PERIMETER INSTITUTE
FOR THEORETICAL PHYSICS

Holographic Entanglement Entropy ^ And Renyi Entropy

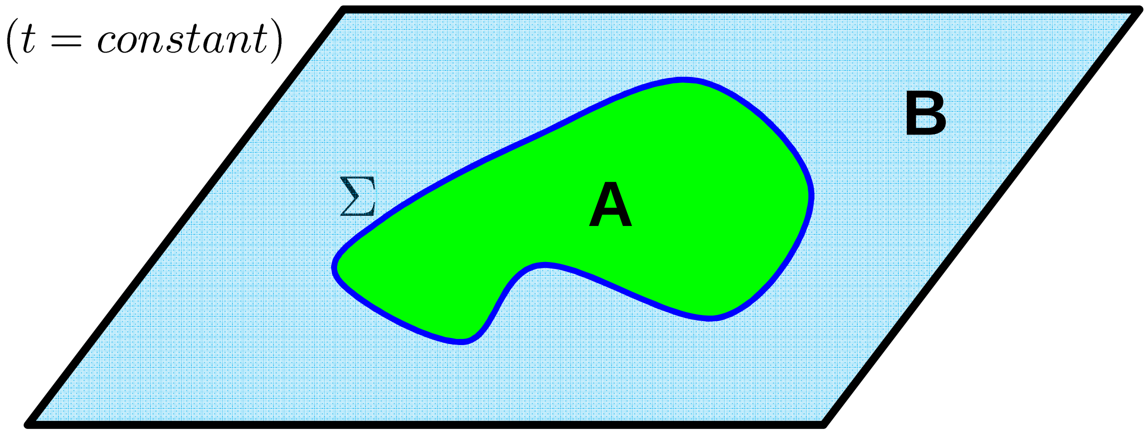
(with H. Casini, M. Huerta, J. Hung, M. Smolkin & A. Yale)

(arXiv:1102.0440, arXiv:1110.1084)

Entanglement Entropy

- what is entanglement entropy?
  general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions
- integrate out degrees of freedom in "outside" region
- remaining dof are described by a density matrix ρ_A

⟶ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



Entanglement Entropy

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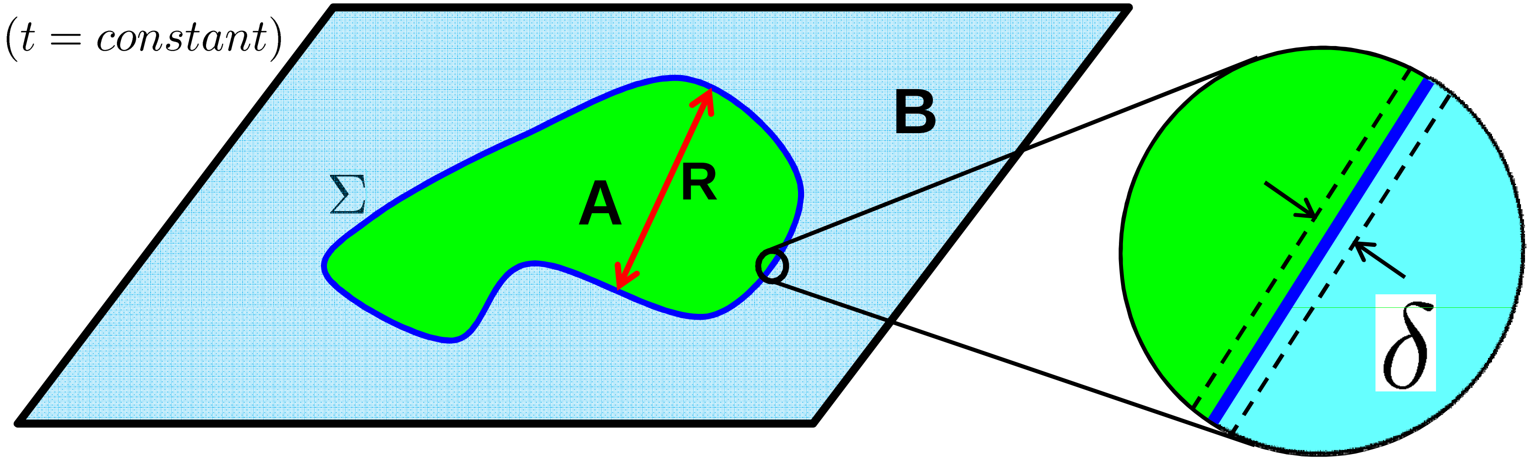

- result is UV divergent!
- must regulate calculation: δ = short-distance cut-off

$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots$$

d = spacetime dimension

- careful analysis reveals geometric structure, eg, $S = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

Entanglement Entropy

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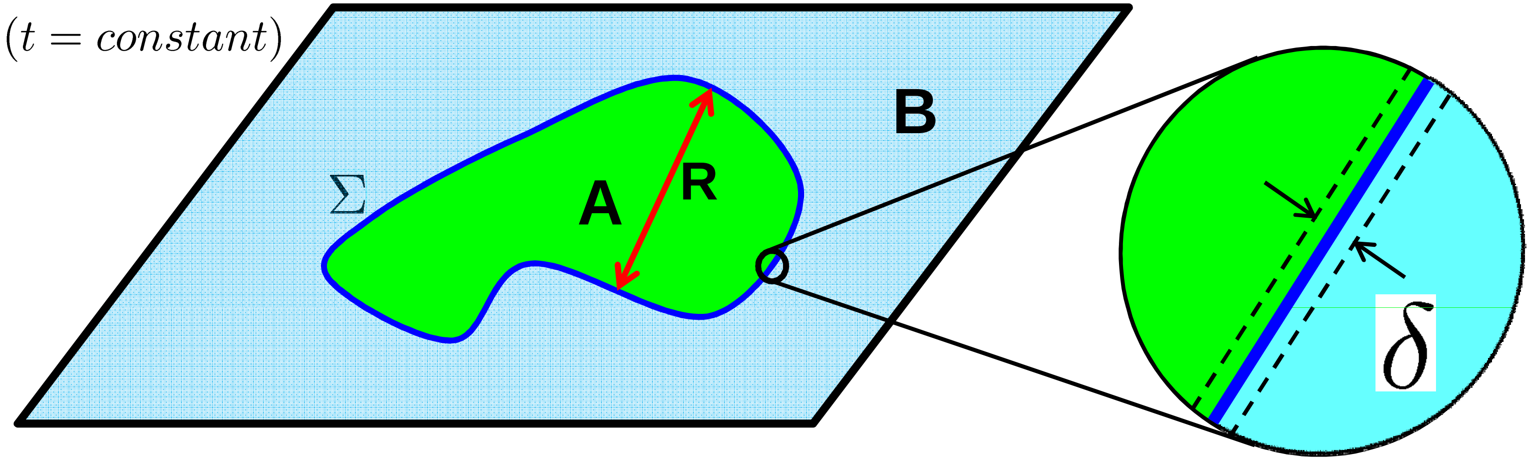
→ calculate von Neumann entropy: $S_{EE} = -Tr\,[\rho_A\,\log\rho_A]$



- must regulate calculation: δ = short-distance cut-off

$$S = c_0\,\frac{R^{d-2}}{\delta^{d-2}} + c_2\,\frac{R^{d-4}}{\delta^{d-4}} + \cdots$$

d = spacetime dimension

- leading coefficients sensitive to details of regulator, eg, $\delta \to 2\delta$
- find universal information characterizing underlying QFT in subleading terms, eg, $S = \cdots + c_d \log(R/\delta) + \cdots$

More general comments on **Entanglement Entropy:**

- nonlocal quantity which is (at best) very difficult to measure

  → no accepted experimental procedure

- in condensed matter theory: diagnostic to characterize quantum critical points or topological phases (eg, quantum hall fluids)

- in quantum information theory: useful measure of quantum entanglement (a computational resource)

- black hole microphysics: leading term obeys "area law" $S \simeq c_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}}$

  → suggested as origin of black hole entropy (eg, $\delta \simeq \ell_P$)

(Bombelli, Koul, Lee & Sorkin `86; Srednicki; Frolov & Novikov; Callan & Wilczek; Susskind; . . . .)

- recently considered in AdS/CFT correspondence

(Ryu & Takayanagi `06)

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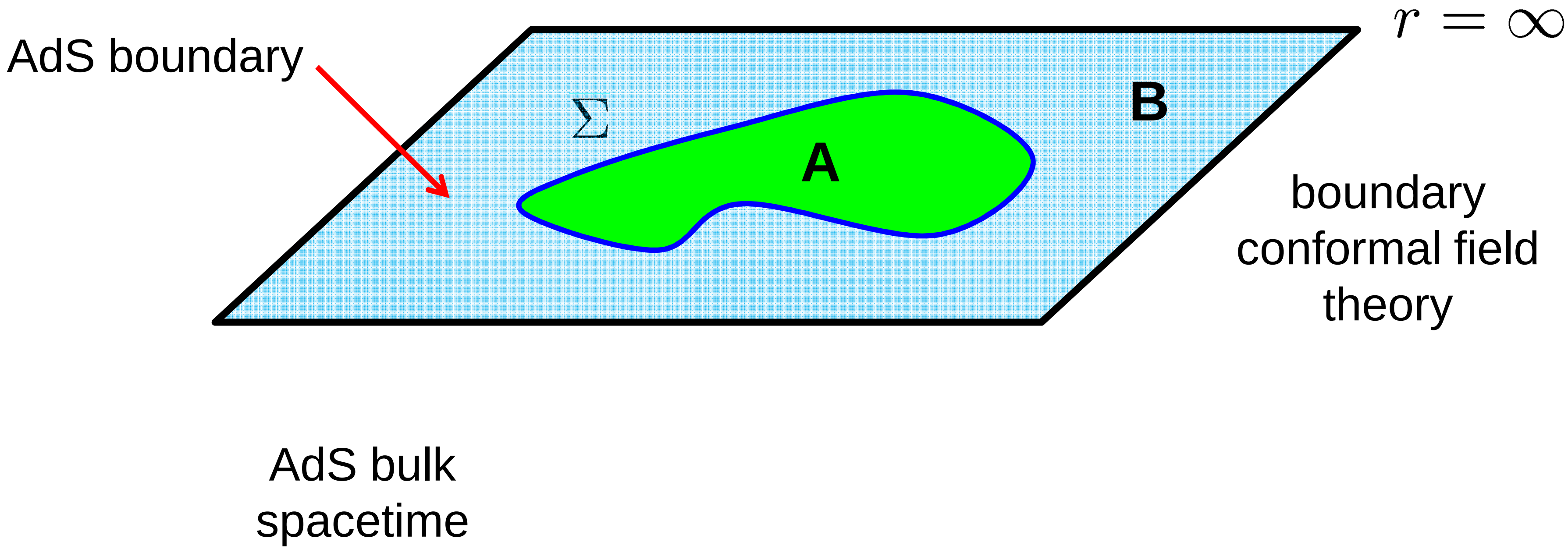
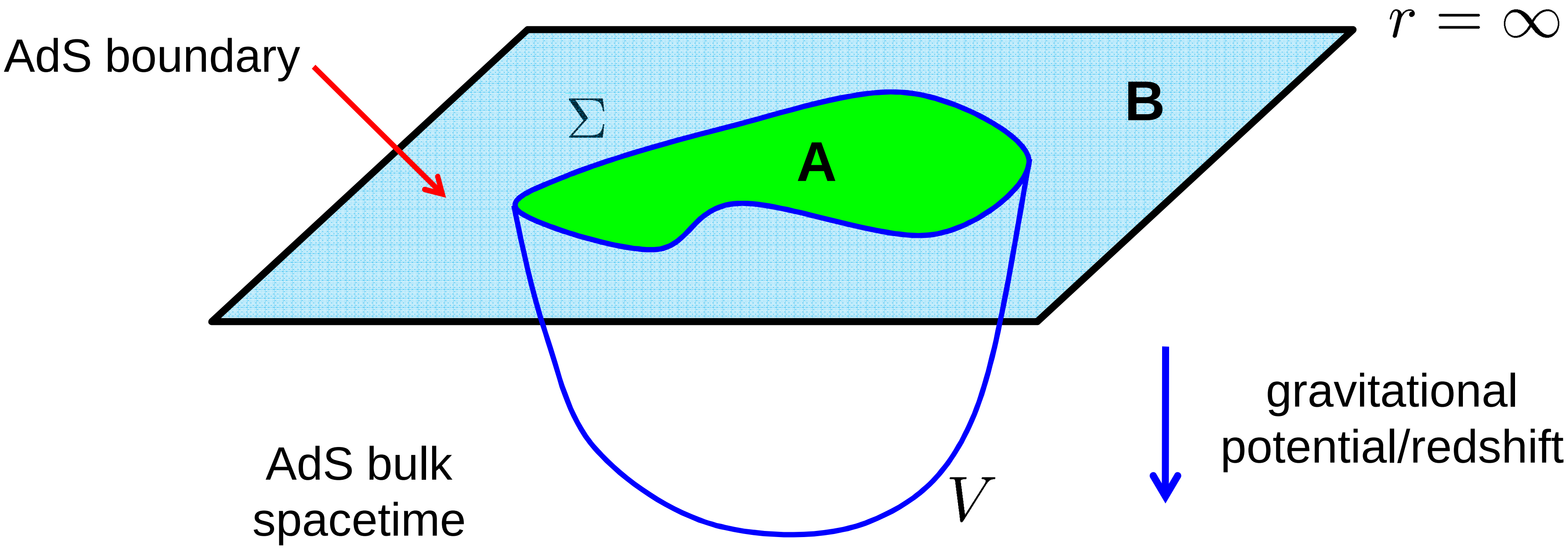
Holographic Entanglement Entropy:



$$S(A) \; = ??$$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



$(d-1)$ dimensional

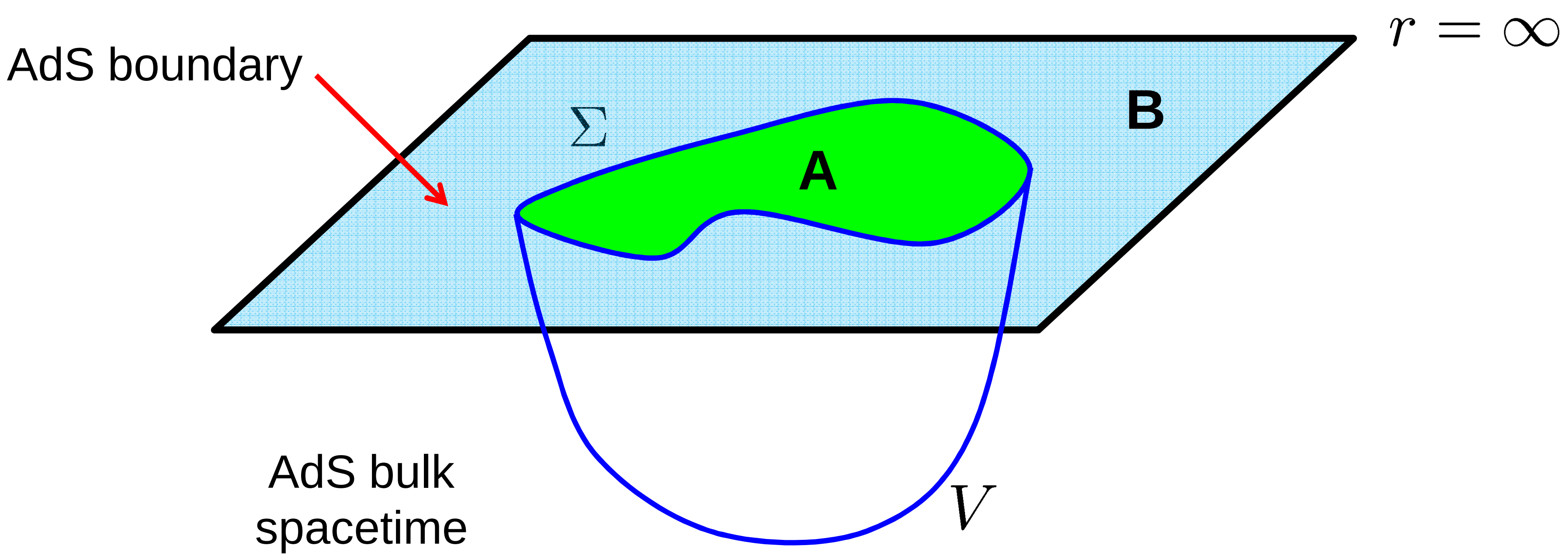
$$S(A) = \underset{\partial V = \Sigma}{\mathrm{ext}} \frac{A_V}{4G_N}$$

$r = \infty$

looks like
BH entropy!

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

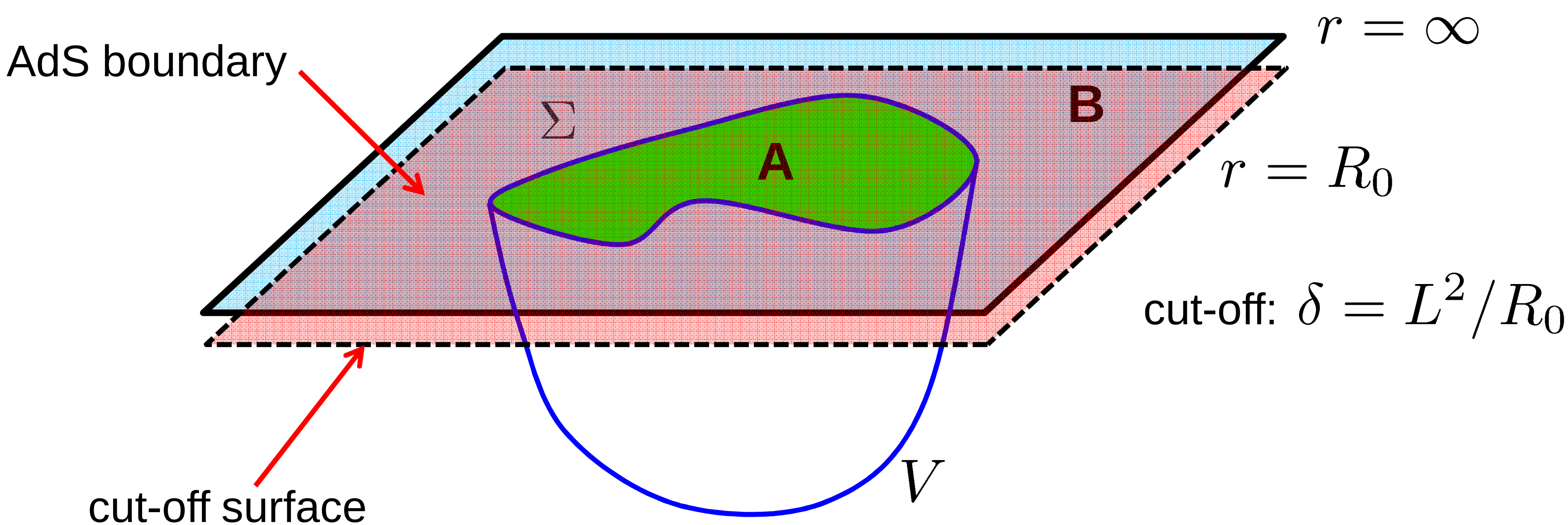


$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N} = \infty!!$$

- "UV divergence" because area integral extends to $r = \infty$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



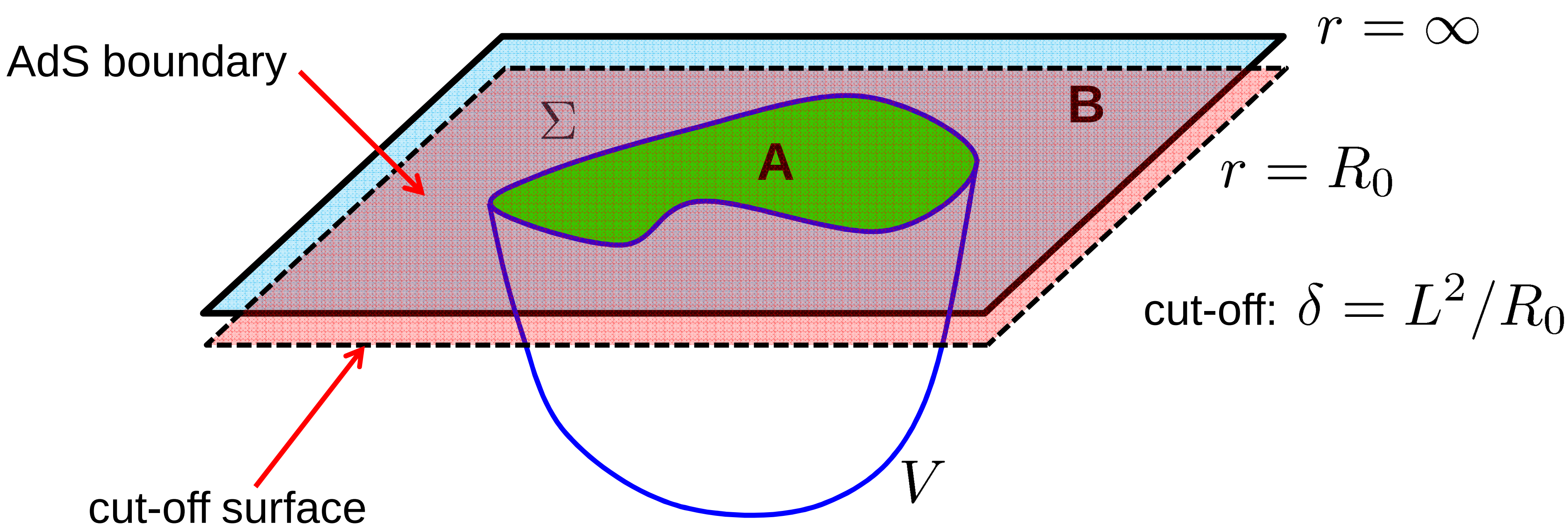
$$S(A) = \underset{\partial V = \Sigma}{\mathrm{ext}} \frac{A_V}{4G_N} \simeq \frac{L^{d-1}}{G_N} \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$$

- "UV divergence" because area integral extends to $r = \infty$
- finite result by stopping radial integral at large radius: $r = R_0$

  ⟶ short-distance cut-off in boundary theory: $\delta = L^2/R_0$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



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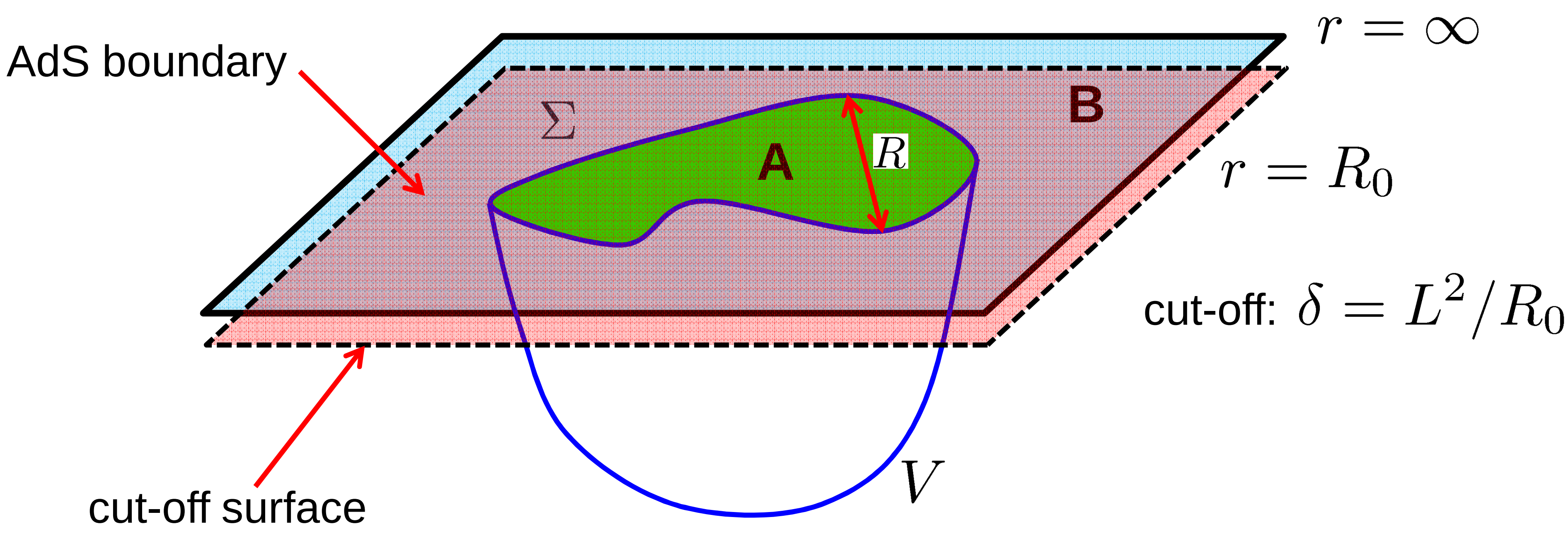
central charge
(counts dof) $(L/\ell_{Planck})^{d-1}$

“Area Law”

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



general expression (as desired):

$$S(A) \simeq c_0(R/\delta)^{d-2} + c_1(R/\delta)^{d-4} + \cdots$$

$$\begin{cases} +c_{d-2}\log(R/\delta) + \cdots & \text{(d even)} \\ +\ c_{d-2}\ + \cdots & \text{(d odd)} \end{cases}$$

universal contributions

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

<u>conjecture</u>

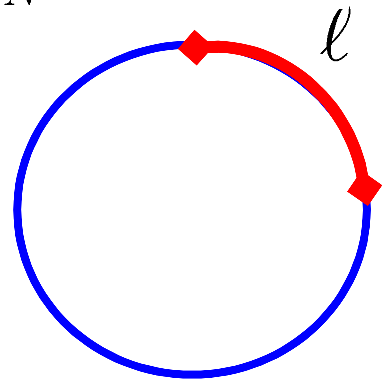
Extensive consistency tests:

1) leading contribution yields “area law”

$$S \simeq \frac{L^{d-1}}{G_N} \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$$

2) recover known results of Calabrese & Cardy for d=2 CFT

$$S = \frac{c}{3} \log \left(\frac{C}{\pi \, \delta} \sin \frac{\pi \ell}{C} \right)$$



$C = \text{circumference}$

(also result for thermal ensemble)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

conjecture

Extensive consistency tests:

1) leading contribution yields "area law" $S \simeq \frac{L^{d-1}}{G_N} \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

. . . . *(lots of interesting tests)* . . . .

7) connection to central charges of CFT for higher even d

(Hung, RCM & Smolkin, arXiv:1101.5813)

8) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RCM, arXiv:1102.044)

(see also: RCM & Sinha, arXiv:1011.5819)

7) connection to central charges of CFT for higher even d

(Hung, RCM & Smolkin, arXiv:1101.5813)

- trace anomaly in CFT (with even d) defines central charges

d=4: $$\langle T_\mu{}^\mu \rangle = \frac{\mathbf{c}}{16\pi^2} I_4 - \frac{\mathbf{a}}{16\pi^2} E_4$$

$I_4 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and $E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$

- universal/logarithmic contribution to entanglement entropy determined by central charges using trace anomaly, eg,

$$S_{uni} = \log(R/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\,\left[\mathbf{c}\left(C^{ijkl}\,\tilde{g}^\perp_{ik}\,\tilde{g}^\perp_{jl} - K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

(Takayanagi & Ryu;Schwimmer & Theisen; Solodukhin)

- R&T proposal for holographic EE exactly reproduces this result
- extends to certain higher curvature theories (eg, GB gravity)

$$S = \underset{\partial V = \Sigma}{\mathrm{ext}}\ \frac{2\pi}{\ell_p^3}\int_V d^3x\sqrt{h}\,\left[1 + \lambda\,L^2\,\mathcal{R}\right]$$

(see also: de Boer, Kulaxizi & Parnachev)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

conjecture

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Calculating Entanglement Entropy:

$$S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$$

- a "standard" approach relies on replica trick, first calculating Renyi entropy and then taking n ⟶ 1 limit

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

- replica trick involves path integral of QFT on singular n-fold cover of background spacetime

- problematic in holographic framework

  ⟶ produce singularity in dual gravity description

  (resolved by quantum gravity/string theory?)

(Fursaev; Headrick)

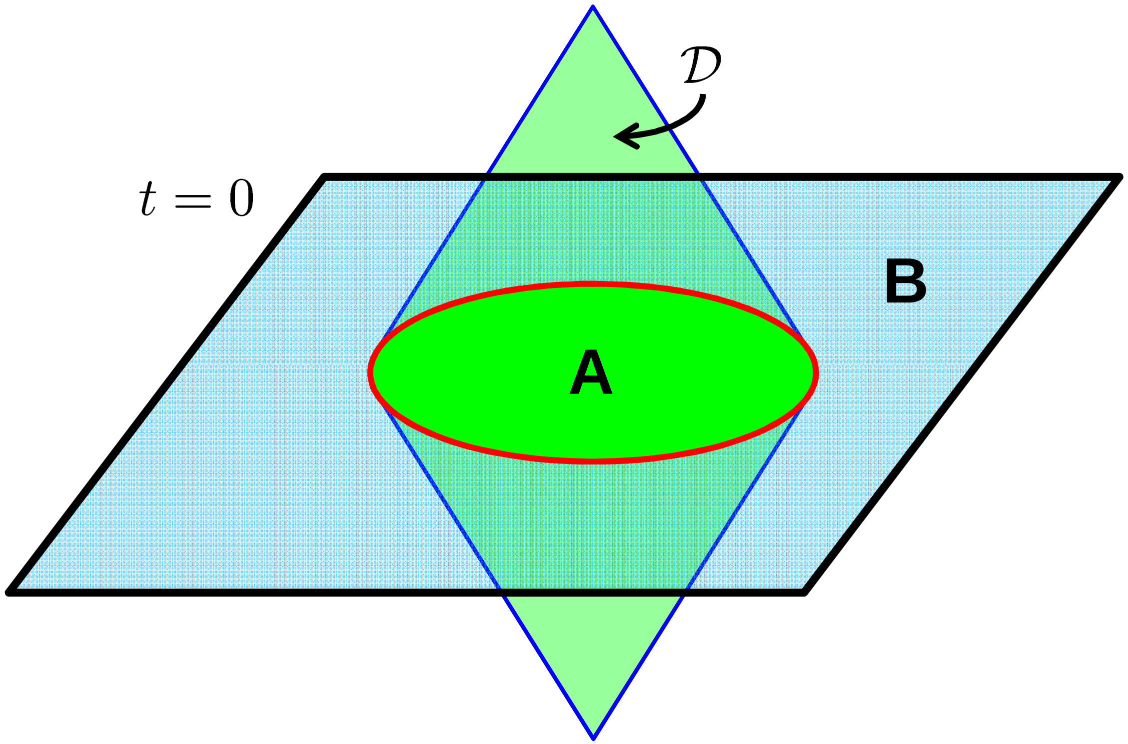
- need another calculation with simpler holographic translation

Calculating Entanglement Entropy:

(Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose $\Sigma = S^{d-2}$ with radius R

→ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$

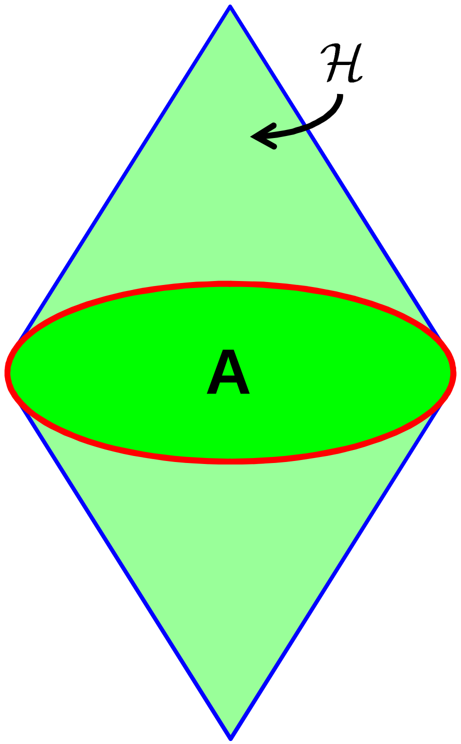


- density matrix ρ_A describes physics in entire causal domain $\mathcal{D}$
- conformal mapping: $\mathcal{D} \rightarrow \mathcal{H} = R_t \times H^{d-1}$

General result for any CFT (Casini, Huerta & RCM)

• take CFT in d-dim. flat space and choose $\Sigma = S^{d-2}$ with radius R

→ entanglement entropy: $S_{EE} = -Tr\,[\rho_A\,\log\rho_A]$



• conformal mapping: $\mathcal{D} \rightarrow \mathcal{H} = R_t \times H^{d-1}$

curvature scale: 1/R   temperature: T=1/2πR !!

• for CFT: $\rho_{thermal} = U\,\rho_A\,U^{-1}$ → $S_{EE} = S_{thermal}$

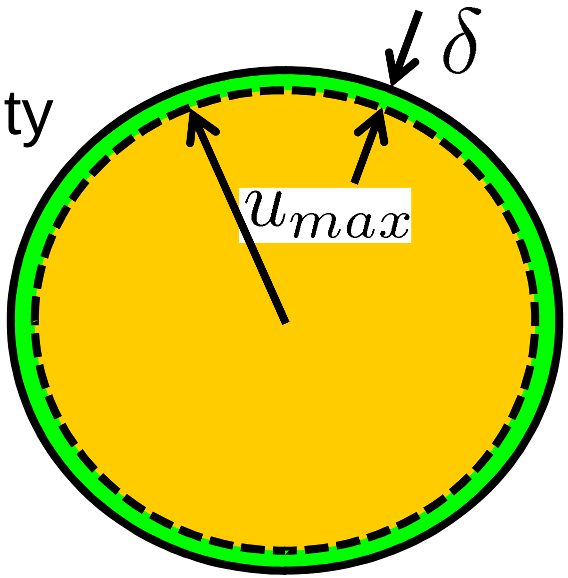
General result for any CFT

(Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose S^{d-2} with radius R
  - → entanglement entropy: $S_{EE} = -Tr\,[\rho_A \log \rho_A]$
  - → by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and T=1/2πR

$$\boxed{S_{EE} = S_{thermal}}$$

- note both sides of equality are divergent
  - → $S_{thermal}$ sums constant entropy density over infinite volume



- conformal map takes original UV cut-off to IR cut-off on H^{d-1}

$$u_{max} \simeq R/\delta$$

General result for any CFT

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→ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$

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$$S_{EE} = S_{thermal}$$

AdS/CFT correspondence:

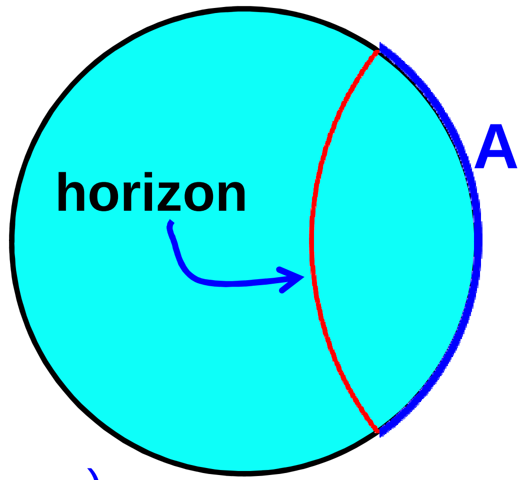
- thermal bath in CFT = black hole in AdS

$$S_{EE} = S_{thermal} = S_{horizon}$$

- only need to find appropriate black hole

→ topological BH with hyperbolic horizon which intersects ∂A on AdS boundary

(Aminneborg et al; Emparan; Mann; . . .)



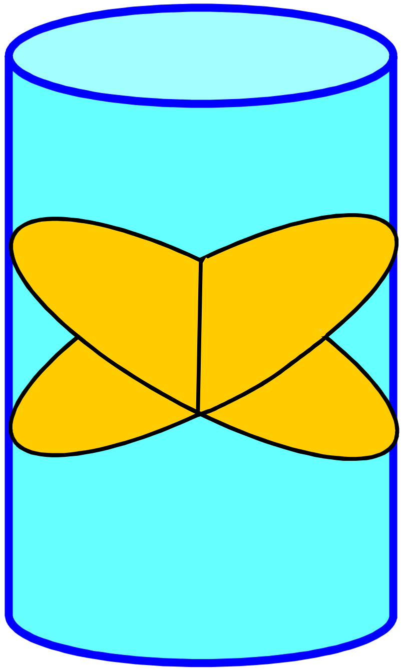
$$S_{EE} = S_{thermal} = S_{horizon}$$

- desired "black hole" is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right) d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \longrightarrow T = \frac{1}{2\pi R}$$

- bulk coordinate transformation implements desired conformal transformation on boundary

- "Rindler coordinates" of AdS space:



$$S_{EE} = S_{thermal} = S_{horizon}$$

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- apply Wald's formula (for any gravity theory) for horizon entropy:

$$S = -2\pi \int d^{d-1}x\sqrt{h}\, \frac{\partial \mathcal{L}}{\partial R^{\mu\nu}{}_{\rho\sigma}}\, \hat{\varepsilon}^{\,\mu\nu}\, \hat{\varepsilon}_{\rho\sigma}$$
$$= \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right)\boxed{a_d^*}\, V\left(H^{d-1}\right)$$

(RCM & Sinha)

where a_d^* contains all of the couplings from the gravity theory

eg, $a_d^* = \frac{\pi^{d/2}}{\Gamma\left(d/2\right)}\frac{L^{d-1}}{\ell_{\mathrm{P}}^{d-1}}$ for Einstein gravity

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(RCM & Sinha)

where a_d^* = central charge for “A-type trace anomaly”

for even d

= entanglement entropy defines effective central charge

for odd d

$$S_{EE} = S_{thermal} = S_{horizon}$$

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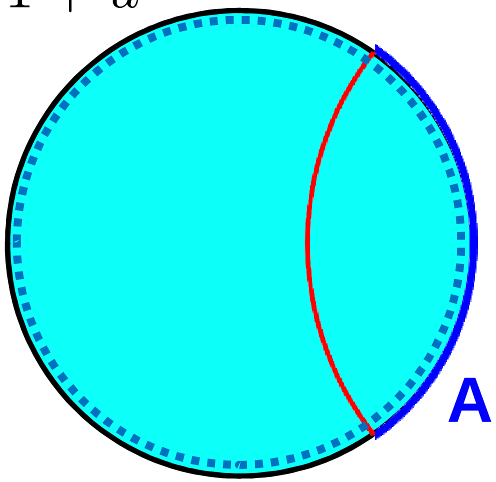
$$ds^2 = \frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2}$$

intersection with standard regulator surface: $z_{min} = \delta$

$$S = a_d^* \frac{4\pi^{\frac{d-3}{2}}}{(d-2)\Gamma\left(\frac{d-1}{2}\right)}\, u_{max}^{d-2} + \cdots$$



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intersection with standard regulator surface: $z_{min} = \delta$

$$S = a_d^* \frac{4\pi^{\frac{d-3}{2}}}{(d-2)\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{R}{\delta}\right)^{d-2} + \cdots$$

"area law" for d-dimensional CFT

$$S_{EE} = S_{thermal} = S_{horizon}$$

- desired "black hole" is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2\, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2}\, d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \quad \longrightarrow \quad T = \frac{1}{2\pi R}$$

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universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^*\ \log\left(2R/\delta\right) + \cdots \quad \text{for even d}$$

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots \quad \text{for odd d}$$

$$S_{EE} = S_{thermal} = S_{horizon}$$

universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \, \log(2R/\delta) + \cdots \quad \text{for even d}$$

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- discussion extends to case with background: $R^{1,d-1} \to R \times S^{d-1}$
- for Einstein gravity, coincides with Ryu & Takayanagi result and horizon (bifurcation surface) coincides with R&T surface

  → no extremization procedure?!?

- applies for classical bulk theories beyond Einstein gravity
- can imagine calculating “quantum” corrections (eg, Hawking rad)

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- recall Renyi entropies (close cousin of entanglement entropy)

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

- universal contribution (for even d)

$$S_n \;=\; \cdots \;+\; constant \times \log\left(R/\delta\right) \;+\; \cdots$$

(Hung, RCM, Smolkin & Yale)

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$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

- universal contribution (for even d)

d=2: $$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right) \log\left(\ell/\delta\right) + \cdots$$

(Calabrese & Cardy)

- few calculations for d > 2

(Metlitski, Fuertes & Sachdev; Hastings, Gonzalez, Kallin & Melko; . . .)

- standard calculation involves singular n-fold cover of spacetime

→ problematic for translation to dual AdS gravity

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- turn to Renyi entropy (close cousin of entanglement entropy)

$$S_n = \frac{1}{1-n}\log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

- recall previous derivation lead to thermal density matrix

$$\rho_A = U^{-1}\frac{e^{-H/T_0}}{Tr\left[e^{-H/T_0}\right]}U \qquad \text{with} \qquad T_0 = \frac{1}{2\pi R}$$

$$\Rightarrow \quad Tr\left[\rho_A^n\right] = \frac{Tr\left[e^{-nH/T_0}\right]}{Tr\left[e^{-H/T_0}\right]^n} \quad \leftarrow$$

partition function at new temperature, $T = T_0/n$

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- turn to Renyi entropy (close cousin of entanglement entropy)

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad S_{EE} = \lim_{n\to 1} S_n$$

- with bit more work, find some convenient formulae:

$$S_n = \frac{n}{1-n}\frac{1}{T_0}\left[F(T_0) - F(T_0/n)\right]$$

where $F(T) = -T\,\log Z(T)$ and $T_0 = 1/(2\pi R)$ or

$$S_n = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S(T)\,dT$$

↑ Renyi entropy for spherical Σ

↑ thermal entropy on hyperbolic space H^{d-1}

- in holographic framework, need to know topological black hole solutions for arbitrary temperature

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- Renyi entropy of CFT for spherical entangling surface:

$$S_n = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S(T)dT \quad \text{where} \quad T_0 = \frac{1}{2\pi R}$$

- need to know topological black holes for arbitrary temperature
- focus on gravity theories where we can calculate: Einstein, Gauss-Bonnet, Lovelock, quasi-topological, …..

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- for example, with Einstein gravity and (boundary) d=4:

$$S_n = \frac{\pi\, n}{4(n-1)} \left(5 + \frac{x_n}{n}\right) \left(1 - \frac{x_n}{n}\right) \frac{L^3}{\ell_P^3} V(H^3)$$

where $x_n = \left(1 + \sqrt{1 + 8n^2}\right) /(4n)$

➡ $$(S_n)_{univ} = \frac{c}{2} \frac{n}{1-n} \left(5 + \frac{x_n}{n}\right) \left(1 - \frac{x_n}{n}\right) \log(2R/\delta)$$

- compare to d=2 result:

$$S_n \;=\; \frac{c}{6} \left(1 + \frac{1}{n}\right) \log\left(\ell/\delta\right) \;+\; \cdots$$

- might suggest simple general form for even d:

$$(S_n)_{univ} = c \times f(d,n) \times \log(2R/\delta)$$

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- for example, with GB gravity and (boundary) d=4:

$$(S_n)_{univ} = \log(2R/\delta)\,\frac{n}{2}\,\frac{1-x^2}{1-n}\left[(5c-a)x^2 - (13c-5a) + 16c\frac{2cx^2-c+a}{(3c-a)x^2-c+a}\right]$$

where $0 = x^3 - \frac{3c-a}{5c-a}\left(\frac{x^2}{n}+x\right) + \frac{1}{n}\frac{c-a}{5c-a}$

- unfortunately indicates no simple universal form:

$$(S_n)_{univ} = a \times f\left(d, n, \frac{c}{a}, t_4, \cdots\right) \times \log(2R/\delta)$$

- further work (with quasi-topological gravity) shows the universal coefficient depends on more CFTdata than central charges

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

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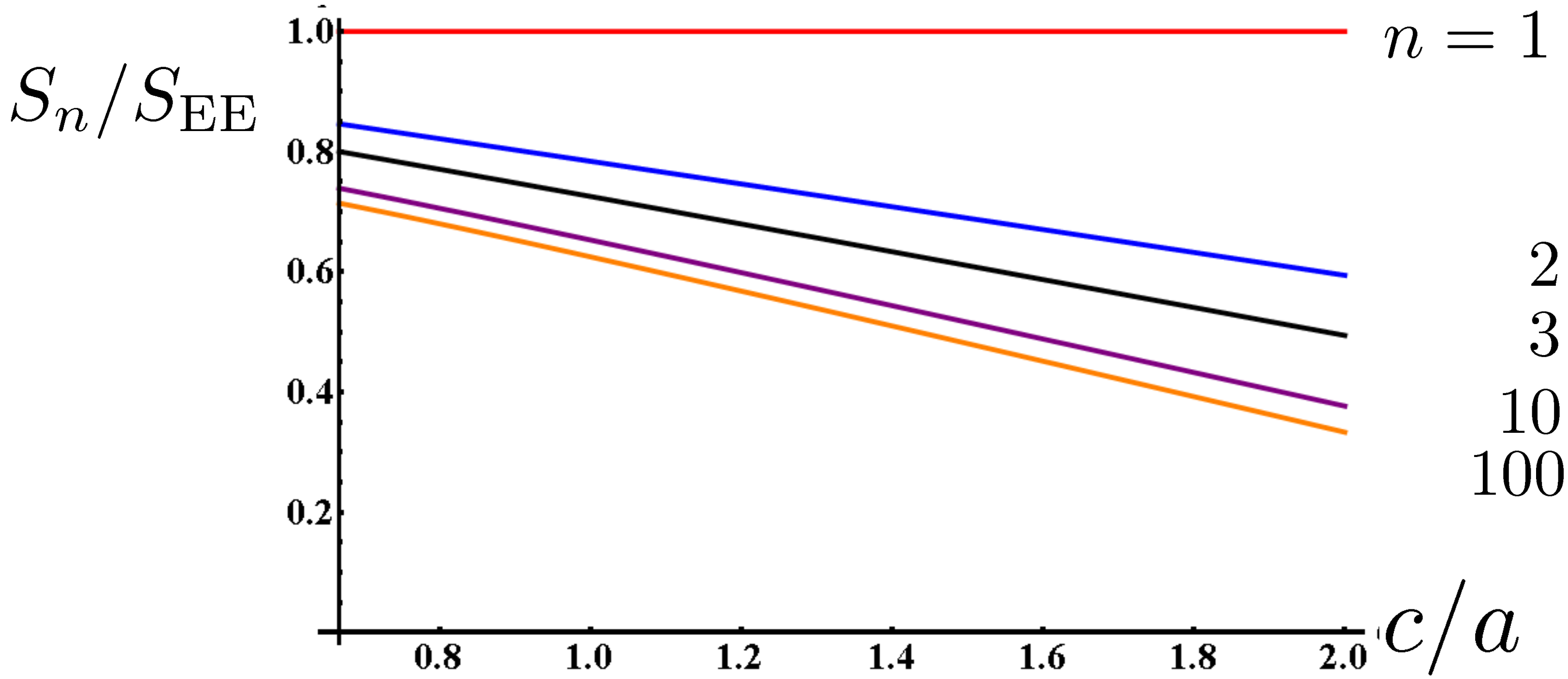
$$(S_n)_{univ} = \log(2R/\delta)\,\frac{n}{2}\,\frac{1-x^2}{1-n}\left[(5c-a)x^2-(13c-5a)\right.$$
$$\left.+16c\frac{2cx^2-c+a}{(3c-a)x^2-c+a}\right]$$

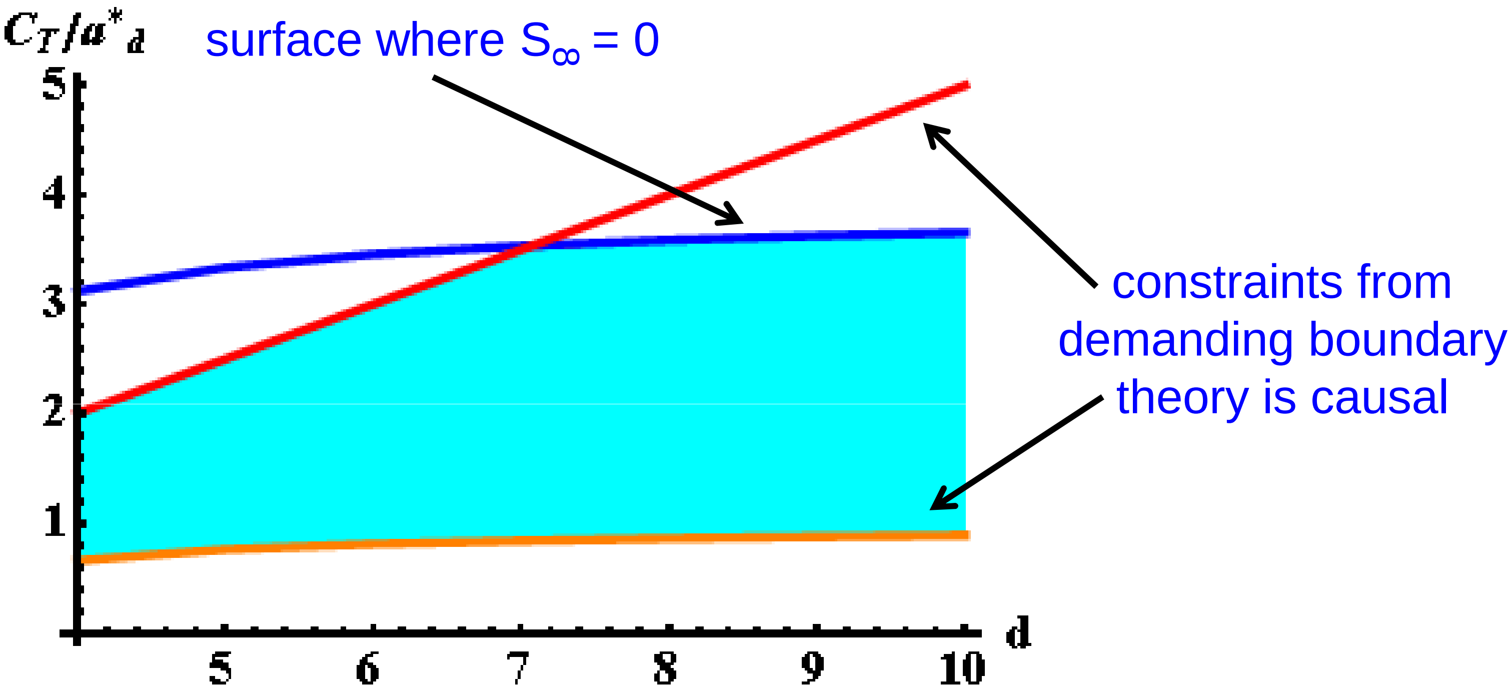
• note despite intimidating expression, results relatively simple:



Phase transitions?

- Problem: holographic S_n goes negative for large n (and large d)

GB gravity:



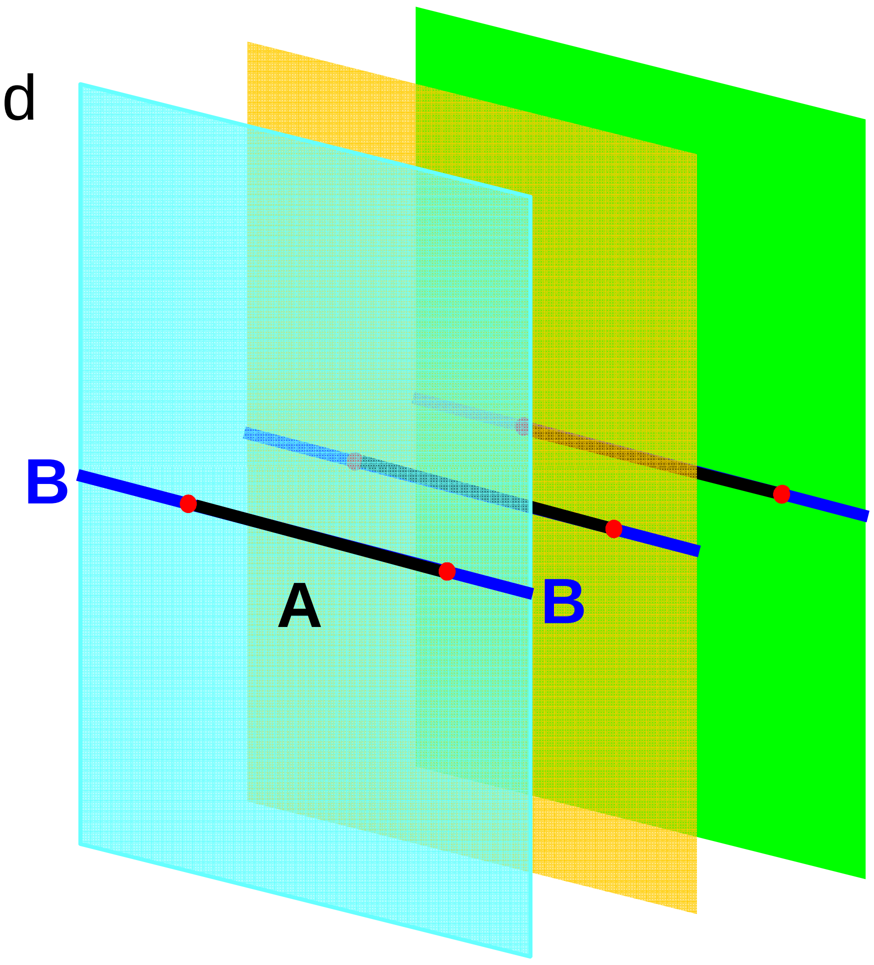
- real problem is that in certain parameter regime, entropy of hyperbolic black hole becomes negative at low temperatures

$\longrightarrow$ identified the wrong saddle-point

- right saddle-point? new constraint on theory space?

Twist Operators:

- $\mathrm{Tr}(\rho_A^n)$ evaluated as Euclidean path integral over n copies of field theory inserting **twist operators** at boundary of region **A**



Twist Operators:

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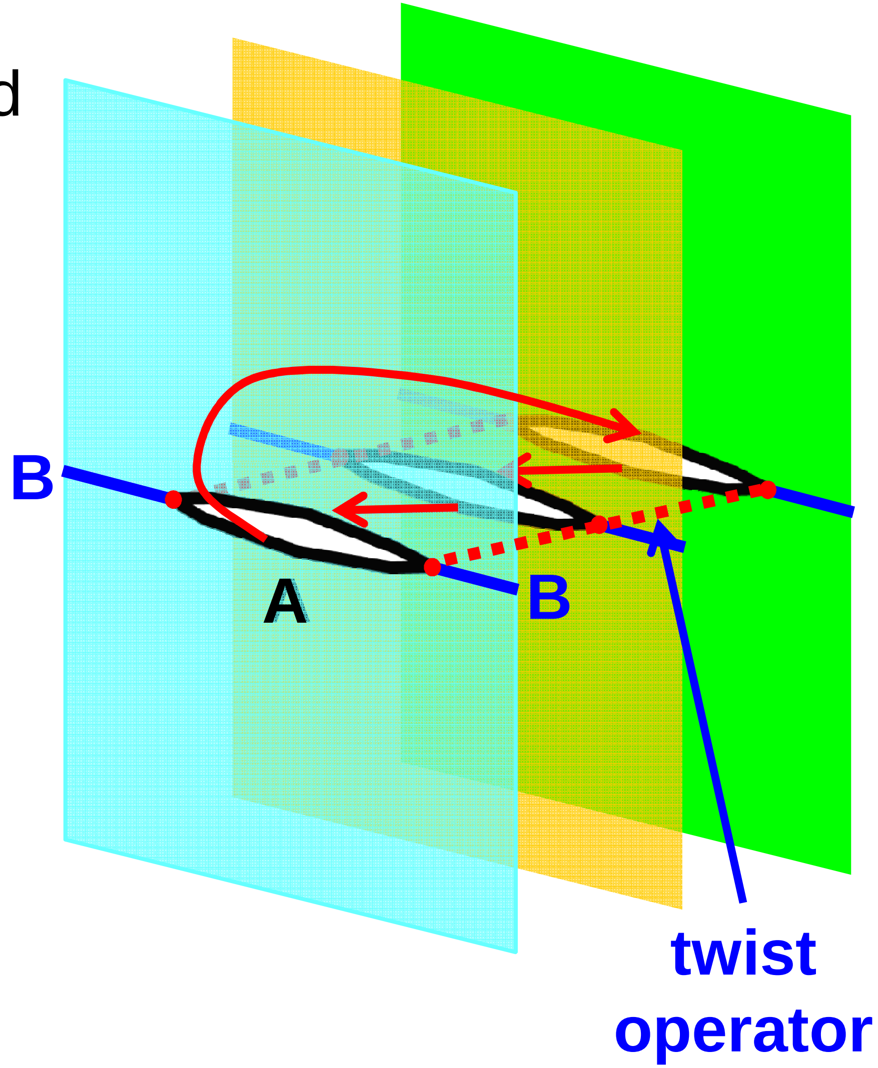
- twist operators introduce n-fold branch cuts where various copies of fields talk to each other



Twist Operators:

- $\mathrm{Tr}(\rho_A^n)$ evaluated as Euclidean path integral over n copies of field theory inserting **twist operators** at boundary of region **A**
- twist operators introduce n-fold branch cuts where various copies talk to each other
- elegant results for d=2, eg, scaling dimension of twist operators

$$h_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

(Calabrese & Cardy)

- in d dimensions, would be (d–2)-dimensional surface operators but little is known about their properties

Twist Operators:

- insertion of stress tensor near planar twist operator for CFT in R^d

$$\langle T_{ab}\,\sigma_n\rangle \;=\; -\frac{h_n}{2\pi}\,\frac{\delta_{ab}}{r_\perp^d}\,, \qquad \langle T_{ai}\,\sigma_n\rangle = 0$$

$$\langle T_{ij}\,\sigma_n\rangle \;=\; \frac{h_n}{2\pi}\,\frac{(d-1)\delta_{ij} - d\,n_i\,n_j}{r_\perp^d}$$

where $a,b \;\|\; \sigma_n$ and $i,j \;\perp\; \sigma_n$

- consider conformal mapping for spherical entangling surface

→ Euclidean version gives one-to-one map: $S^1\times H^{d-1}\rightarrow R^d$

→ with $\beta = n/T_0 = 2\pi R\,n \;\; (n\in \mathbb{Z})$ get n-fold cover of R^d

→ have spherical twist operator σ_n on S^{d-2}

Twist Operators:

- evaluate $\langle T_{\alpha\beta}\, \sigma_n \rangle$ correlator by mapping from thermal bath

$$\langle T_{\alpha\beta}\, \sigma_n \rangle = \Omega^{d-2}\, \frac{\partial X^\mu}{\partial x^\alpha}\, \frac{\partial X^\nu}{\partial x^\beta} \left(\langle T_{\mu\nu}(T_0/n) \rangle - \mathcal{A}_{\mu\nu}(T_0) \right)$$

creates singularity near twist operator

uniform thermal bath

anomalous bit

(compare: Marolf, Rangamani & Van Raamsdonk)

- read off h_n from short distance singularity

$$h_n = 2\pi \frac{n\, R^d}{d-1} \left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n) \right)$$

Twist Operators:

- evaluate $\langle T_{\alpha\beta}\,\sigma_n\rangle$ correlator by mapping from thermal bath

$$\langle T_{\alpha\beta}\,\sigma_n\rangle = \underbrace{\Omega^{d-2}\frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial X^\nu}{\partial x^\beta}}_{\text{creates singularity near twist operator}}\left(\underbrace{\langle T_{\mu\nu}(T_0/n)\rangle}_{\text{uniform thermal bath}} - \mathcal{A}_{\mu\nu}\big(T_0)\rangle\right)$$

creates singularity near twist operator

uniform thermal bath

anomalous bit

(compare: Marolf, Rangamani & Van Raamsdonk)

- for example, with GB gravity and (boundary) d=4:

$$h_n = \frac{n}{4\pi}\left(x^2-1\right)\left[c-a-x^2(5c-a)\right]$$

where $$0 = x^3 - \frac{3c-a}{5c-a}\left(\frac{x^2}{n}+x\right) + \frac{1}{n}\frac{c-a}{5c-a}$$

- no simple universal form can be expected
- again, CFT data beyond central charges also appears

Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories

- holographic entanglement entropy is part of an interesting dialogue has opened between string theorists and physicists in a variety of fields (eg, condensed matter, nuclear physics, . . .)

- potential to learn lessons about issues in boundary theory
  - eg, readily calculate Renyi entropies for wide class
    - of theories in higher dimensions

- potential to learn lessons about issues in bulk gravity theory
  - eg, holographic entanglement entropy may give new
    - insight into quantum gravity or emergent spacetime

(eg, van Raamsdonk)

Lots to explore!