\xi] = \frac{1}{2C} \int dt' \, \xi(t')^2 \,.$$

Step 3 Recall $\delta(f(x)) \sim \delta(x - x_0)$, where x_0 solves $f(x_0) = 0$. So

$$\int \mathcal{D}q \, J \, \delta(EoM) \, q(t_1) \, q(t_2) ... = \underbrace{q_\xi(t_1) \;\; q_\xi(t_2)}_{\text{satisfy } EoM(q,\xi) = 0} ...$$

Path integral for Brownian particle

Step 1 Write Langevin equation as $EoM \equiv (\dot{q} + \frac{\partial F}{\partial q} - \xi) = 0$

Step 2 Gaussian noise:

$$\langle ... \rangle = \int \mathcal{D}\xi \, e^{-W[\xi]} (...) \,, \quad \text{where} \quad W[\xi] = \frac{1}{2C} \int dt' \, \xi(t')^2 \,.$$

Step 3 Recall $\delta(f(x)) \sim \delta(x - x_0)$, where x_0 solves $f(x_0) = 0$. So

$$\int \mathcal{D}q \, J \, \delta(EoM) \, q(t_1) \, q(t_2) ... = \underbrace{q_\xi(t_1)} \; \underbrace{q_\xi(t_2)} ...$$

$$\text{satisfy } EoM(q, \xi) = 0$$

Step 4 Write $\delta(EoM) = \int \mathcal{D}p \, e^{i \int p \, EoM}$, do the integral over $\xi(t)$.

Path integral for Brownian particle (2)

When the dust settles:

$$\langle\, q(t_1) \dots q(t_n) \,\rangle = \int \mathcal{D}q \, \mathcal{D}p \, J \, e^{iS[q,p]} \, q(t_1) \dots q(t_n)$$

where

$$S[q,p] = \int dt \, \left(p\dot{q} + p\frac{\partial F}{\partial q} + \frac{iC}{2} p^2 \right) \, .$$

Path integral for Brownian particle (2)

When the dust settles:

$$\langle\, q(t_1) \ldots q(t_n)\,\rangle = \int \mathcal{D}q\, \mathcal{D}p\, J\, e^{iS[q,p]}\, q(t_1) \ldots q(t_n)$$

where

$$S[q,p] = \int dt\, \left(p\dot{q} + p\frac{\partial F}{\partial q} + \frac{iC}{2} p^2 \right) .$$

For the simple Langevin equation $F(q) = \frac{1}{2}\gamma q^2$,

$$S(\omega) = \text{FT of } \langle q(t) q(t')\rangle = \frac{C}{\omega^2 + \gamma^2} ,$$

as expected.

Bottomline:

In the stochastic model

$$\dot{q}(t) + \underbrace{\frac{\partial F(q)}{\partial q}}_{\text{relaxation term}} = \underbrace{\xi(t)}_{\text{noise term}}$$

correlation functions can be derived from field theory with

$$S_{\text{eff}}[q,p] = \int dt \left(p\dot{q} + p\frac{\partial F}{\partial q} + \frac{iC}{2}p^2 \right)$$

See e.g. J.Zinn-Justin's QFT book

- This effective action is **not** meant to reproduce the classical equation of motion for a particle subject to friction force.
- Rather, it is to be used to construct the generating functional for the correlation functions of $q(t)$

Comment:

- Can do the same for field theory $q_i(t) \to \varphi(\boldsymbol{x}, t)$ Martin+Siggia+Rose, 1973
- Can apply to dynamic critical phenomena Hohenberg+Halperin, 1977
- Can study fluctuations in 2+1 dim fluids See e.g. Forster+Nelson+Stephen, 1977 Khalatnikov+Lebedev+Sukhorukov, 1983
- Can rewrite the equations of hydrodynamics as a quantum field theory, with T playing the role of $\hbar$.

Outline

A simple toy model

- Incompressible fluid: impose $\boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0$ — Forster+Nelson+Stephen, 1977
- Momentum conservation:

$$\partial_t \pi_i = -\partial_j T_{ij} + \xi_i \,, \qquad T_{ij} = P\delta_{ij} - \gamma_\eta(\partial_i \pi_j + \partial_j \pi_i) + \frac{\pi_i \pi_j}{\bar{w}}$$

- Current conservation:

$$\partial_t n = -\partial_i J_i + \theta \,, \qquad J_i = -D\partial_i n + \frac{n\pi_i}{\bar{w}}$$

- Stochastic model:

$$\partial_t \pi_i = -\partial_i P + \gamma_\eta \boldsymbol{\nabla}^2 \pi_i - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})\pi_i}{\bar{w}} + \xi_i \,,$$

$$\partial_t n = D\boldsymbol{\nabla}^2 n - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})n}{\bar{w}} + \theta \,,$$

A simple toy model

- Incompressible fluid: impose $\boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0$ Forster+Nelson+Stephen, 1977
- Momentum conservation:

$$\partial_t \pi_i = -\partial_j T_{ij} + \xi_i \,, \qquad T_{ij} = P\delta_{ij} - \gamma_\eta(\partial_i\pi_j + \partial_j\pi_i) + \frac{\pi_i\pi_j}{\bar{w}}$$

- Current conservation:

$$\partial_t n = -\partial_i J_i + \theta \,, \qquad J_i = -D\partial_i n + \frac{n\pi_i}{\bar{w}}$$

- Stochastic model:

$$\partial_t \pi_i = -\partial_i P + \gamma_\eta \boldsymbol{\nabla}^2 \pi_i - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})\pi_i}{\bar{w}} + \xi_i \,,$$

$$\partial_t n = D\boldsymbol{\nabla}^2 n - \frac{(\boldsymbol{\pi}\cdot\boldsymbol{\nabla})n}{\bar{w}} + \theta \,,$$

Note that the convective term couples charge density fluctuations to momentum density fluctuations

Effective action for the toy model

$$S_{\rm eff} = \int dt\, d^d x \Big(\mathcal{L}^{(2)} + \mathcal{L}^{(int)} \Big)$$

$$\mathcal{L}^{(2)} = -\frac{\sigma}{2}\rho\nabla^2\rho - \frac{\tilde\sigma}{2}\lambda_i\nabla^2\lambda_i - i\rho(\partial_t n - D\nabla^2 n) - i\lambda_i(\partial_t\pi_i - \Gamma\nabla^2\pi_i)$$
$$+\bar\psi_i(\partial_t - \Gamma\nabla^2)\psi_i + \bar\psi_{\rm n}(\partial_t - D\nabla^2)\psi_{\rm n}\,,$$

$$\mathcal{L}^{(int)} = -\frac{i}{w}\rho\pi_i\partial_i n - \frac{i}{w}\lambda_i\pi_j\partial_j\pi_i$$
$$+\frac{1}{w}\bar\psi_i\partial_k\pi_i\,\psi_k + \frac{1}{w}\bar\psi_i\pi_k\partial_k\psi_i + \frac{1}{w}\bar\psi_{\rm n}\partial_i n\,\psi_i + \frac{1}{w}\bar\psi_{\rm n}\pi_k\partial_k\psi_{\rm n}\,,$$

plus the constraints $\partial_i\pi_i = 0$, $\partial_i\lambda_i = 0$, $\partial_i\bar\psi_i = 0$, $\partial_i\psi_i = 0$.
The constants are $\sigma = 2TD\chi$, $\tilde\sigma = 2T\Gamma w$, $\Gamma = \eta/w$.

One-loop correlation functions in the toy model

As $\boldsymbol{k}\to 0$:

$$\langle T_{0i}T_{0j}\rangle = \frac{2Tw\Gamma(\omega)\boldsymbol{k}^2}{\omega^2+\Big(\Gamma(\omega)\boldsymbol{k}^2\Big)^2}\,,\qquad \langle J_0 J_0\rangle = \frac{2T\chi D(\omega)\boldsymbol{k}^2}{\omega^2+\Big(D(\omega)\boldsymbol{k}^2\Big)^2}\,.$$

This looks like the familiar linear response functions, except D and η now depend on ω.

One-loop correlation functions in the toy model

As $\boldsymbol{k}\to 0$:

$$\langle T_{0i}T_{0j}\rangle = \frac{2Tw\Gamma(\omega)\boldsymbol{k}^2}{\omega^2+\Big(\Gamma(\omega)\boldsymbol{k}^2\Big)^2}\,,\qquad \langle J_0 J_0\rangle = \frac{2T\chi D(\omega)\boldsymbol{k}^2}{\omega^2+\Big(D(\omega)\boldsymbol{k}^2\Big)^2}\,.$$

This looks like the familiar linear response functions, except D and η now depend on ω.

In $d{=}3$ dimensions:

$$\Gamma(\omega) = \Gamma - \frac{23}{30\pi s}\frac{\sqrt{|\omega|}}{(4\Gamma)^{3/2}}\,,\qquad D(\omega) = D - \frac{1}{3\pi s}\frac{\sqrt{|\omega|}}{[2(\Gamma{+}D)]^{3/2}}\,.$$

Conventional Kubo formulas make sense:

$$D = \frac{1}{2T\chi}\lim_{\omega\to 0}\lim_{\boldsymbol{k}\to 0}\frac{\omega^2}{\boldsymbol{k}^2}G_{nn}(\omega,\boldsymbol{k})$$

One-loop correlation functions in the toy model

As $\boldsymbol{k}\to 0$:

$$\langle T_{0i}T_{0j}\rangle = \frac{2Tw\Gamma(\omega)\boldsymbol{k}^2}{\omega^2+\Big(\Gamma(\omega)\boldsymbol{k}^2\Big)^2}\,, \qquad \langle J_0 J_0\rangle = \frac{2T\chi D(\omega)\boldsymbol{k}^2}{\omega^2+\Big(D(\omega)\boldsymbol{k}^2\Big)^2}\,.$$

This looks like the familiar linear response functions, except D and η now depend on ω.

One-loop correlation functions in the toy model

As $\boldsymbol{k}\to 0$:

$$\langle T_{0i}T_{0j}\rangle = \frac{2Tw\Gamma(\omega)\boldsymbol{k}^2}{\omega^2+\Big(\Gamma(\omega)\boldsymbol{k}^2\Big)^2}\,, \qquad \langle J_0 J_0\rangle = \frac{2T\chi D(\omega)\boldsymbol{k}^2}{\omega^2+\Big(D(\omega)\boldsymbol{k}^2\Big)^2}\,.$$

This looks like the familiar linear response functions, except D and η now depend on ω.

In $d{=}2$ dimensions:

$$\Gamma(\omega) = \Gamma(\mu) + \frac{1}{32\pi s}\frac{1}{\Gamma(\mu)}\ln\frac{\mu}{\omega}\,, \quad D(\omega) = D(\mu) + \frac{1}{8\pi s}\frac{1}{\Gamma(\mu)+D(\mu)}\ln\frac{\mu}{\omega}\,.$$

One-loop correlation functions in the toy model

As $\boldsymbol{k}\to 0$:

$$\langle T_{0i}T_{0j}\rangle = \frac{2Tw\Gamma(\omega)\boldsymbol{k}^2}{\omega^2 + \Big(\Gamma(\omega)\boldsymbol{k}^2\Big)^2}\,, \qquad \langle J_0 J_0\rangle = \frac{2T\chi D(\omega)\boldsymbol{k}^2}{\omega^2 + \Big(D(\omega)\boldsymbol{k}^2\Big)^2}\,.$$

This looks like the familiar linear response functions, except D and η now depend on ω.

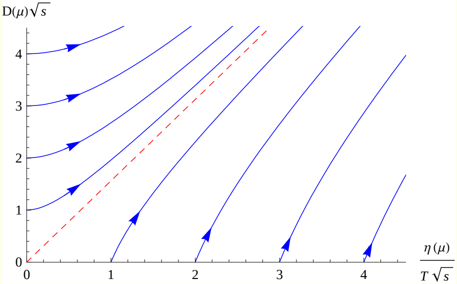
In $d{=}2$ dimensions:

$$\Gamma(\omega) = \Gamma(\mu) + \frac{1}{32\pi s}\frac{1}{\Gamma(\mu)}\ln\frac{\mu}{\omega}\,, \quad D(\omega) = D(\mu) + \frac{1}{8\pi s}\frac{1}{\Gamma(\mu)+D(\mu)}\ln\frac{\mu}{\omega}\,.$$

Now $\eta(\mu)$ and $D(\mu)$ are running "masses" obeying the RG equations

$$\mu\frac{\partial\Gamma}{\partial\mu} = -\frac{1}{32\pi s}\frac{1}{\Gamma}\,, \qquad \mu\frac{\partial D}{\partial\mu} = -\frac{1}{8\pi s}\frac{1}{\Gamma+D}\,.$$

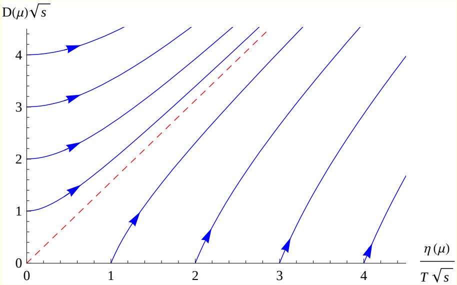
RG flow diagram in $d=2$



In the extreme low-frequency limit $\mu \to 0$:

$$DT = \frac{\sqrt{17}-1}{2}\,\frac{\eta}{s} \approx 1.56\,\frac{\eta}{s}$$

RG flow diagram in $d=2$

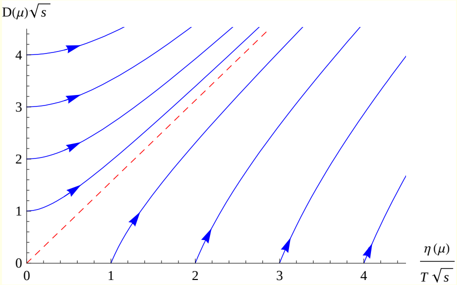


In the extreme low-frequency limit $\mu \to 0$:

$$DT = \frac{\sqrt{17}-1}{2}\,\frac{\eta}{s} \approx 1.56\,\frac{\eta}{s}$$

D and η are not independent transport coefficients in extreme IR

RG flow diagram in $d=2$



In the extreme low-frequency limit $\mu \to 0$:

$$DT = \frac{\sqrt{17} - 1}{2} \frac{\eta}{s} \approx 1.56 \frac{\eta}{s}$$

D and η are not independent transport coefficients in extreme IR

I was excited to derive this result, but then I saw it in V.Lebedev's lectures as an "exercise for the reader"

What happens in the full hydro?

- It gets more messy!
- It gets even messier in relativistic hydro!
- And even messier in relativistic hydro with parity-breaking!

However, the same qualitative conclusions will presumably hold

Outline

Lessons

- When Parity is not a symmetry, hydrodynamics looks different from what one would naively expect from Landau-Lifshitz, vol.6.
- When Parity is not a symmetry, hydro equations get modified by non-dissipative terms. A similar phenomenon happens in 3+1 dim as well. Son+Surowka, 2009
- Vorticity is a gravitational analogue of the magnetic field, and needs to be treated as a thermodynamic variable.
- Thermodynamics of “axionic” AdS back holes is consistent with vortical subtractions
- Regardless of whether Parity is a symmetry or not, hydro fluctuations are important in 2+1 dimensions

I would like to understand:

- I have only talked about thermal states with $\Omega = 0$. Transport in states with $\Omega \neq 0$?
- Parity-broken correlation functions from gravity?
- Systematic treatment of fluctuation effects in relativistic hydro. Work in progress with Guy Moore and Paul Romatschke.
- Effective action for dissipative hydro from AdS/CFT? Perhaps the linearized action is ok, but the full action requires ghosts.
- Effective action for relativistic superfluids?
- The flow of transport coefficients in 2+1 dim at non-zero density? In external magnetic field?

THE END!