Critical Fermi surface states in 2+1 dimensions.

Part I

Max Metlitski

KITP, September 9, 2011

Collaborators

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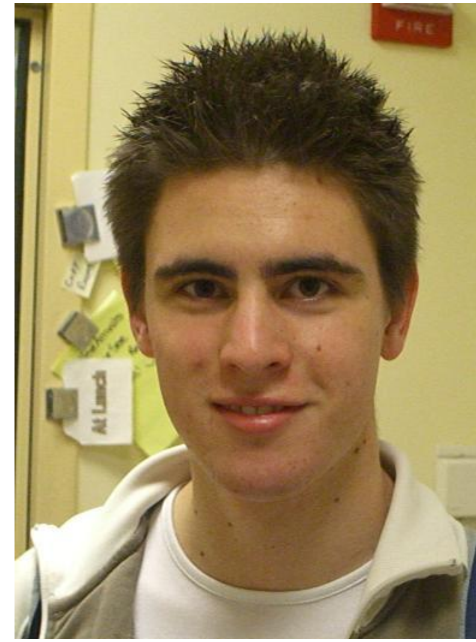


David Mross
(MIT)

Critical Fermi surface states in 2+1 d. Part I.

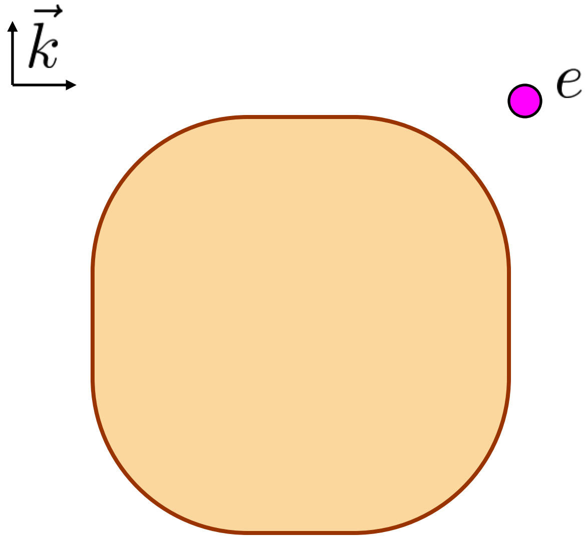
- Introduction to critical FS states and phase transitions in metals
- Experimental motivation
- Other proposed realizations of critical Fermi surface states
  - spinon Fermi-surface state of Mott-insulators
  - composite-fermion liquid of half-filled Landau-level
- Brief review of the RG treatment of Fermi-liquid theory
- “Theory” of critical Fermi surface states in 2+1d

Critical Fermi surface states in 2+1 d. Part II.

- "Theory" of critical Fermi surface states in 2+1d (continued)
  - ε – expansion (aka Nayak-Wilczek expansion)
  - the MIT double scaling limit

- Pairing instabilities of critical Fermi surface states

Landau Fermi-liquid theory



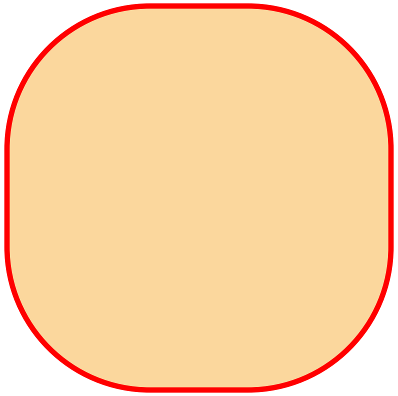
- low energy excitations have the same quantum numbers as for a non-interacting Fermi gas

$$G(\vec{k},\omega) = \frac{Z}{i\omega - v_F(|\vec{k}| - k_F)}$$

- Quasiparticle residue Z and Fermi velocity v_F can renormalize

L. D. Landau (1956)

Critical Fermi surface states



- Are there states with

  - a sharp Fermi-surface
  - no Landau quasiparticles

$$G(\vec{k},\omega) \neq \frac{Z}{i\omega - v_F(|\vec{k}| - k_F)}$$

- 1d: Luttinger liquids: $G(k,\omega) \sim \frac{1}{i\omega - k}(k^2+\omega^2)^{(K-1)^2/4K}$

- d > 1 ?

Some candidate critical Fermi surface states

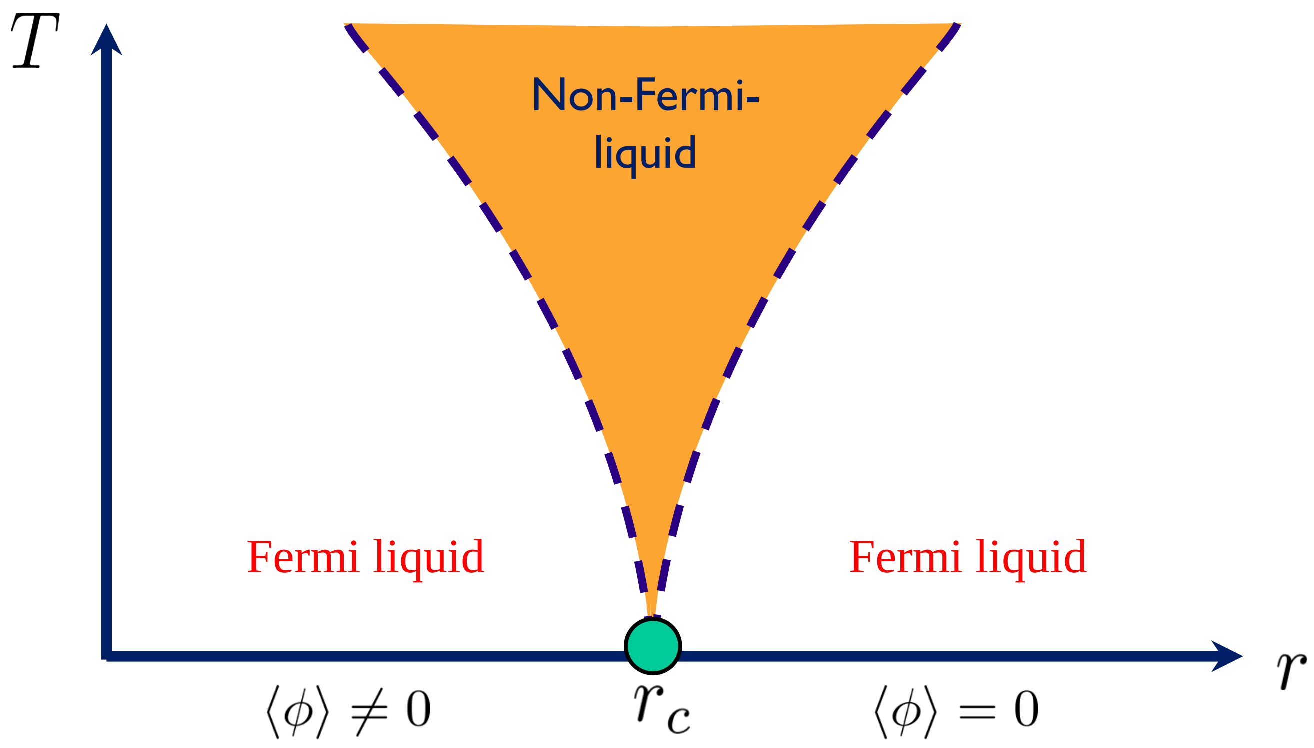
- Phase transitions in metals

- Spinon Fermi-surface state of Mott-insulators

- Composite-fermion liquid of QHE system

Phase transitions in metals

- Order parameter: ϕ



Phase transitions in metals

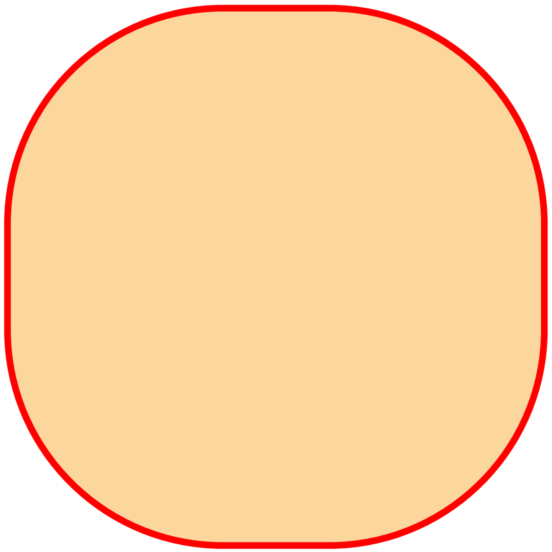
$\vec{Q} = 0$

ferromagnet:

$$\langle \vec{S}(\vec{x}) \rangle \sim \langle \vec{\phi} \rangle$$

nematic:

$$\langle \psi^\dagger (\partial_x^2 - \partial_y^2) \psi(\vec{x}) \rangle \sim \langle \phi \rangle$$



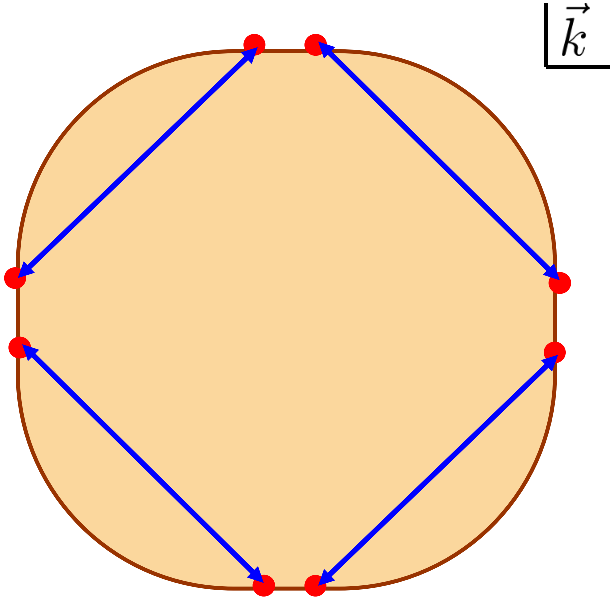
$\vec{Q} \neq 0$

charge-density wave:

$$\langle \rho(\vec{x}) \rangle \sim \langle \phi \rangle e^{i\vec{Q}\cdot\vec{x}}$$

spin-density wave:

$$\langle \vec{S}(\vec{x}) \rangle \sim \langle \vec{\phi} \rangle e^{i\vec{Q}\cdot\vec{x}}$$



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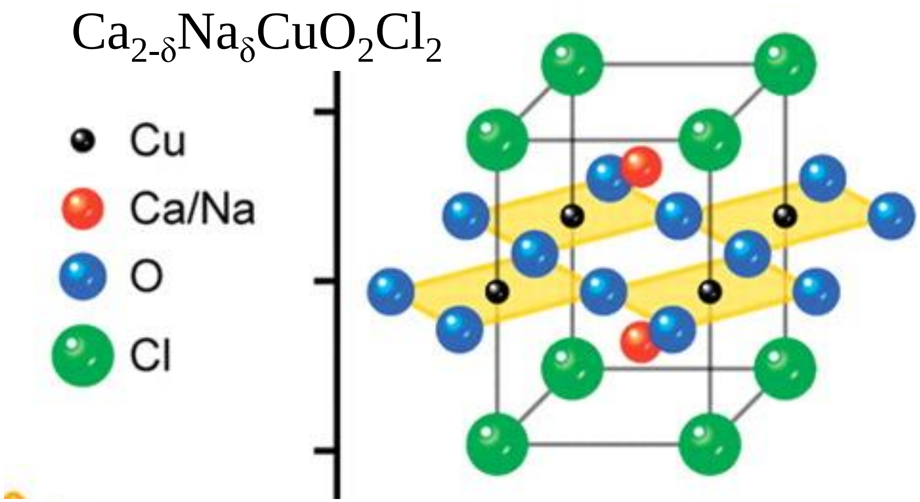
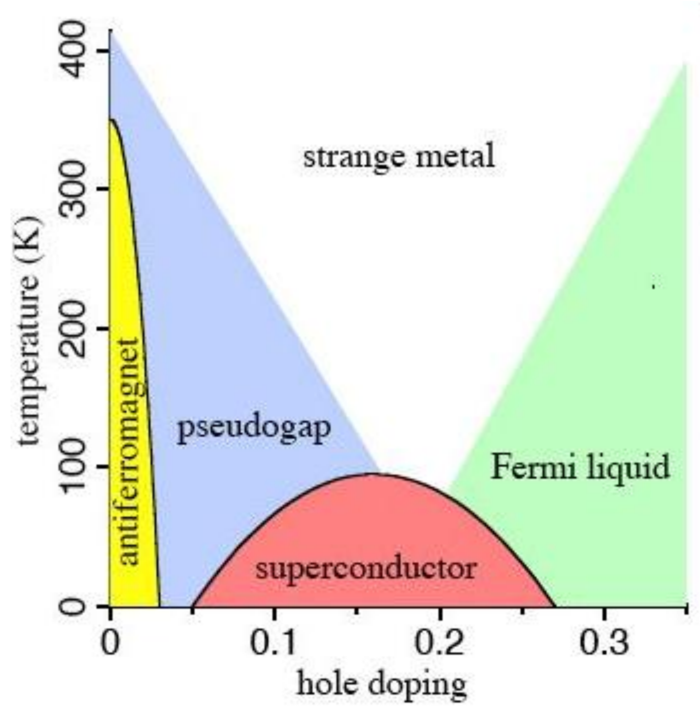
Experimental motivation: "strange" metal physics





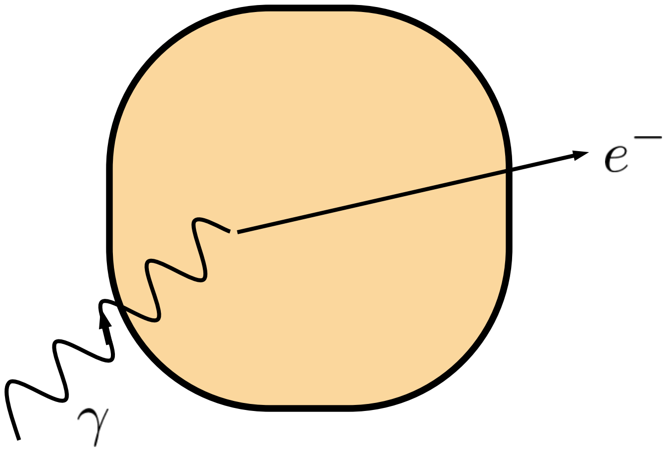
- Strange metal regime characterized by $\rho \sim T$

  (compare to $\rho \sim T^2$ in a Fermi-liquid with umklapps)

- Optical conductivity: $\sigma(\omega) \sim \omega^{-\alpha}, \quad \omega \gg T, \quad \alpha \approx 2/3$

  (compare to $\sigma(\omega) \sim const$ in a Fermi-liquid with umklapps)

Evidence for disappearance of quasiparticles from photoemission experiments



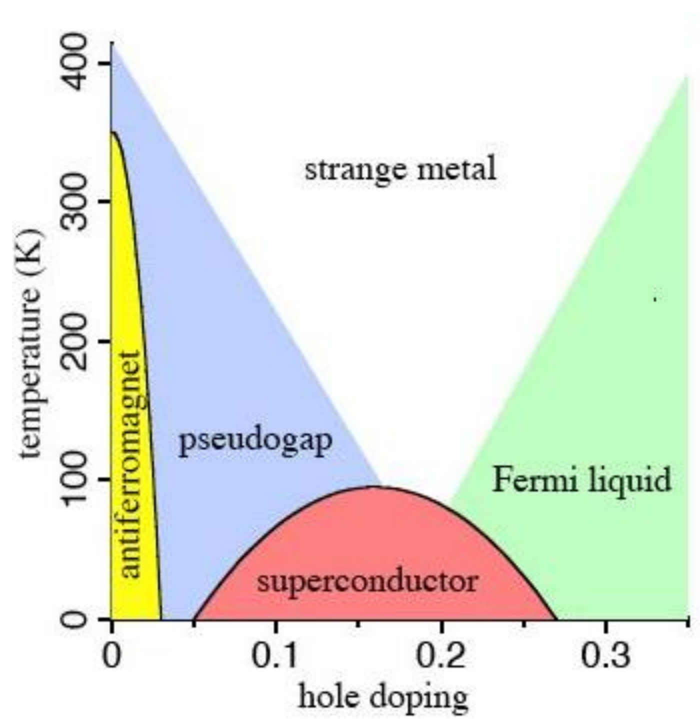
$$I(\vec{k},\omega) \propto n_F(\omega) Im G^R(\vec{k},\omega)$$

- Strange metal: very broad peaks

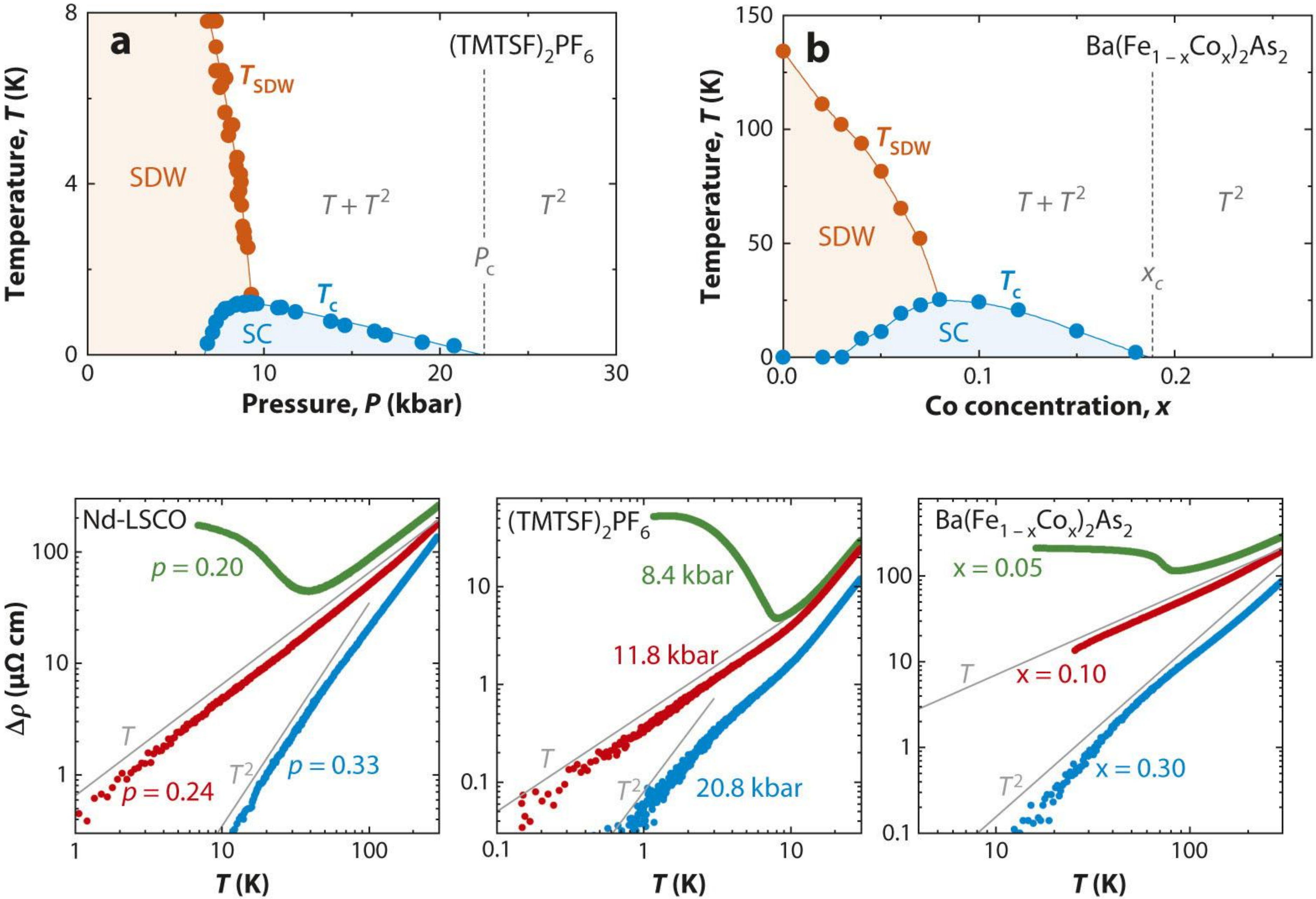
$$\Gamma \sim A + B max(\omega, T)$$

- Fermi-liquid to strange metal crossover

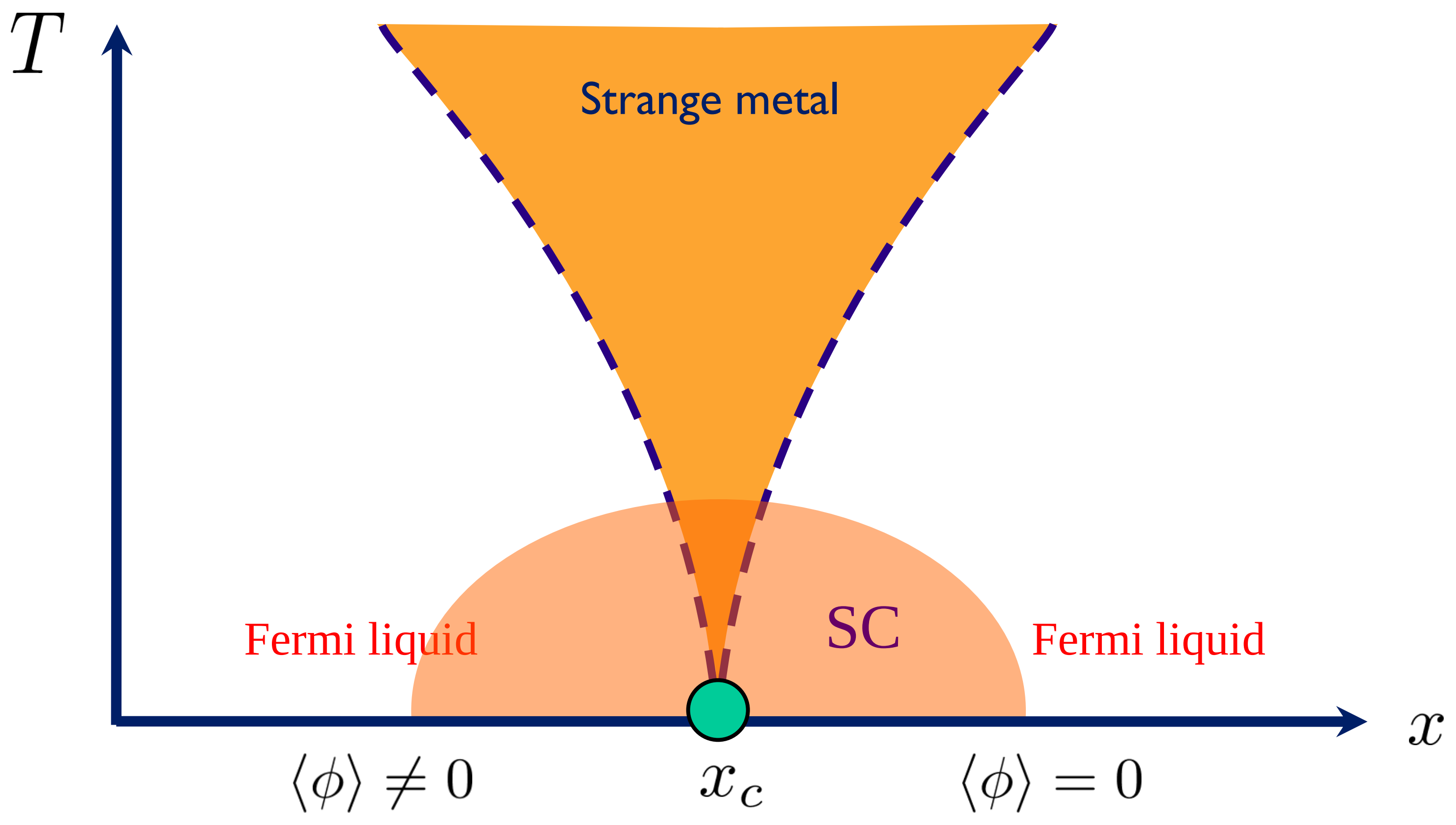
$$Z \to 0$$



A ubiquitous phase diagram



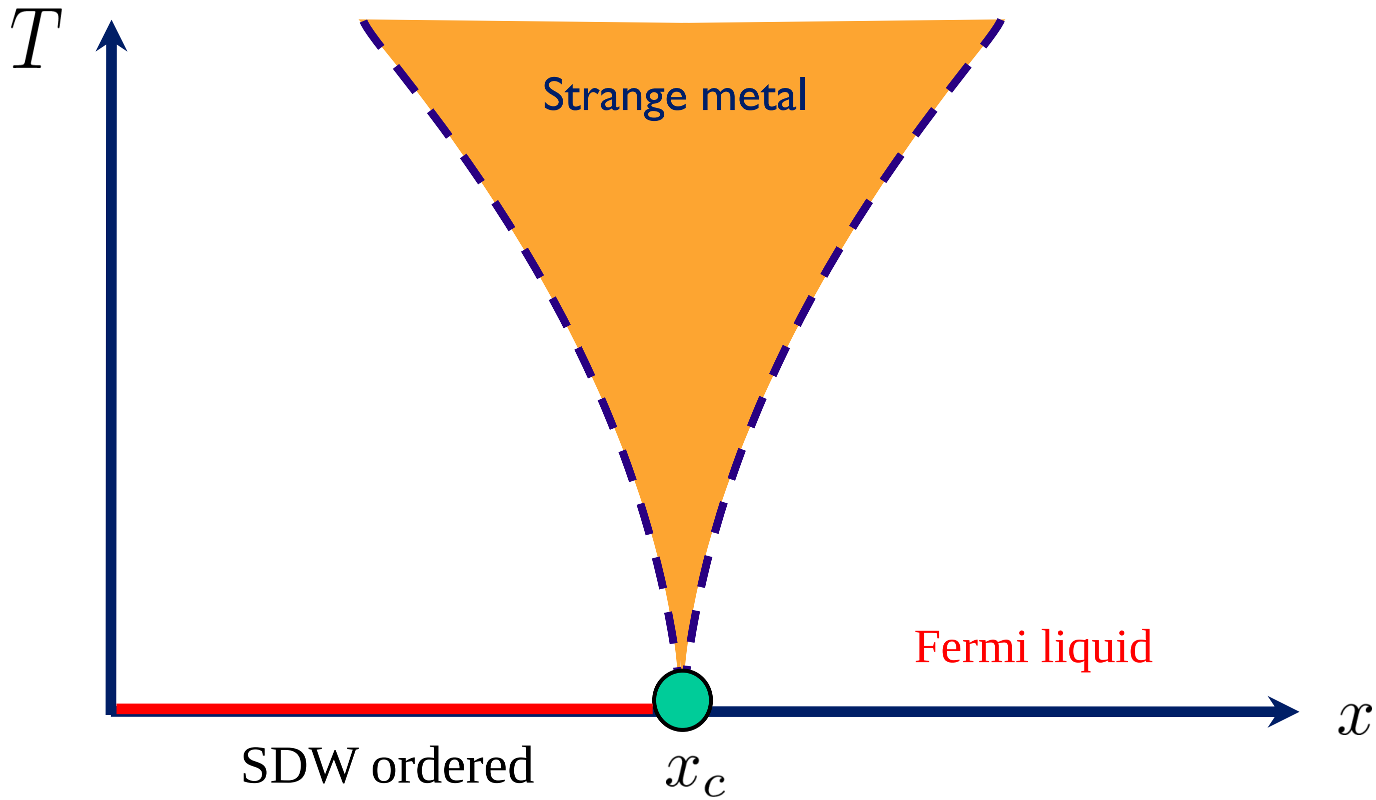
QCP scenario



What order onsets at QCP?

- Antiferromagnetic spin-density wave (SDW)
  - very natural for electron-doped cuprates, pnictides, organics
  - in hole-doped cuprates might be realized via the competing orders scenario

E. Demler, S. Sachdev and Y. Zhang, PRL (2001);
S. Sachdev, Physica Status Solidi B (2010)



What order onsets at QCP?

- Another candidate for QCP (cuprates/pnictides) – nematic order

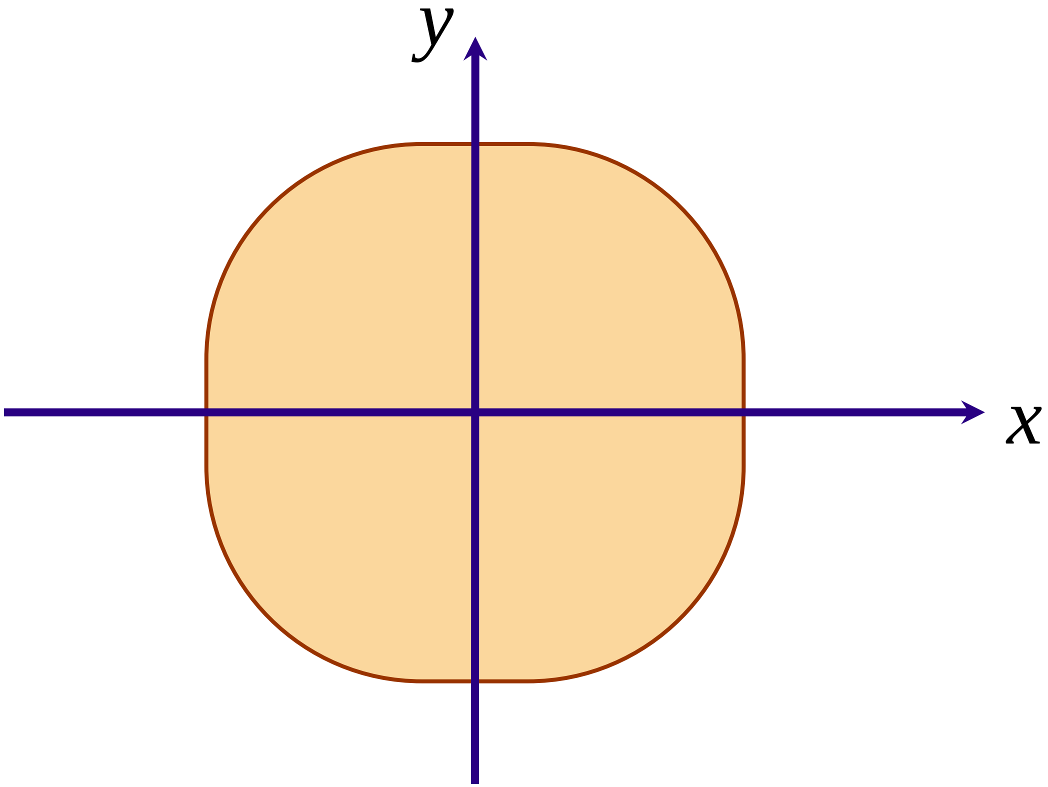
  S. A. Kivelson, E. Fradkin, and V. J. Emery, Nature (1998).

- Breaking of point-group (rotation) symmetry of the lattice
- Translational symmetry preserved
- Work in 2D, square lattice
- Introduce an Ising order parameter ϕ

  under 90 degree rotations, $R_{\pi/2} : \phi \to -\phi$

- Transition out of a metallic state (Pomeranchuk instability)

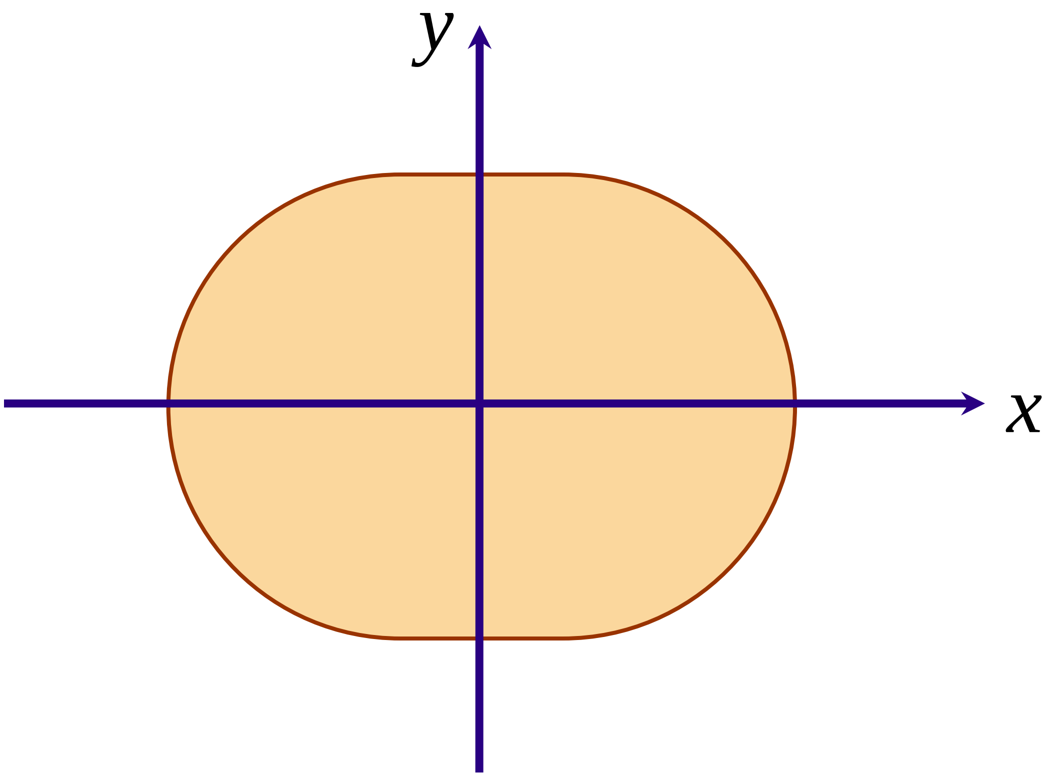
Nematic transition



Fermi surface with full square lattice symmetry

$$\langle \phi \rangle = 0$$

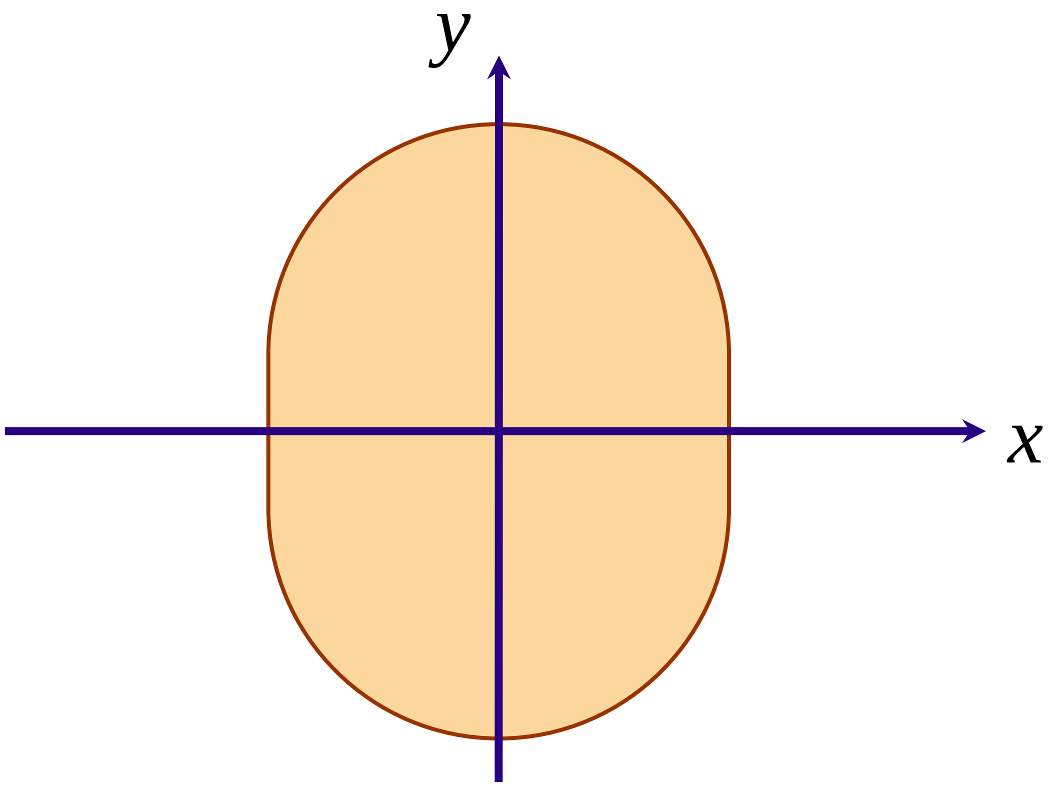
Nematic state



Spontaneous elongation along the x direction

$d_{x^2-y^2}$ Ising-nematic order parameter $\langle \phi \rangle > 0$

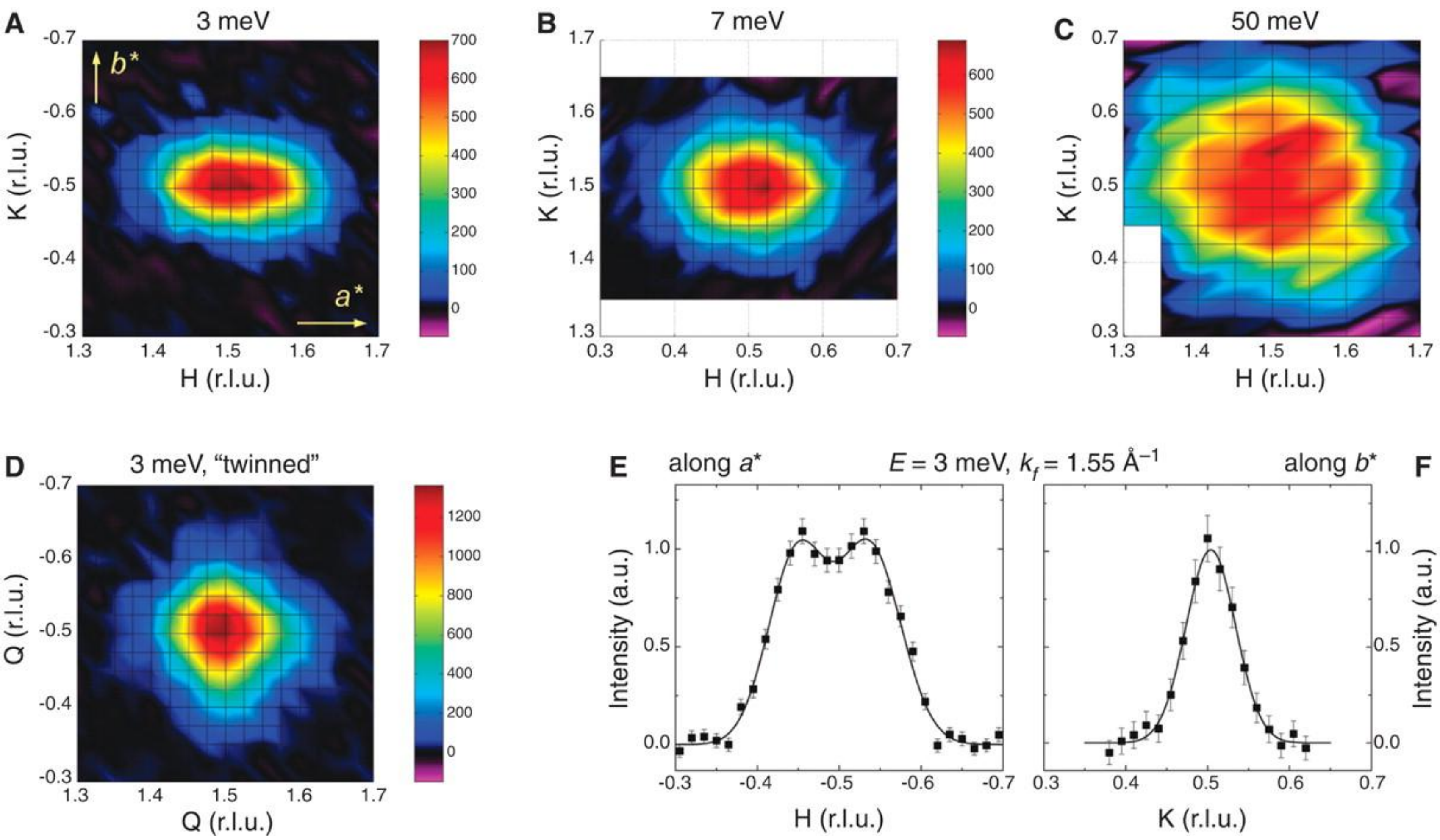
Nematic state



Spontaneous elongation along the y direction

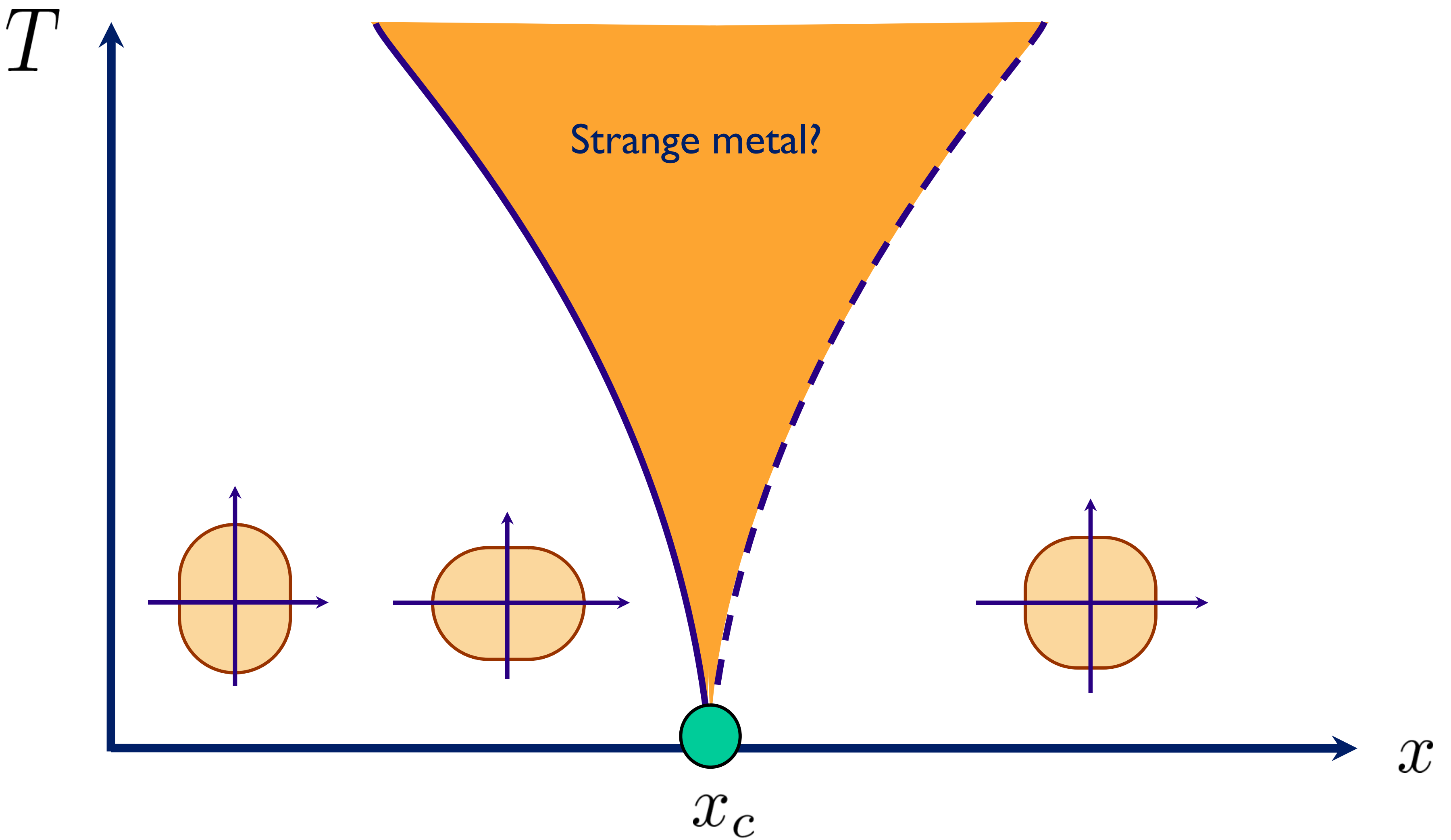
$d_{x^2-y^2}$ Ising-nematic order parameter $\langle \phi \rangle < 0$

Nematic order in YBCO revealed by neutron scattering

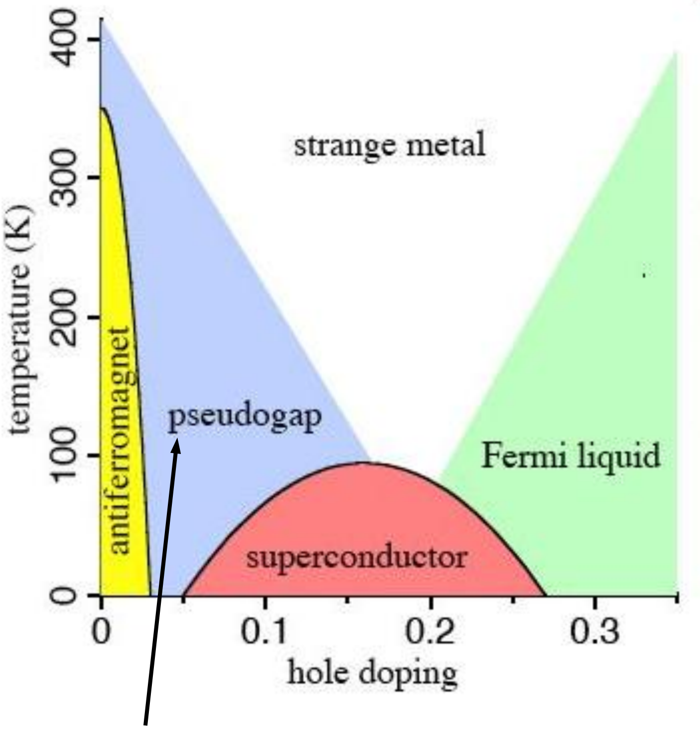


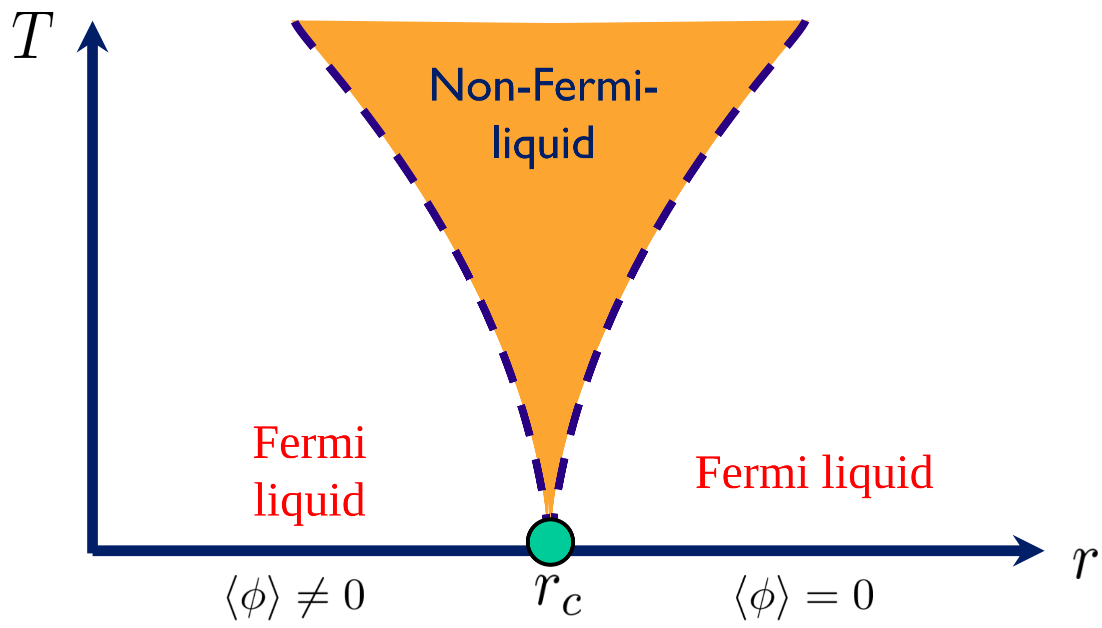
V. Hinkov, D. Haug, B. Fauqué, P. Bourges, Y. Sidis, A. Ivanov, C. Bernhard, C. T. Lin, and B. Keimer , Science 319, 597 (2008)

Nemetic transition



Caveats!





- Pseudogap is not a Fermi-liquid either!
  - proximity to the Mott insulator
  - more drastic changes in the Fermi-surface
- Similar considerations for heavy-fermion compounds.
- Simple metallic QCP scenario on better footing for less strongly correlated systems like electron-doped cuprates, pnictides and organics.

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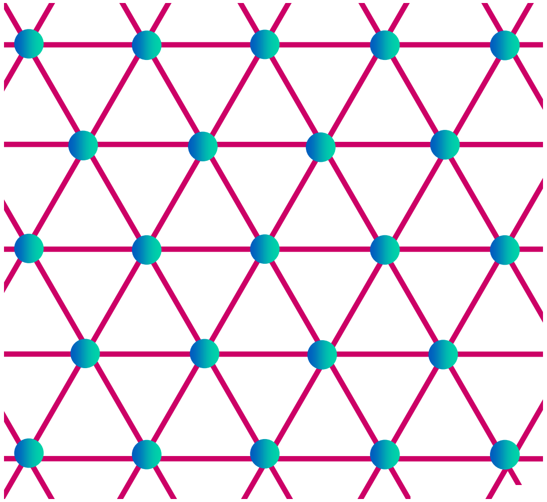
Spin-liquids



- Mott insulators with one electron per unit cell and no symmetry breaking.

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + \ldots$$

- Fractionalize

$$S_i^a = \frac{1}{2} f_{i\alpha}^{\dagger} \sigma_{\alpha\beta}^a f_{i\beta}$$

- $f_\alpha \quad - \quad S = 1/2$ fermionic quasiparticles (spinons)
- U(1) gauge invariance:

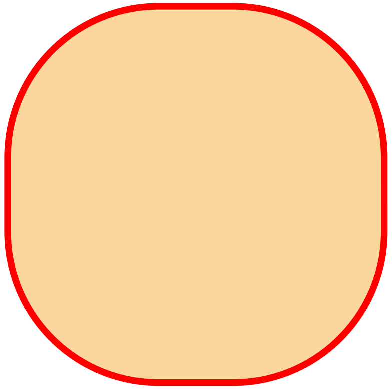
$$f_i(\tau) \to e^{i\varphi_i(\tau)} f_i(\tau)$$

- The low-energy description involves an emergent U(1) gauge field a_μ .

Spinon Fermi-surface

$$S_i^a = \frac{1}{2} f_{i\alpha}^\dagger \sigma_{\alpha\beta}^a f_{i\beta}$$

- Imagine that the spinons form a Fermi-surface



- Fluctuations of the emergent gauge field make the spinon Fermi-surface critical.

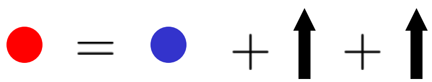
- Candidate for the "spin-liquid" state observed by exact diagonalization of triangular lattice Hubburd model at intermediate U/t.

- Might be relevant for a quasi-2d organic material $EtMe_3Sb[Pd(dmit)_2]_2$.

Half-filled Landau level

B. I. Halperin, P. A. Lee, N. Read (1993)

- Form composite fermions by attaching two flux-quanta to each electron



- Composite fermions see no net magnetic field and form a Fermi-surface
- A U(1) Chern-Simons gauge field is used to attach flux:

$$L = f^\dagger(\partial_\tau - i a_\tau) f + \frac{1}{2m} |(\nabla + i\frac{e}{c}\vec{A} - i\vec{a})|^2 f + \frac{i\nu}{4\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

$$\nabla \times \vec{a} = 2(2\pi) f^\dagger f$$

- Fluctuations of a_μ make the Fermi-surface of the composite fermion critical.

Candidate critical Fermi surface states

- Phase transitions in metals

- Spinon Fermi-surface state of Mott-insulators

- Composite-fermion liquid of QHE system

- All three states involve a Fermi surface interacting with a gapless boson

- Difficult problem, due in part to an absence of a full RG program.

- “Solved” in $N \to \infty$ limit in early 90’s

  Declared open again after work of S. S. Lee (2009).

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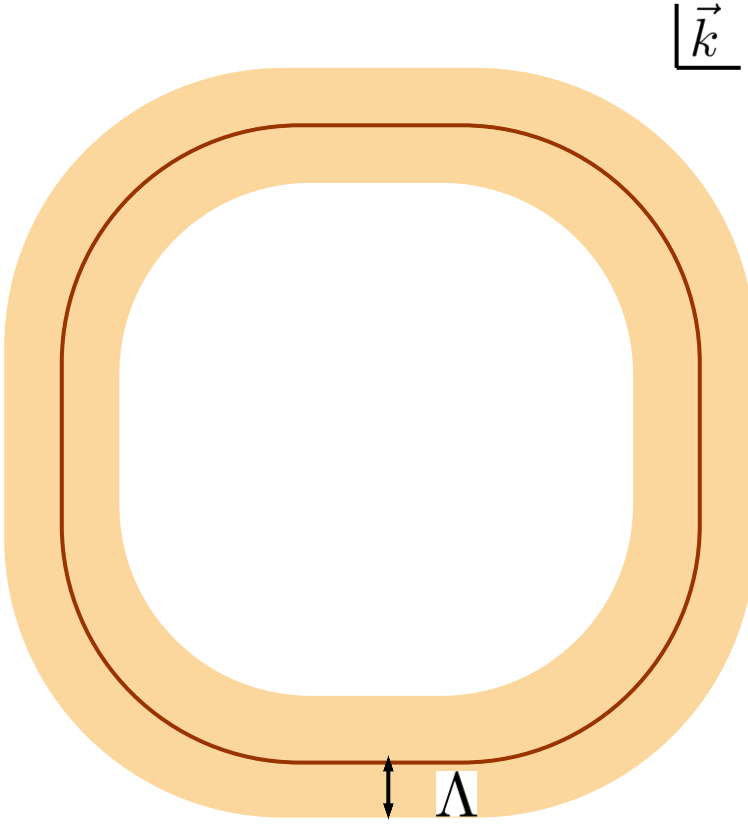
RG treatment of a Fermi-liquid

$$S_2 = \int \frac{d^2\vec{k}d\omega}{(2\pi)^3}\psi^\dagger_\alpha(\vec{k},\omega)(-i\omega + v^*(\hat{k})k)\psi_\alpha(\vec{k},\omega)$$

k - distance to the Fermi-surface

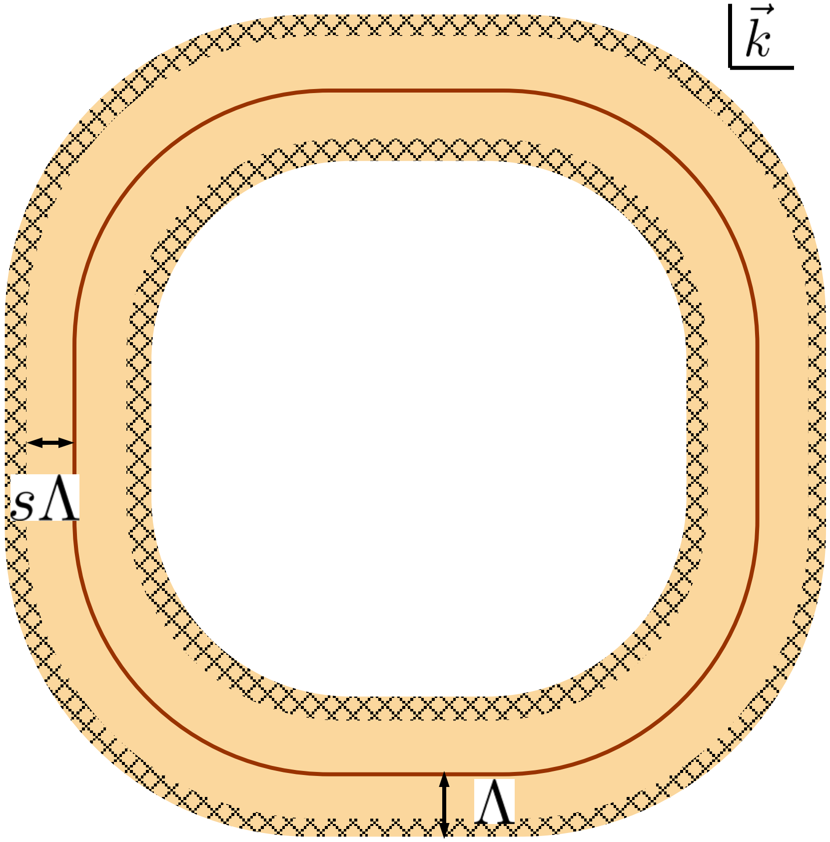


$\Lambda \ll k_F$

R. Shankar, Physica A (1991); J. Polchinski, (1992).

RG treatment of a Fermi-liquid

$$S_2 = \int \frac{d^2\vec{k}d\omega}{(2\pi)^3}\psi^\dagger_\alpha(\vec{k},\omega)(-i\omega + v^*(\hat{k})k)\psi_\alpha(\vec{k},\omega)$$



Shrink cut-off: $\Lambda \to s\Lambda$

Rescale fields and momenta:

$$\psi(k,\omega,\hat{k}) \to s^{-3/2}\psi'(k/s,\omega/s,\hat{k})$$

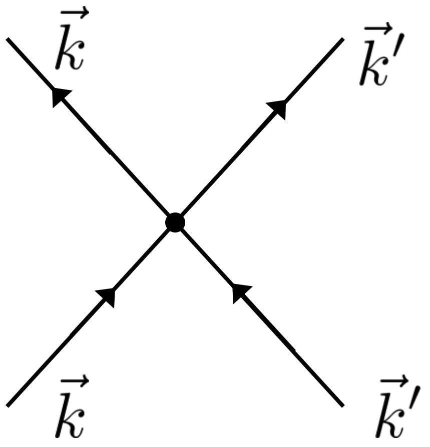
Leaves S_2 invariant.

R. Shankar, Physica A (1991); J. Polchinski, (1992).

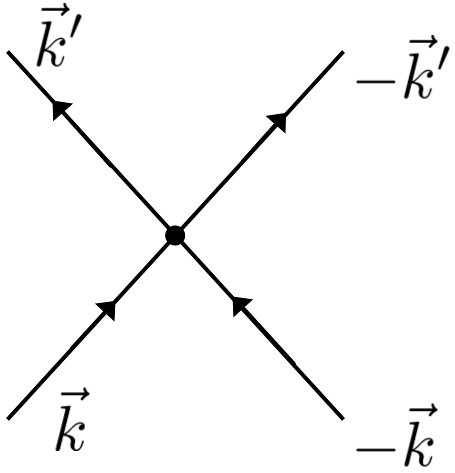
Perturbations

$$S_4 = -\frac{1}{4}\int\prod_{i=1}^{4}\frac{d^3k_i}{(2\pi)^3}U_{\alpha\beta;\gamma\delta}(\hat{k}_1,\hat{k}_2;\hat{k}_3,\hat{k}_4)\psi^\dagger_\alpha(k_1)\psi^\dagger_\beta(k_2)\psi_\gamma(k_3)\psi_\delta(k_4)$$
$$\times(2\pi)^3\delta^3(k_1+k_2-k_3-k_4)$$

- Only two types of momentum conserving processes keep fermions on the FS



Forward-scattering



BCS scattering

Perturbations

$$S_4 = -\frac{1}{4}\int \prod_{i=1}^{4} \frac{d^3k_i}{(2\pi)^3} U_{\alpha\beta;\gamma\delta}(\hat{k}_1, \hat{k}_2; \hat{k}_3, \hat{k}_4)\psi^\dagger_\alpha(k_1)\psi^\dagger_\beta(k_2)\psi_\gamma(k_3)\psi_\delta(k_4)$$
$$\times(2\pi)^3\delta^3(k_1 + k_2 - k_3 - k_4)$$

$$U = U^{FS} + U^{BCS}$$

$$U^{FS}_{\alpha\beta;\gamma\delta}(\hat{k}, \hat{k}'; \hat{k}, \hat{k}') = \delta_{\alpha\gamma}\delta_{\beta\delta}F^c(\hat{k}, \hat{k}') + (2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta})F^s(\hat{k}, \hat{k}')$$

$$U^{BCS}_{\alpha\beta;\gamma\delta}(\hat{k}, -\hat{k}; \hat{k}', -\hat{k}') = (\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})V^s(\hat{k}, \hat{k}')$$
$$+ (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})V^a(\hat{k}, \hat{k}')$$

RG flow

- Both forward and BCS scattering interactions are marginal at tree level
- Flow equations simplify for a rotationally invariant system

$$F^{c,s}(\theta,\theta') = \sum_m F_m^{c,s} e^{im(\theta-\theta')}; \qquad V^{c,s}(\theta,\theta') = \sum_m V_m^{c,s} e^{im(\theta-\theta')}$$

- Flow equations:

$$\frac{dF_m}{d\ell} = 0$$

forward-scattering – exactly marginal

$$\frac{dV_m}{d\ell} = -N(0)V_m^2 \qquad V_m(\ell) = \frac{V_m}{1 + N(0)V_m \ell}$$

- Repulsive BCS interaction – marginally irrelevant

Attractive BCS interaction – marginally relevant; $\Delta \sim \Lambda \exp\left(-\frac{1}{N(0)V}\right)$

R. Shankar, Physica A (1991); J. Polchinski, (1992).

Fixed-point theory

- The fixed point theory is exactly solvable
- Fermion Green's function unmodified from the free-form

$$G(\vec{k},\omega) = \frac{1}{i\omega - vk}$$

- Quasiparticles infinitely long-lived in the fixed point theory
- Quasiparticle life-time given by irrelevant operators

$$\Gamma \sim \omega^2 \log(\Lambda/\omega), \quad \Gamma \ll \omega$$

(Simple Shankar RG might not be enough to understand corrections to scaling!)

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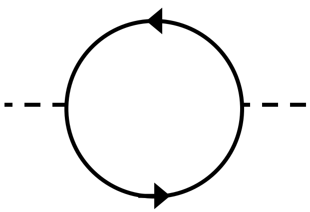
Theory of the critical Fermi surface

- Theory of a gapless boson field interacting with the Fermi surface
- U(1) spin-liquid:

$$L = f^\dagger(\partial_\tau - i a_\tau + \epsilon(-i\vec{\nabla} - \vec{a}))f + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda)^2$$

- Integrate the fermions out at the RPA level (work in $\nabla \cdot \vec{a} = 0$ gauge):



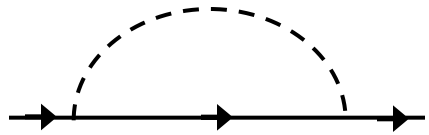
$$\delta S = \frac{1}{2} a_\mu \Pi_{\mu\nu} a_\nu$$

$$\Pi_{\tau\tau}(\omega = 0, \vec{q} \to 0) = N(0)$$ - Debye screened

$$\Pi_{ij}(\omega, \vec{q}) = \left(\gamma(\hat{q})\frac{|\omega|}{|\vec{q}|} + C\vec{q}^2\right)\left(\delta_{ij} - \frac{q_i q_j}{\vec{q}^2}\right)$$ - Landau damped

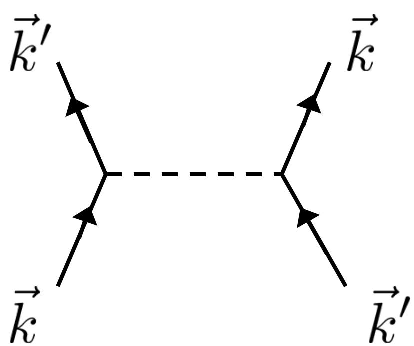
- Magnetic fluctuations are very soft: $\omega \sim |\vec{q}|^z$ $z = 3$

Feedback on the fermions

$\Sigma(\omega, \vec{k}) =$  $= -ic_f sgn(\omega)|\omega|^{2/3}$

$$G_s^{-1}(\omega, \vec{k}) = -\cancel{i\omega} - ic_f sgn(\omega)|\omega|^{2/3} + v_F(|k| - k_F)$$

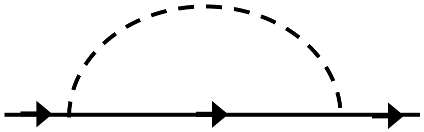
- Non Fermi-liquid!
- Reason: singular forward scattering at small angle

$F(\theta, \theta') =$ 

$\sim \frac{1}{|\vec{k} - \vec{k}'|^2} \sim \frac{1}{(\theta - \theta')^2}$

P. A. Lee, N. Nagaosa (1992); J. Polchinski (1993)

A note of caution!

$\Sigma(\omega, \vec{k}) =$  $= -ic_f sgn(\omega)|\omega|^{2/3}$

• As we will see below, higher loop corrections do lead to momentum dependence.

• Even at one loop order, only correct for $|\omega| \gg |k|^3$

- ok, as $|\omega| \sim |k|^{3/2}$

$$G_s^{-1}(\omega, \vec{k}) = -ic_f sgn(\omega)|\omega|^{2/3} + v_F|k|$$

• For $|\omega| \ll |k|^3$ $\quad \Sigma(\omega, k) \sim -i\frac{\omega}{|k|} + i\frac{\omega^2 sgn(\omega)}{|k|^4}$

B. I. Halperin, P. A. Lee, N. Read (1993)

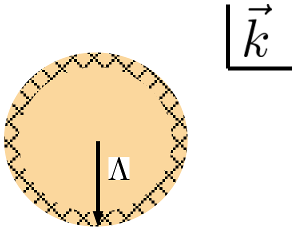
How to scale?

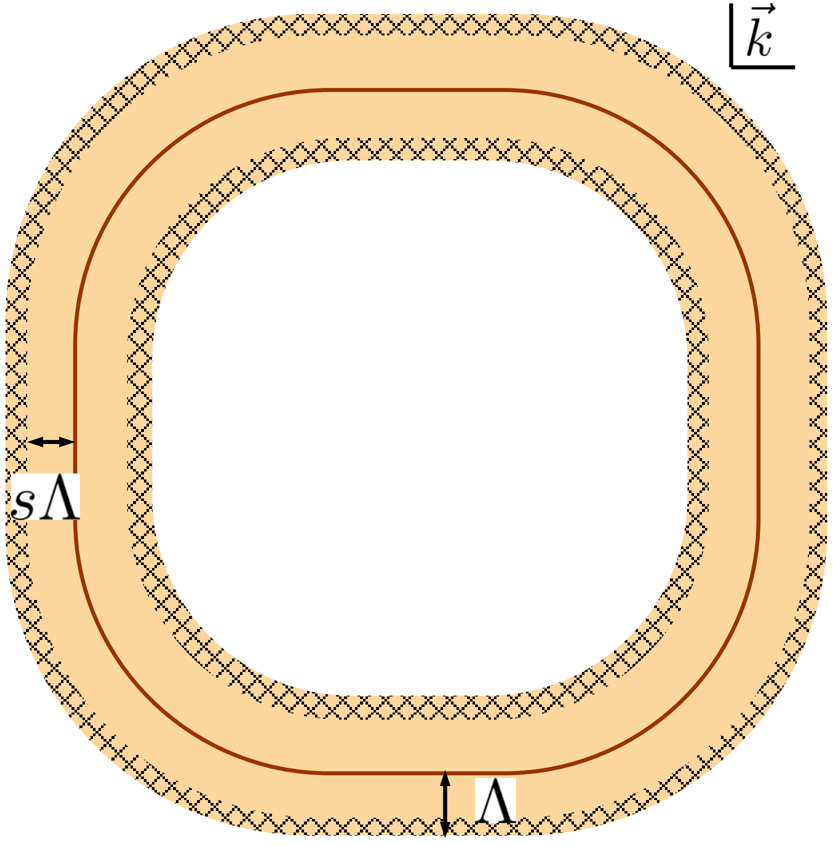
Gauge field: $\vec{a}(\vec{k},\omega)$

$$\vec{k} \to s\vec{k}$$

Fermions $\psi(k,\hat{k},\omega)$

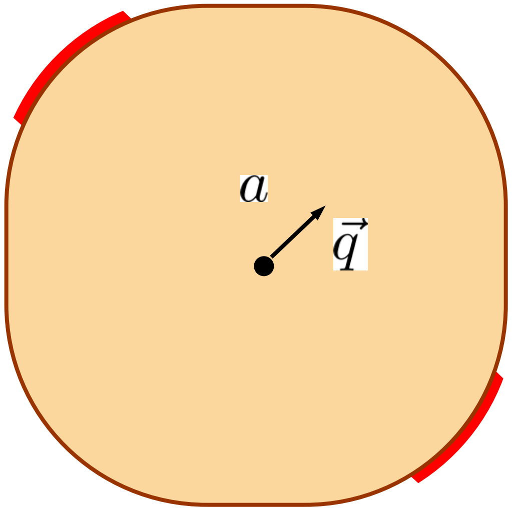
$$k \to sk,\ \ \hat{k} \to \hat{k}$$





Two-patch regime

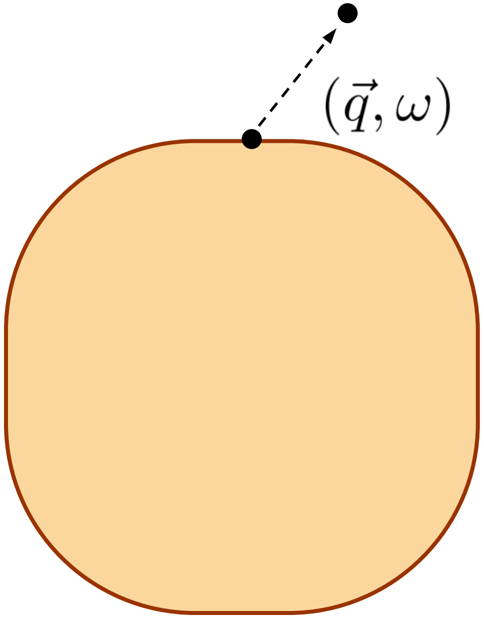
- Most singular kinematic regime: two-patch



J. Polchinski (1993); B. Altshuler, L. Ioffe, A. Millis (1994).

Why two-patch regime?

• Order parameter fluctuations are very soft:



$$\omega \sim |\vec{q}|^3$$

$$\Delta\epsilon \sim v_F|\vec{q}| \sim \omega^{1/3}$$

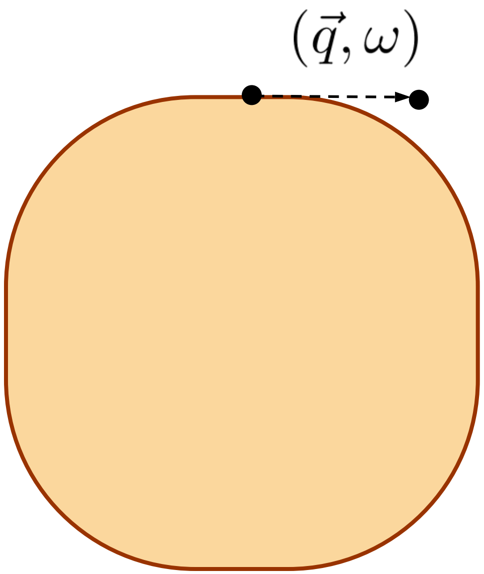
$$\Sigma(\omega) \sim \omega^{2/3}$$

• Cannot effectively absorb a Landau-damped bosonic mode.

B. Altshuler, L. Ioffe, A. Millis (1994).

Why two-patch regime?

• Order parameter fluctuations are very soft:



$$\omega \sim |\vec{q}|^3$$

$$\Delta\epsilon \sim \frac{v_F |\vec{q}|^2}{2k_F} \sim \omega^{2/3}$$

$$\Sigma(\omega) \sim \omega^{2/3}$$

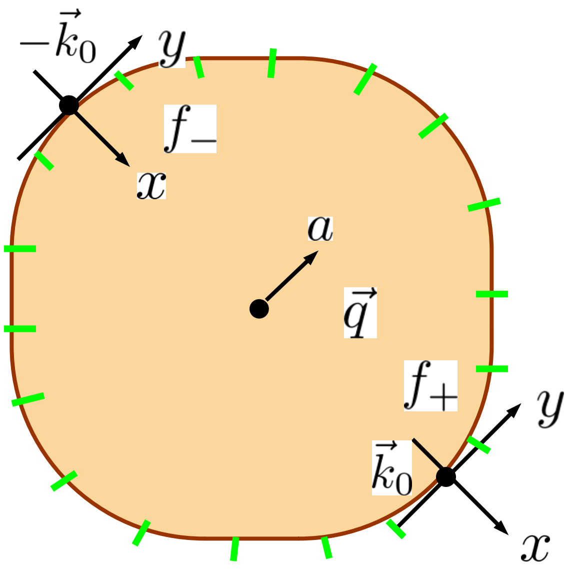
• Cannot effectively absorb a Landau-damped bosonic mode unless its momentum is nearly tangent to the Fermi surface.

• This "conspiracy" between fermions and bosons is special to 2+1 dimensions.

B. Altshuler, L. Ioffe, A. Millis (1994).

Two-patch theory

- For each $\hat{q}$ expand the fermion fields about two opposite points on the Fermi surface, $\vec{k}_0$ and $-\vec{k}_0$.



$$L_f = f^\dagger_{+\sigma}\left(\partial_\tau + v_F(-i\partial_x - \frac{\partial_y^2}{2K})\right) f_{+\sigma} + f^\dagger_{-\sigma}\left(\partial_\tau + v_F(i\partial_x - \frac{\partial_y^2}{2K})\right) f_{-\sigma}$$

$$L_a = \frac{1}{2e^2}(\partial_y a)^2, \qquad a_i(\vec{q},\omega) = \epsilon_{ij}\frac{q_j}{|\vec{q}|}a(\vec{q},\omega)$$

$$L_{int} = v_F a(f^\dagger_{+\sigma}f_{+\sigma} - f^\dagger_{-\sigma}f_{-\sigma})$$

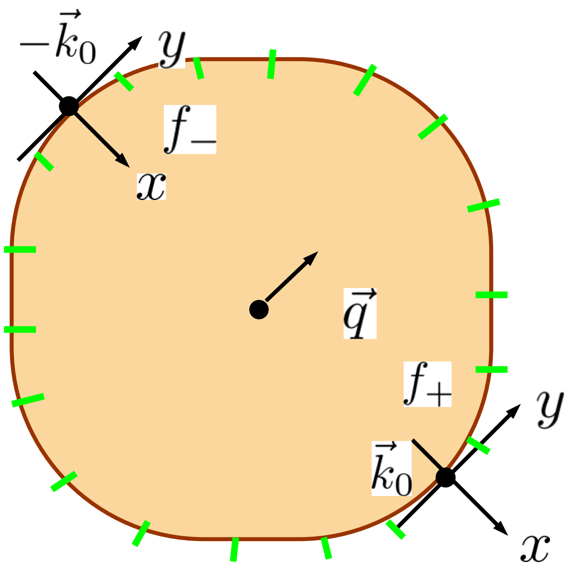
- Crucial to keep the Fermi-surface curvature radius K.

J. Polchinski (1993); S. S. Lee (2009)

Two-patch theory

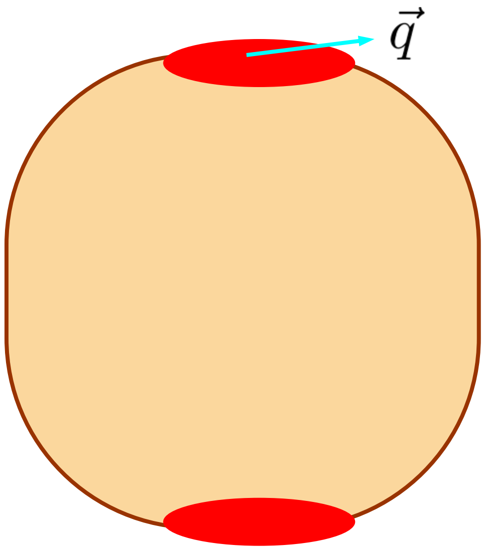
$$S = \int d\tau dx dy \; L$$

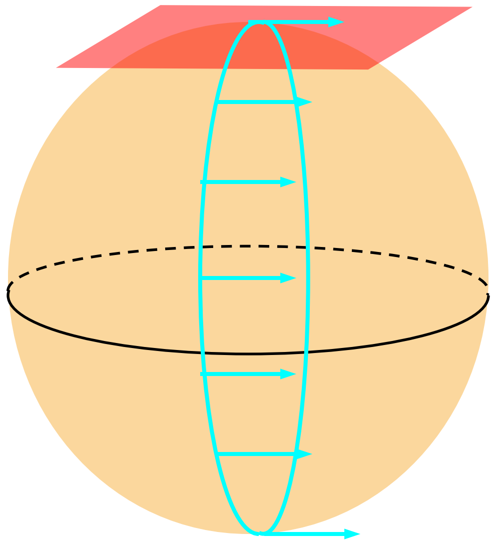
- Have an infinite set of 2+1 dimensional theories labeled by points on the Fermi-surface ($\hat{q}$)



- Extra-dimension $\hat{q}$
- Key assumption: can neglect coupling between patches.

Two dimensions are unique





Strongly coupled local field theory

Bosons and fermions enter on the same footing

No local description

Effective theory weakly coupled

Anisotropic scaling

$$L = \sum_s f_s^\dagger (\partial_\tau + v_F(-is\partial_x - \frac{\partial_y^2}{2K}))f_s + v_F a \sum_s s f_s^\dagger f_s + \frac{1}{2e^2}(\partial_y a)^2$$

- Fermion kinetic term dictates:

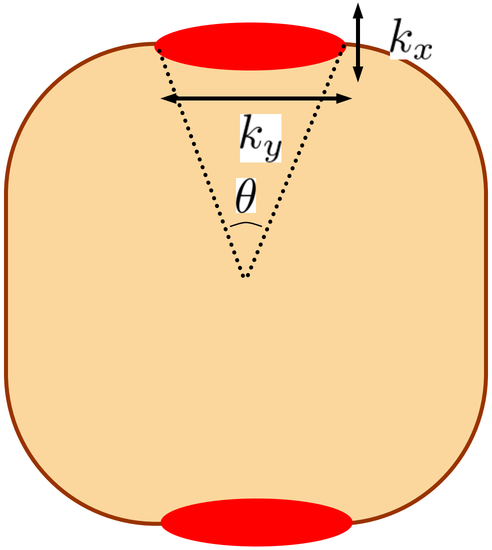
$$k_y \to s k_y, \quad k_x \to s^2 k_x$$

J. Polchinski (1993); B. Altshuler, L. Ioffe, A. Millis (1994).

Two-patch scaling

<u>Critical Fermi surface</u>

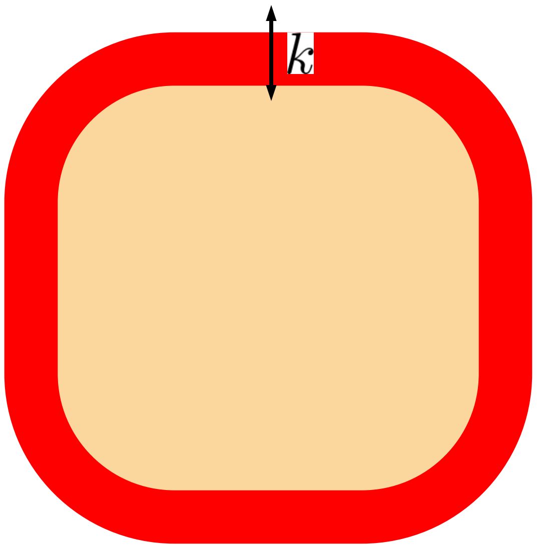
$$k_y \to s k_y, \quad k_x \to s^2 k_x$$



$$\theta \to s\theta$$

<u>Fermi-liquid</u>

$$k \to sk$$



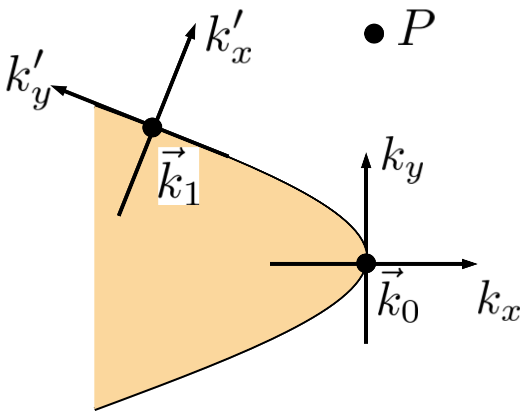
θ does not flow

J. Polchinski (1993); B. Altshuler, L. Ioffe, A. Millis (1994), S. S. Lee (2008).

Shift symmetry

$$L = \sum_s f_s^\dagger (\partial_\tau + v_F(-is\partial_x - \frac{\partial_y^2}{2K}))f_s + v_F a \sum_s s f_s^\dagger f_s + \frac{1}{2e^2}(\partial_y a)^2$$

- What if we expand about a different point on the Fermi surface?



$$\vec{k}_0 = (0,0) \quad \vec{k}_1 = (\kappa_x, \kappa_y)$$

Rotate coordinates by $\theta = \frac{\kappa_y}{K}$

$$\begin{aligned} k_x' &= k_x - \kappa_x + \theta(k_y - \kappa_y) \\ k_y' &= k_y - \kappa_y \end{aligned}$$

- Exact symmetry of the low-energy theory
- Formally akin to a Galilean transformation

Consequences of shift symmetry

$$D(\omega, q_x, q_y) = D(\omega, q_y)$$

$$G_s(\omega, k_x, k_y) = G(\omega, \underbrace{sk_x + \frac{k_y^2}{2K}}_{\text{distance to Fermi surface}})$$

- Fermi surface curvature K does not renormalize.
- Tree level scaling relation is exact!

$$k_y \to sk_y, \;\; k_x \to s^2 k_x$$

Dynamical scaling

$$L = \sum_s f_s^\dagger(\partial_\tau + v_F(-is\partial_x - \frac{\partial_y^2}{2K}))f_s + v_F a \sum_s s f_s^\dagger f_s + \frac{1}{2e^2}(\partial_y a)^2$$

- How to scale time?

$$k_y \to s k_y,\ k_x \to s^2 k_x,\ \tau \to s^z \tau$$

- Choose z to leave the gauge-fermion coupling invariant (marginal)

$$z = 3$$

- Fermion kinetic term is irrelevant under such scaling

- Define the theory via $\eta \to 0^+$ limit $f_s^\dagger \partial_\tau f_s \to \eta f_s^\dagger \partial_\tau f_s$

Problem

$$L = \sum_s f_s^\dagger (\eta \partial_\tau + (-is\partial_x - \partial_y^2)) f_s + a \sum_s s f_s^\dagger f_s + \frac{1}{2e^2} (\partial_y a)^2$$

- No expansion parameter

$$[e^2] = \frac{q_y^3}{\omega}$$ - dimensionfull

- Theory is strongly coupled

- Usual approach: large- N expansion

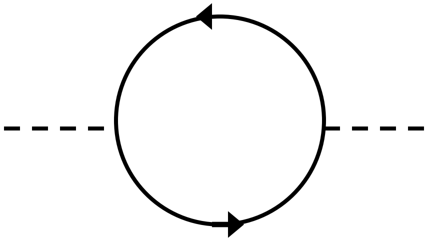
$$L = \sum_s f_s^\dagger (\eta \partial_\tau + (-is\partial_x - \partial_y^2)) f_s + a \sum_s s f_s^\dagger f_s + \frac{N}{2e^2} (\partial_y a)^2$$

$$S_{\text{eff}}[a] \sim N\Gamma[a]$$ - use saddle point approximation

- Actually, fails for this problem *S. S. Lee (2009)*

Sanity check: one loop results

$$\Pi^0(\omega, \vec{q}) =$$



$$= c_b N \frac{|\omega|}{|q_y|}$$

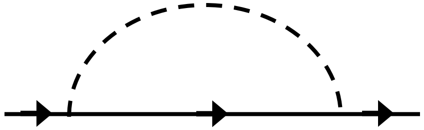
$$D^{-1}(\omega, q_y) = N\left(c_b \frac{|\omega|}{|q_y|} + \frac{q_y^2}{e^2}\right)$$ - Landau damping $z = 3$

- Landau damping comes from the two-patch regime

Sanity check: one loop results

- Fermion self-energy at criticality

$$\Sigma(\omega, \vec{k}) =$$



$$= -i\frac{c_f}{N} sgn(\omega)|\omega|^{2/3}$$

$$G_s^{-1}(\omega, \vec{k}) = -i\frac{c_f}{N} sgn(\omega)|\omega|^{2/3} + s k_x + k_y^2$$

- Respects the scaling

$$k_y \to s k_y,\ k_x \to s^2 k_x,\ \tau \to s^z \tau \qquad z = 3$$

Scaling properties

$$L = \sum_s f_s^\dagger (\eta \partial_\tau + (-is\partial_x - \partial_y^2)) f_s + a \sum_s s f_s^\dagger f_s + \frac{1}{2e^2} (\partial_y a)^2$$

- Shift symmetry + Ward-Identities constrain the RG properties severely
- Only two anomalous dimensions

η_f - fermion anomalous dimension

z - dynamical critical exponent

$$f = Z_f^{1/2} f_r, \quad e^2 = Z_e e_r^2$$

$$\eta_f = -\Lambda \frac{\partial}{\partial \Lambda} \log Z_f \qquad z - 3 = -\Lambda \frac{\partial}{d\Lambda} \log Z_e$$

Scaling forms: gauge field

- $$D^{-1}(\omega, \vec{q}) \sim |\vec{q}|^{z-1} f\left(\frac{|\omega|}{|\vec{q}|^{z}}\right)$$

- Simple Landau-damped frequency dependence consistent with scaling form

$$D^{-1}(\omega, \vec{q}) - D^{-1}(\omega = 0, \vec{q}) \sim \frac{|\omega|}{|\vec{q}|}$$

- Static behaviour

$$D^{-1}(\omega = 0, \vec{q}) \sim |\vec{q}|^{z-1}$$

Scaling forms: fermions

- $$G^{-1}(\omega, \vec{k}) \sim k^{1-\eta_f/2} g\left(\frac{|\omega|}{|k|^{z/2}}\right), \quad k = k_x + \frac{k_y^2}{2K}$$

- "Fermionic dynamical exponent" is half the "bosonic dynamical exponent"

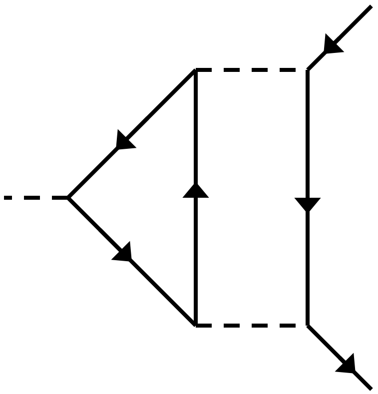
- Static behaviour: $G^{-1}(0, \vec{k}) \sim k^{1-\eta_f/2}$

- Dynamic behaviour: $G^{-1}(\omega, 0) \sim \omega^{(2-\eta_f)/z}$

M.M and S. Sachdev (2010)

Failure of large-N expansion

- Can we systematically compute anomalous dimensions in large-N limit?
- Large-N expansion fails at higher loops:



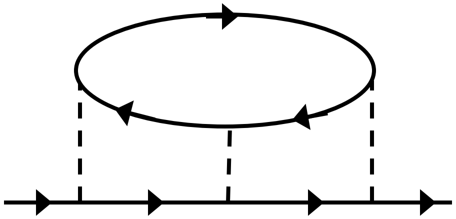
$$\sim \frac{1}{\eta\omega^{1/3}}$$

-infra-red divergence

$$G_0(\omega, \vec{k}) = \frac{1}{-i\eta\omega + k_x + k_y^2}$$

S. S. Lee (2009)

Failure of large-N expansion



$$\sim \frac{1}{\eta\omega^{1/3}} \times \omega^{2/3}$$

-infra-red divergence

$$G_0(\omega, \vec{k}) = \frac{1}{-i\eta\omega + k_x + k_y^2}$$

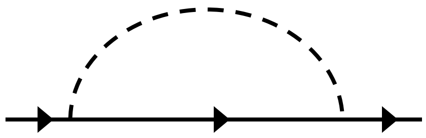
S. S. Lee (2009)

Failure of large-N expansion

- Wrong dynamical scaling of bare fermion Green's function

$$G_0(\omega, \vec{k}) = \frac{1}{-i\eta\omega + k_x + k_y^2}$$

- Solution: dress by one-loop self-energy

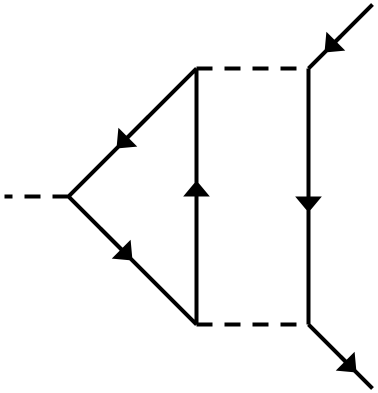


$$G_1(\omega, \vec{k}) = \frac{1}{-i\frac{c_f}{N} sgn(\omega)|\omega|^{2/3} + k_x + k_y^2}$$

- Traded small parameter $\eta \rightarrow \frac{1}{N}$

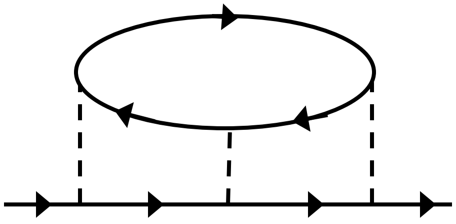
S. S. Lee (2009)

Violation of large-N counting



$$\sim \frac{1}{N\eta} \to O(1)$$

same as leading order!



$$\sim \frac{1}{N^2\eta} \to O\left(\frac{1}{N}\right)$$

S. S. Lee (2009)

Violation of large-N counting

$$G^{-1}(\omega, 0) = -i\eta\omega - i\frac{c_f}{N}|\omega|^{2/3} sgn(\omega)$$

- Crossover scale: $\Lambda \sim \left(\frac{c_f}{\eta N}\right)^3 \overset{N\to\infty}{\to} 0$

- Limits $N \to \infty$ and $\omega \to 0$ do not commute.

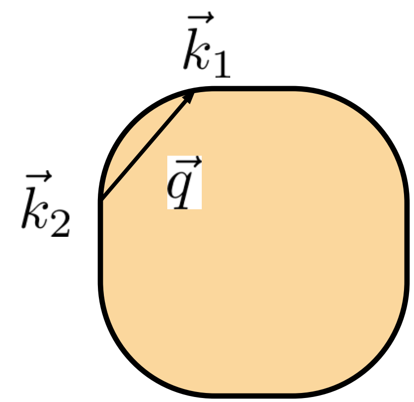
S. S. Lee (2009)

Genus expansion

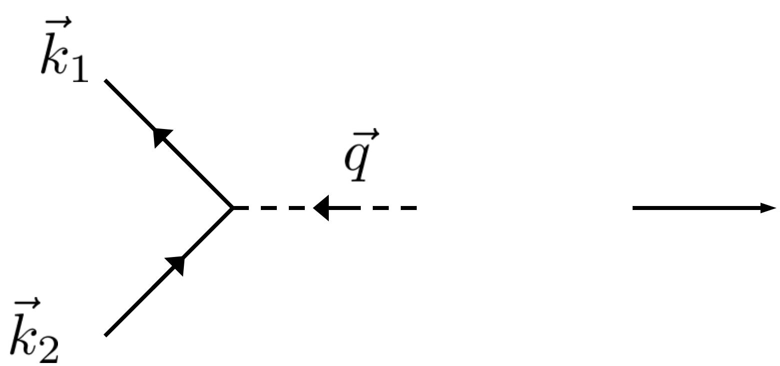
- A systematic way to count the power of N
- Where do extra powers of N come from?

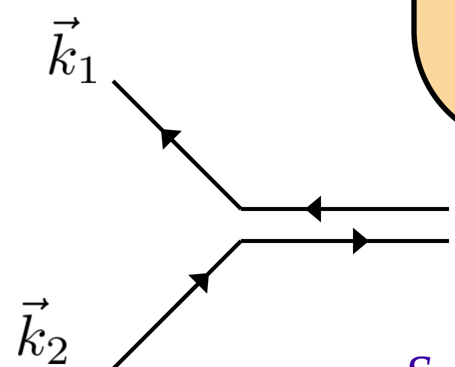
$$G_1(\omega, \vec{k}) = \frac{1}{-i\frac{c_f}{N} sgn(\omega)|\omega|^{2/3} + k_x + k_y^2}$$

- Need to find the phase space for all fermions to be on the Fermi-surface
- Double-line representation



$$\vec{q} = \vec{k}_1 - \vec{k}_2$$

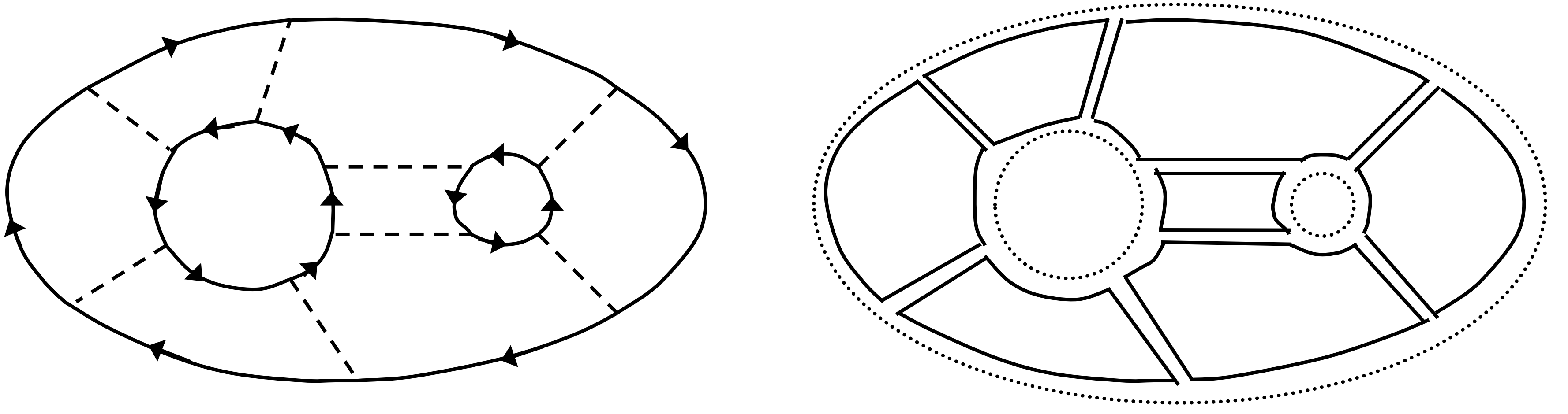




S. S. Lee (2009)

Genus expansion

- Go to double-line representation and classify diagrams by their topology



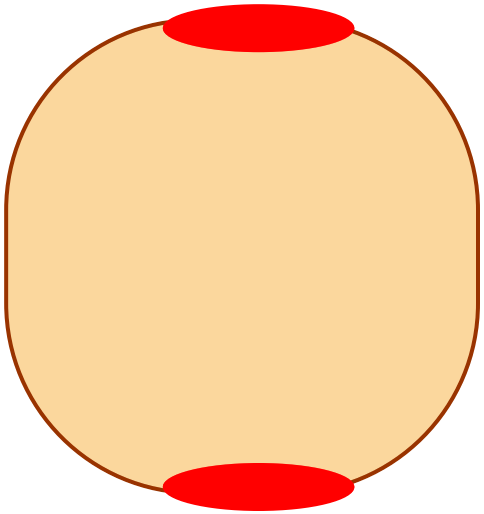
- Degree of a diagram in N is related to the genus of the surface on which it can be drawn

- At $N = \infty$ have to sum an infinite set of planar diagrams

S. S. Lee (2009)

One patch vs two patches

- For a theory with one patch all planar diagrams are finite due to kinematics



$$z = 3, \quad \eta_f = 0, \qquad N = \infty$$

S. S. Lee (2009)

- For a theory with two patches we find:

  i) divergences appear in planar graphs

  ii) large-N genus counting is violated

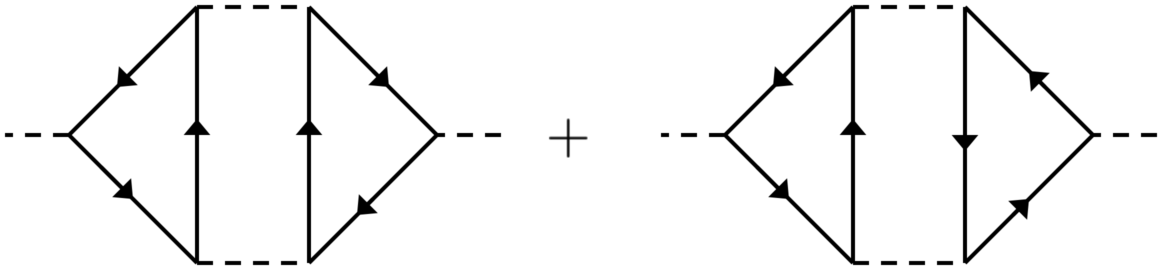
To three loop order:

$$z = 3, \quad \eta_f \neq 0$$

η_f is not suppressed for $N \to \infty$

Dynamical exponent at three loops

$$\delta^3\Pi(\omega = 0, q_y) =$$



Genus counting: planar $\sim O(N)$ non-planar $\sim O(1)$

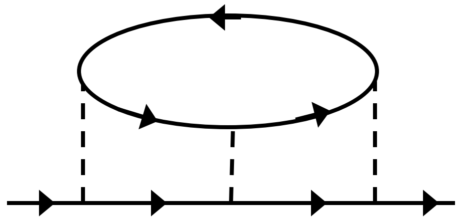
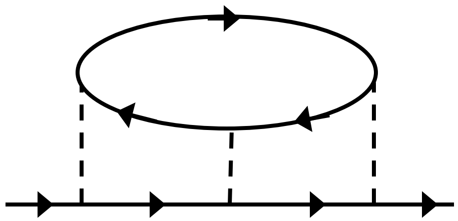
Actually: $\sim N q_y^{3/2}$ $\sim -N q_y^{3/2}$

Cancellation: $\sim N^{3/2} q_y^2$

- Divergences coming from outside of two-patch regime cancel!
- The contribution of the two-patch regime is finite. $\boxed{z = 3}$
- Violation of genus expansion.

J. Rech, C. Pepin and A. Chubukov (2006)

Fermion anomalous dimension at three loops

$$\delta^3\Sigma(\omega = 0, \vec{k}) =$$  $+$ 

Genus counting: planar non-planar

$$= (k_x + k_y^2)\log\left(\frac{\Lambda_y}{|k_x + k_y^2|^{1/2}}\right) \times \left[O(1) \quad + \quad O\left(\frac{1}{N^2}\log^3 N\right)\right]$$

$$\begin{aligned} \eta_f &= -0.06824, \quad N = 2 \\ \eta_f &= -0.10619, \quad N = \infty \end{aligned}$$

Remarks

Three loops: $z = 3$

$$\begin{aligned} \eta_f &= -0.06824, \quad N = 2 \\ \eta_f &= -0.10619, \quad N = \infty \end{aligned}$$

- Fermion anomalous dimension is not suppressed for large-N
- Anomalous dimension numerically small
- Is $z = 3$ to all orders?
- Further diagrams with singular contributions from outside two patch region? Do these always cancel?
- Does a sensible large N limit exist?

Extension: nematic transition

- $\phi -$ nematic order parameter

$$L_\psi = f^\dagger_{+\sigma}\left(\partial_\tau - iv_F\partial_x - \frac{1}{2K}\partial_y^2\right)f_{+\sigma} + f^\dagger_{-\sigma}\left(\partial_\tau + iv_F\partial_x - \frac{1}{2K}\partial_y^2\right)f_{-\sigma}$$

$$L_\phi = \frac{1}{2}(\partial_y\phi)^2 + \frac{r}{2}\phi^2$$

To three loops:

$\eta_f \rightarrow -\eta_f$

$$L_{int} = d_{k_0}\,\phi(f^\dagger_{+\sigma}f_{+\sigma} + f^\dagger_{-\sigma}f_{-\sigma})$$

Conclusion I

• Boson coupled to a Fermi surface is a strongly coupled problem

• Some exact statements about scaling forms can be made assuming the two-patch field theory

• Future directions

- explore $d > 2$ in more detail
- explore elements of two-patch physics in Fermi-liquids more

• Next time:

- A better controlled deformation of the problem
- Pairing instability and physics beyond the two-patch theory

To be continued…