DOUBLE DIFFUSIVE CONVECTION ~~SEMI-CONVECTION~~ IN STELLAR INTERIORS: INSIGHT FROM 3D SIMULATIONS

Pascale Garaud, UC Santa Cruz
Giovanni Mirouh, ENS Cachan
Erica Rosenblum, Stony Brooks
Adrienne Traxler, UC Santa Cruz
Stephan Stellmach, U Muenster
(with help from Douglas Gough)

Double-diffusive convection

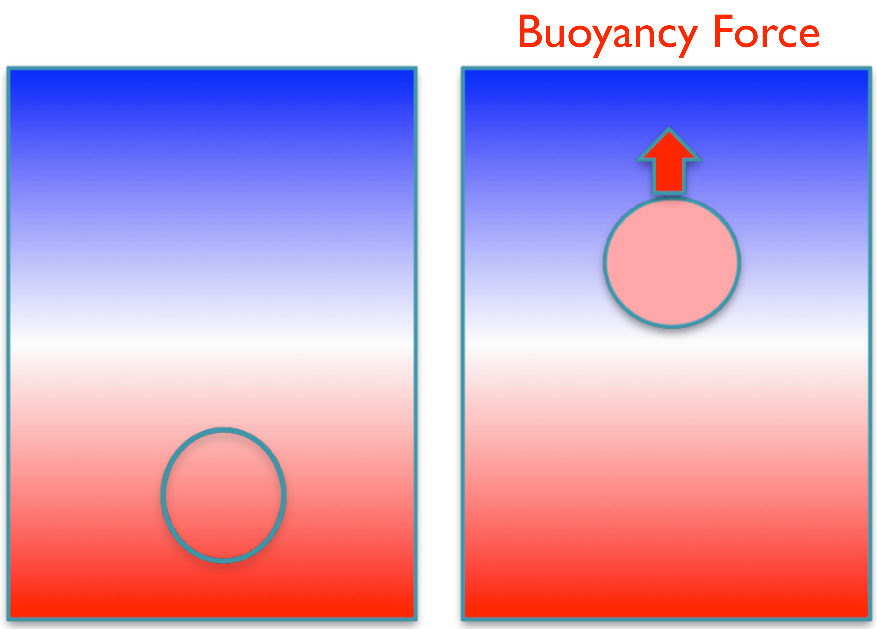
- What we expect naively (overturning convection)
- Double-diffusive convection
- Layered convection vs. non-layered convection
  - Numerical model and sample results
  - Criterion for layer formation
  - Transport through a staircase
- A new model for double-diffusive convection

What we expect naively

- *Overturning* convection is a linear instability of stratified fluids with "top-heavy" density profiles, and occurs whenever $\rho_z > 0$
  - If density depends on temperature only, then we have thermal convection for fluids heated at the bottom.

Instability criterion:

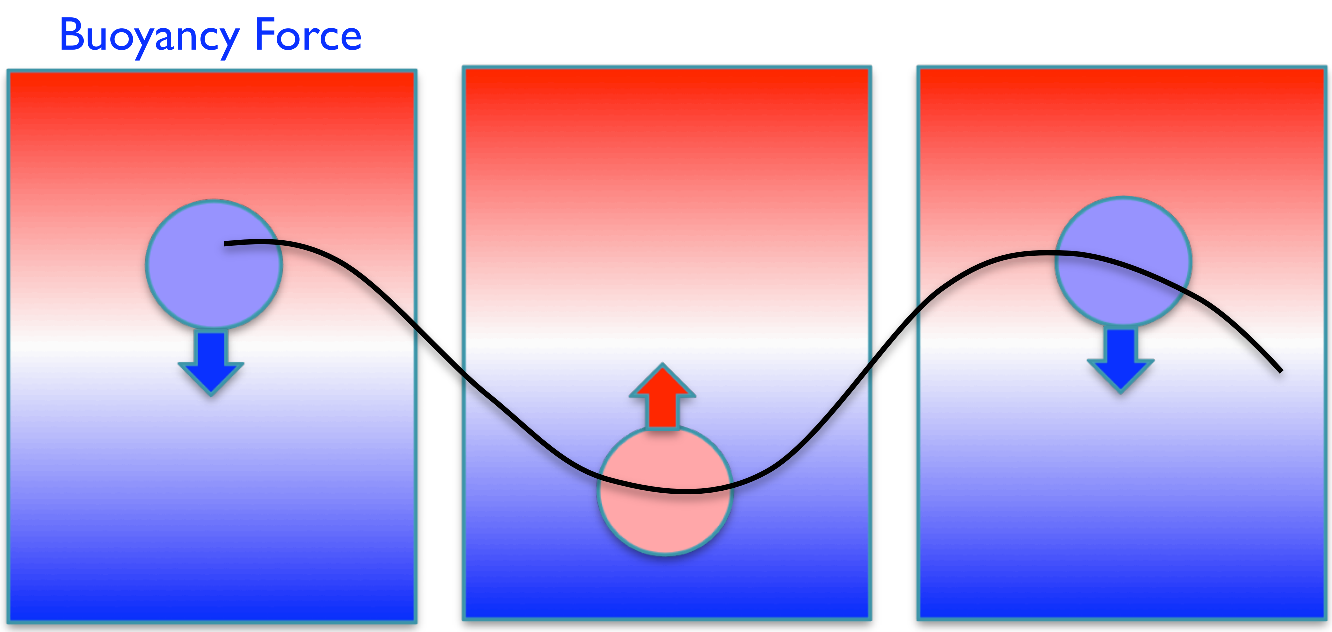
$$\rho \propto -T \Rightarrow$$
$$\rho_z > 0 \Leftrightarrow T_z < 0$$



  - This is a VERY efficient kind of convection.

What we expect naively

- *Overturning* convection is a linear instability of stratified fluids with "top-heavy" density profiles, and occurs whenever $\rho_z > 0$
  - Fluids hotter at the top are stable against overturning convection.



What we expect naively

Aside: In the previous argument, the system is assumed incompressible. In most astrophysical systems, it is not.

The correct criterion for **instability** is

$$\left(\frac{\partial\rho}{\partial p}\right)_{ad} > \left(\frac{\partial\rho}{\partial p}\right)$$

which translates, in terms of temperature, into the Schwarzchild criterion:

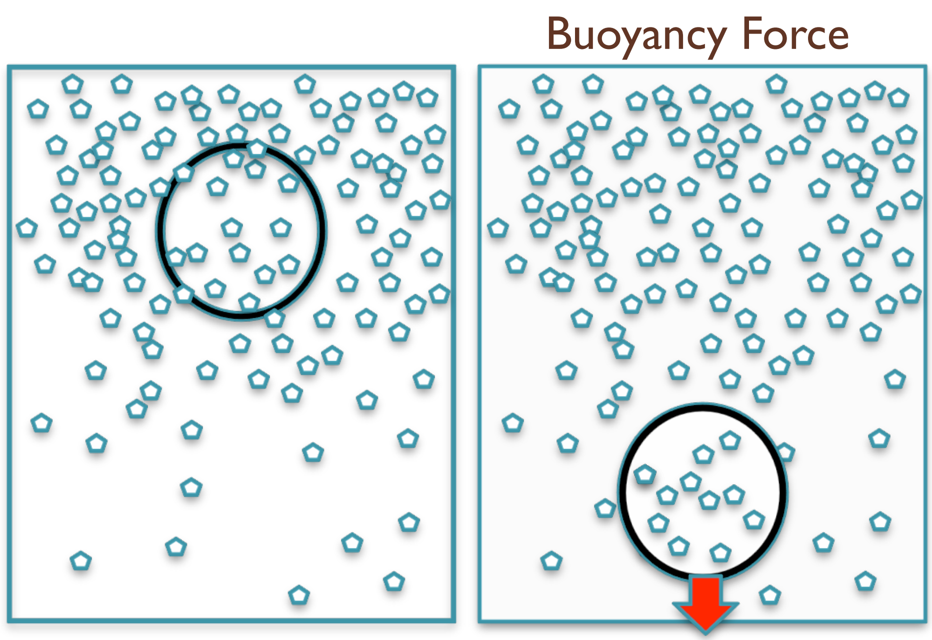
$$\left(\frac{\partial \ln T}{\partial \ln p}\right)_{ad} < \left(\frac{\partial \ln T}{\partial \ln p}\right)$$

What we expect naively

- *Overturning* convection is a linear instability of stratified fluids with "top-heavy" density profiles, and occurs whenever $\rho_z > 0$
  - If density depends on composition only, then we have overturning convection for fluids with top-heavy composition.

Instability criterion:

$$\rho = \beta S \Rightarrow$$
$$\rho_z > 0 \Leftrightarrow S_z > 0$$



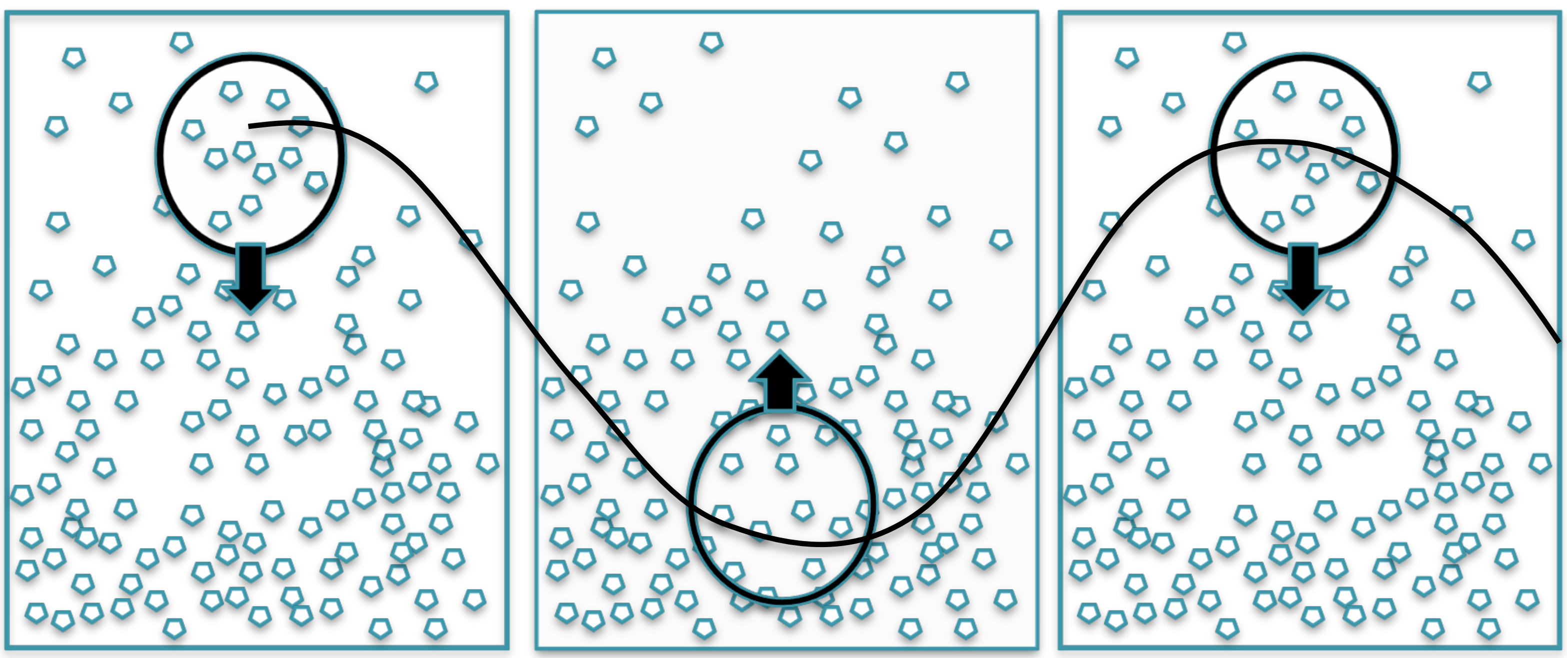
  - This is a VERY efficient kind of convection
  - Fluids with bottom-heavy composition are stable

What we expect naively

- In compositionally stably stratified fluids

$$\overline{T}_z = 0, \ \overline{S}_z < 0 \text{ and } \overline{\rho}_z < 0$$

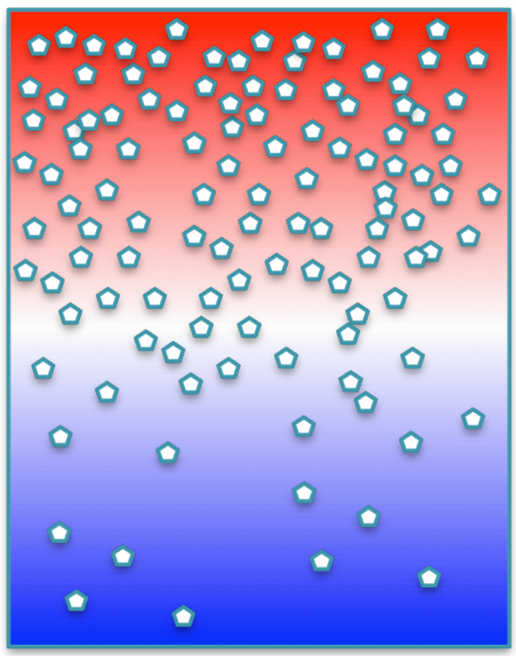
displaced blobs of fluid oscillate with the buoyancy frequency.

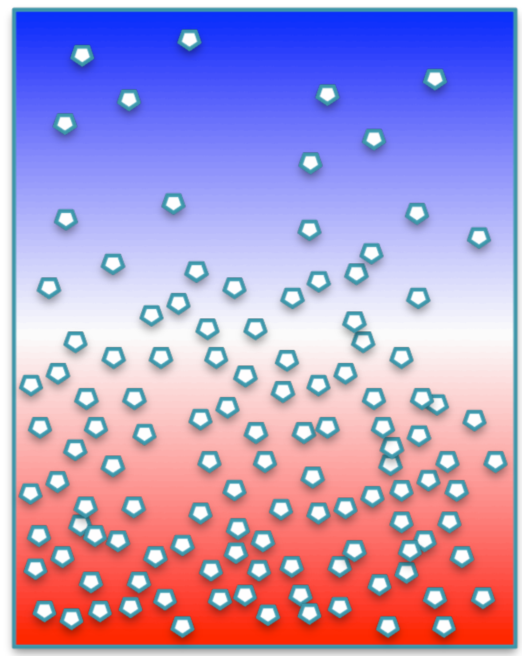


What we expect naively

- What happens when both types of stratification compete?
  - (Stable temperature gradient with unstable composition?)
  - **Unstable temperature gradient with stable composition?**

Instability criterion ?





What we expect naively

The answer is superficially simple:

- With $\rho = -\alpha T + \beta S \Rightarrow \rho_z = -\alpha T_z + \beta S_z$

the new criterion for instability for overturning convection is

$$\rho_z > 0 \quad \rightarrow \quad -\alpha T_z + \beta S_z > 0$$

- For compressible fluids, the equivalent criterion for instability is called the Ledoux criterion

$$\nabla - \nabla_{ad} + \nabla_{\mu} > 0 \Leftrightarrow$$

$$\left(\frac{\partial \ln T}{\partial \ln p}\right) - \left(\frac{\partial \ln T}{\partial \ln p}\right)_{ad} > \left(\frac{\partial \ln \mu}{\partial \ln p}\right)$$

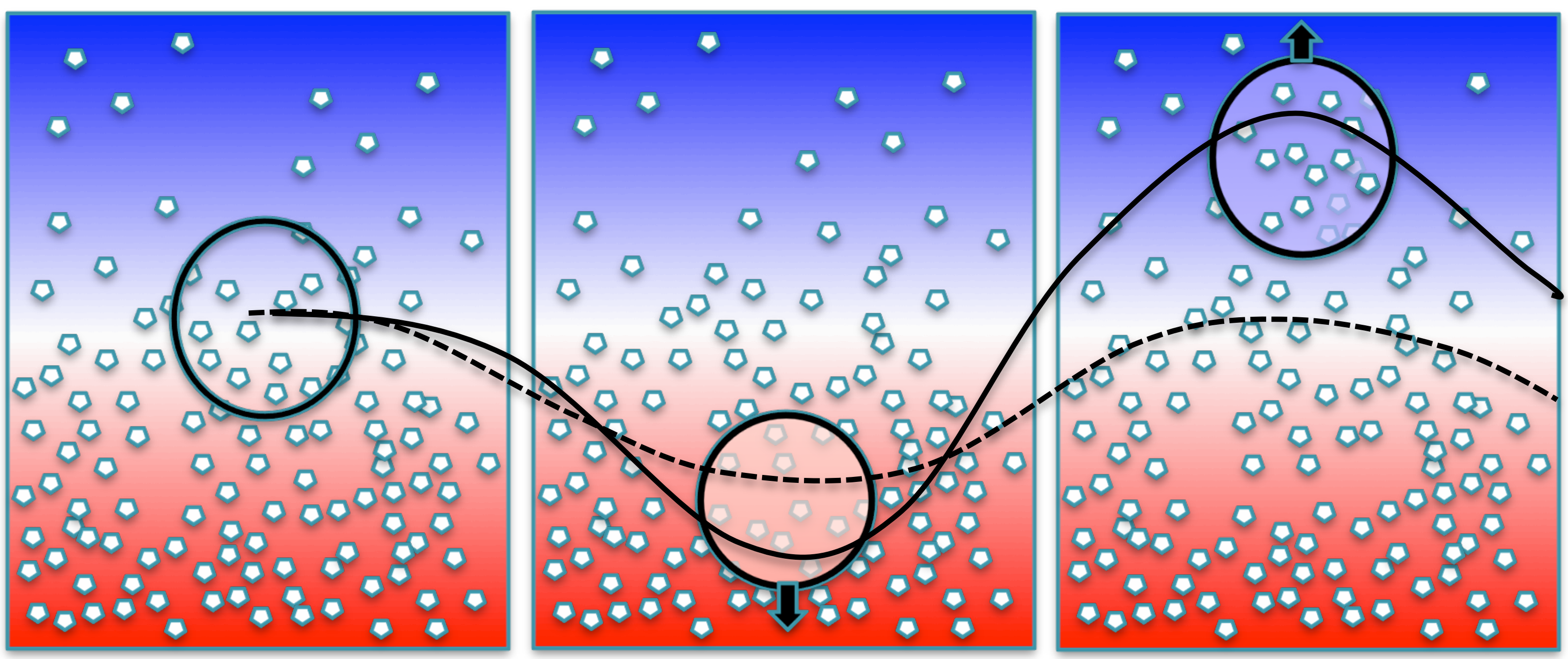
Double-diffusive convection

- What we expect naively (overturning convection)
- Double-diffusive convection
- Layered convection vs. non-layered convection
  - Numerical model and sample results
  - Criterion for layer formation
  - Transport through a staircase
- A new model for double-diffusive convection

Double-diffusive convection

- The presence of even a small (but unstable) temperature gradient can cause a double-diffusive instability instability in a system that is stable to overturning convection.

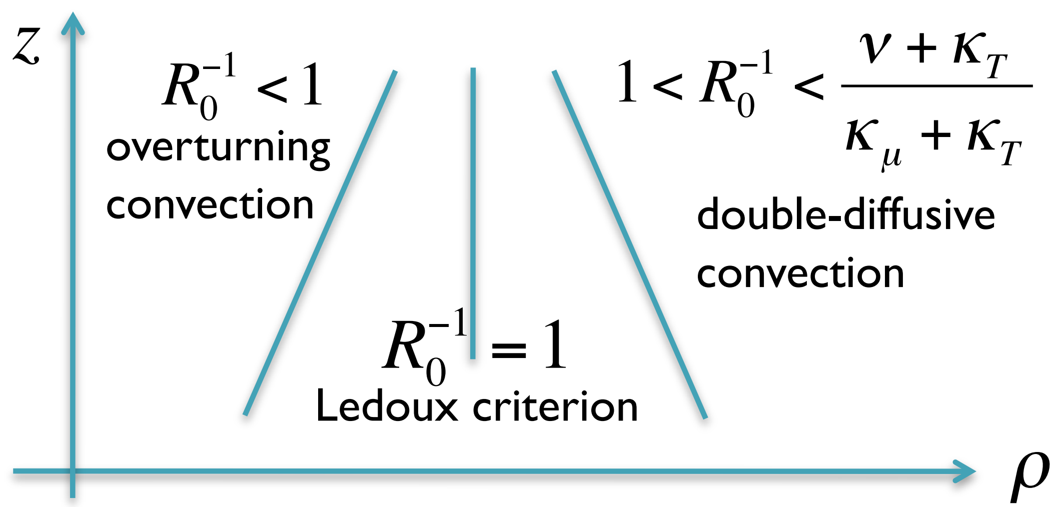
$$\overline{T}_z, \overline{S}_z < 0 \text{ and } \overline{\rho}_z < 0$$



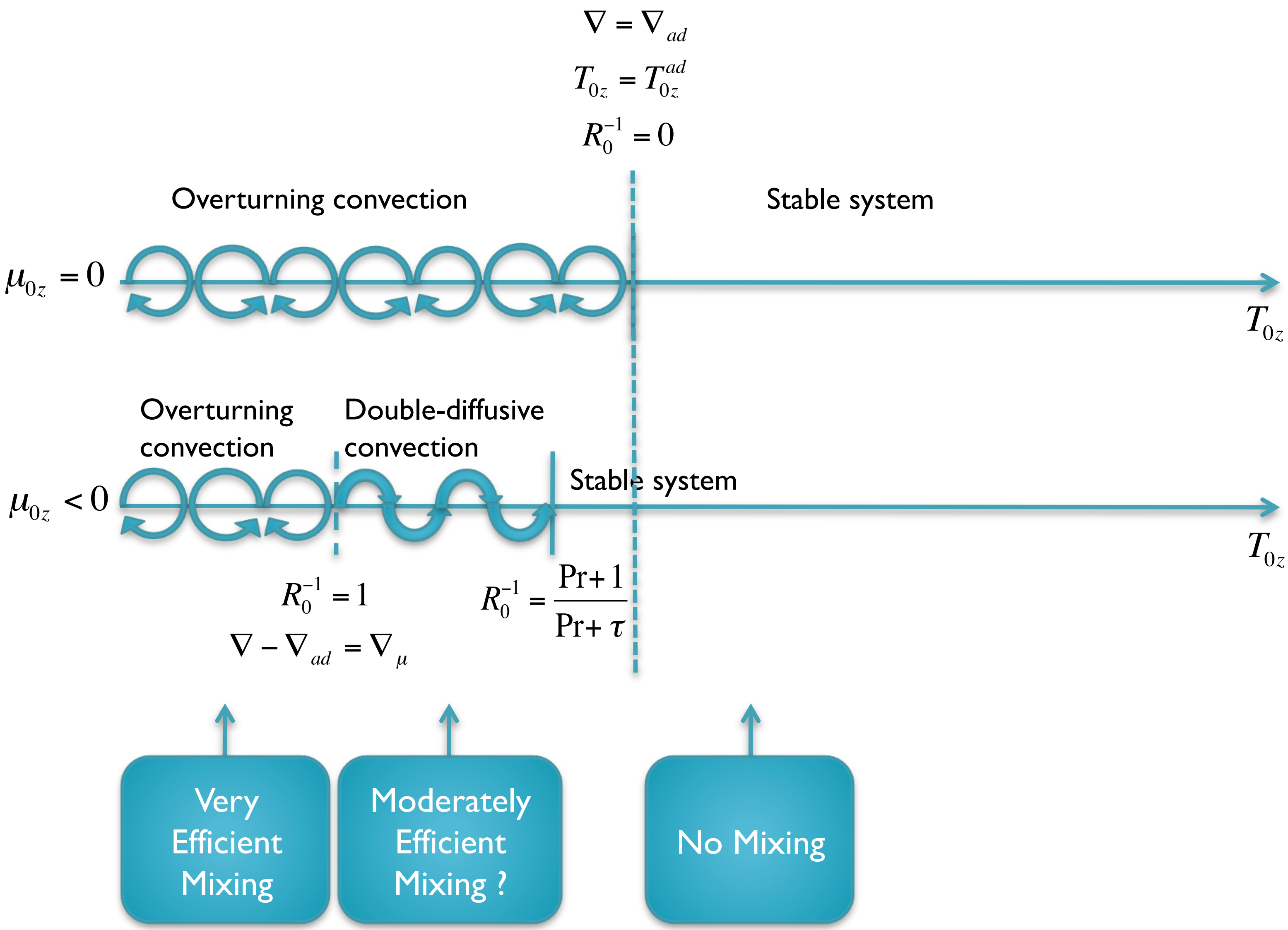
Double-diffusive convection

- Double-diffusive stability depends principally on the non-dimensional **inverse density ratio:**

$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha (T_{0z} - T_{0z}^{ad})} = \frac{\nabla_\mu}{\nabla - \nabla_{ad}} = \frac{\text{Stabilizing compositional stratification}}{\text{Destabilizing thermal stratification}}$$

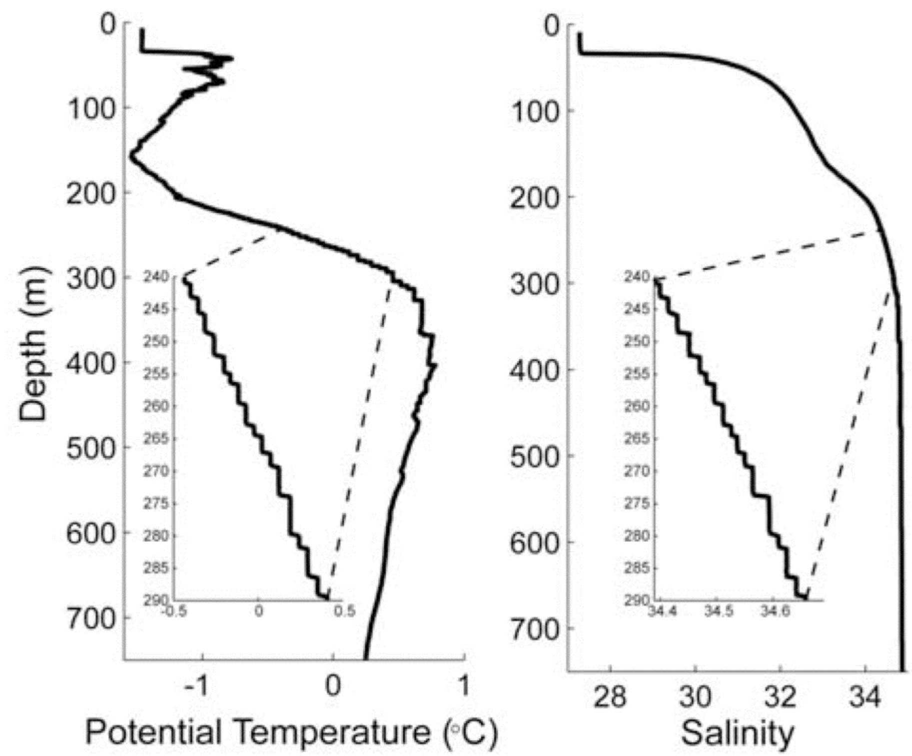


Double-diffusive convection



Double-diffusive staircases

- Double-diffusive staircases are often observed in the polar ocean
  - Layers are typically 10s of meters deep
  - Can have large horizontal extent, and persist for months or more
  - Layered convection also seen in laboratory experiments.
  - **Transport through staircase larger than through standard DD convection**



Timmermans et al. 2010

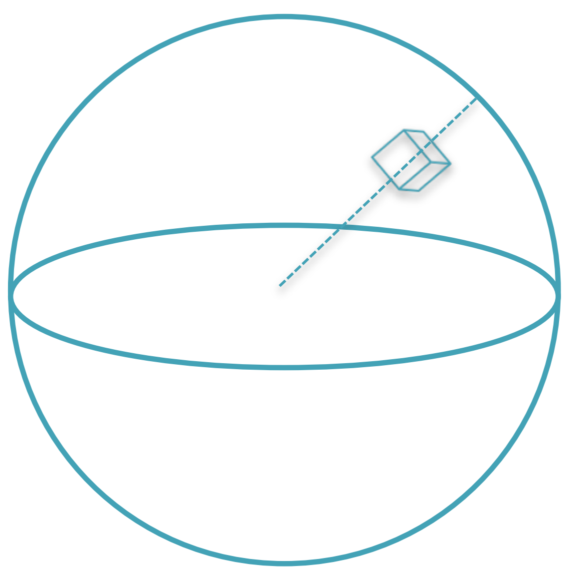
Double-diffusive convection

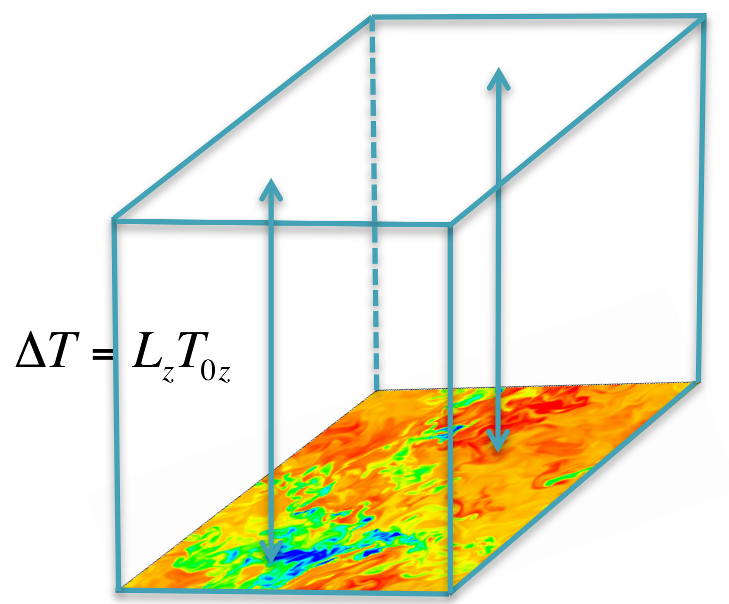
- What we expect naively (overturning convection)
- Double-diffusive convection
- Layered convection vs. non-layered convection
  - Numerical model and sample results
  - Criterion for layer formation
  - Transport through a staircase
- A new model for double-diffusive convection

Numerical model

Goal: to study phenomenon in astrophysical systems.

- Linear theory ➔ eddy scale much, much smaller than system scale.
- Model considered here:
  - Assume **background** temperature and concentration profiles are linear (constant gradients T_{0z}, T_{0z}^{ad}, μ_{0z})
  - Assume that all **perturbations** are triply-periodic in domain (L_x,L_y,L_z):





Mathematical model

- Governing non-dimensional equations:

$$\frac{1}{\Pr}\left(\frac{\partial u}{\partial t} + u\cdot\nabla u\right) = -\nabla p + (T-\mu)\mathbf{e_z} + \nabla^2 u$$

$$\frac{\partial T}{\partial t} + u\cdot\nabla T - w = \nabla^2 T$$

$$\frac{\partial \mu}{\partial t} + u\cdot\nabla \mu - R_0^{-1} w = \tau\nabla^2 \mu$$

$$\nabla\cdot u = 0$$

$$\Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta\mu_{0z}}{\alpha\left|T_{0z} - T_{0z}^{ad}\right|}$$

$$[l] = d = \left(\frac{\kappa_T \nu}{\alpha g\left|T_{0z} - T_{0z}^{ad}\right|}\right)^{1/4}, \quad [t] = \frac{d^2}{\kappa_T}, \quad [T] = d\left|T_{0z} - T_{0z}^{ad}\right|, \quad [\mu] = \frac{\alpha}{\beta} d\left|T_{0z} - T_{0z}^{ad}\right|$$

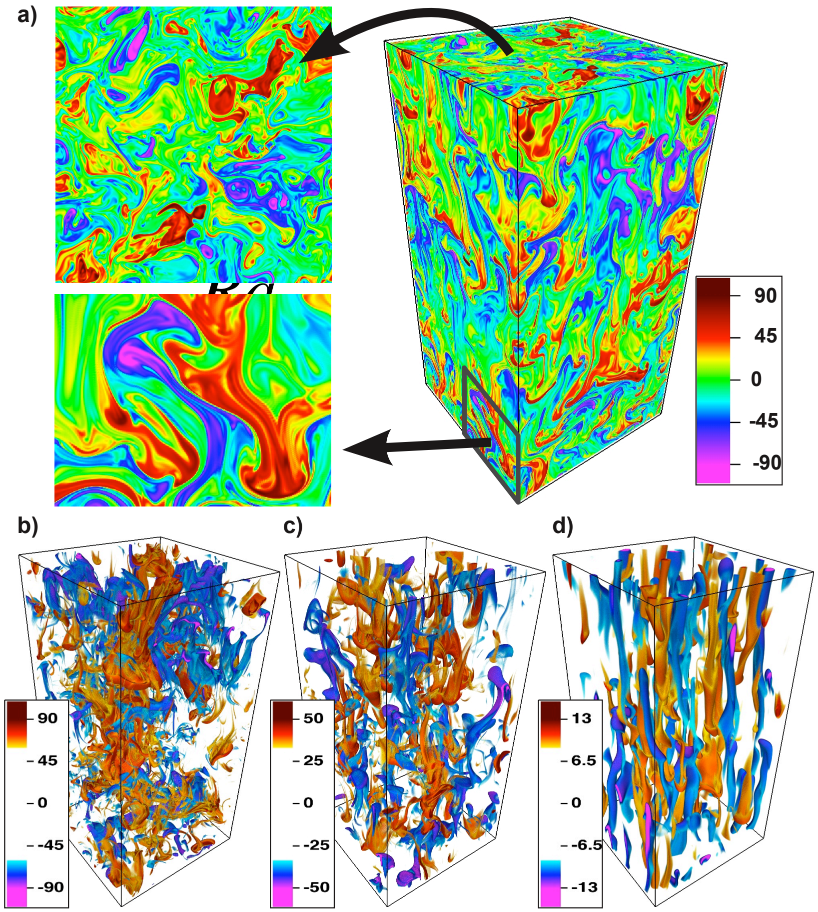
Numerical model

- Stephan Stellmach developed high-performance 3D code to study double-diffusive convection
- Code solves non-dimensional equations described earlier, for input parameters:

$$\mathrm{Pr},\ \tau,\ R_0^{-1}$$

$$L_x, L_y, L_z$$

- Code is pseudo-spectral, triply periodic, DNS.



Example of fingering convection in salt water.

$\mathrm{Pr} = 7$

$\tau = 0.01$

Numerical model

- In astrophysical systems, typical parameters Pr and τ are <<1 because thermal diffusion increased by photon transport while other diffusion coefficients are not.
  - Planetary interiors: $\mathrm{Pr},\ \tau \approx 10^{-2}$
  - Stellar interiors: $\mathrm{Pr},\ \tau \approx 10^{-6}$

- The stellar parameter regime is not achievable numerically. Planetary regime on the other hand is accessible to DNS.

- We ran a series of numerical experiments with decreasing Pr, τ

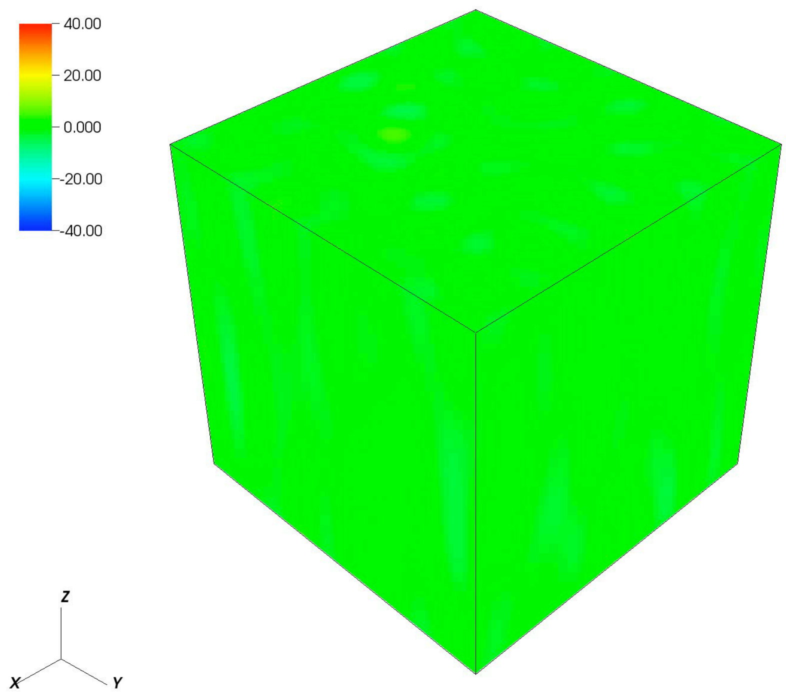
Set	1	2	3	4	5	6	7
Pr	1/3	1/10	1/30	1/100	1/3	1/10	1/3
τ	1/3	1/10	1/30	1/100	1/10	1/3	1/30

$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha \left| T_{0z} - T_{0z}^{ad} \right|}$$

Numerical model

Sample results for double-diffusive convection



$$\mathrm{Pr} = 0.03, \tau = 0.03, R_0^{-1} = 1.5$$

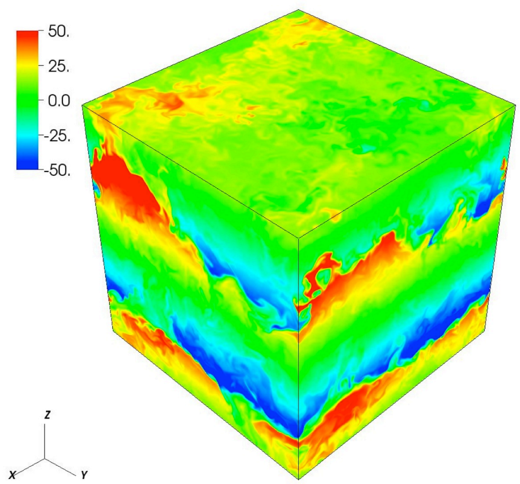
$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

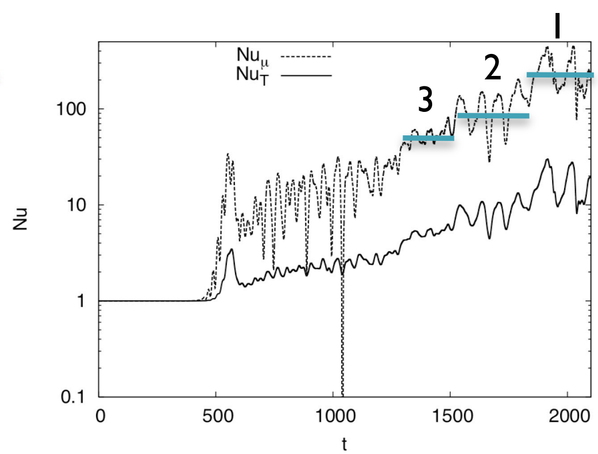
$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha \left| T_{0z} - T_{0z}^{ad} \right|}$$

Numerical model

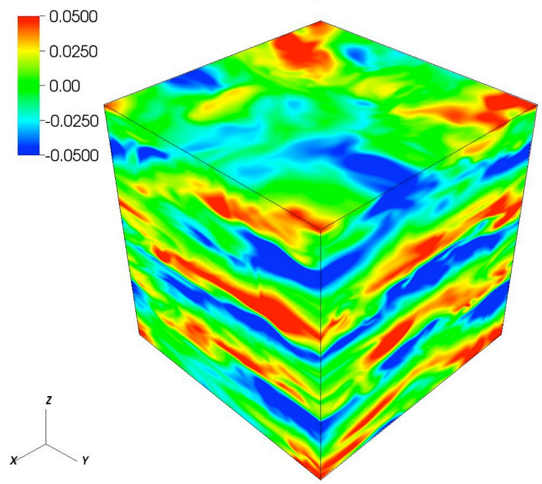
Sample results for double-diffusive convection: two possible outcomes

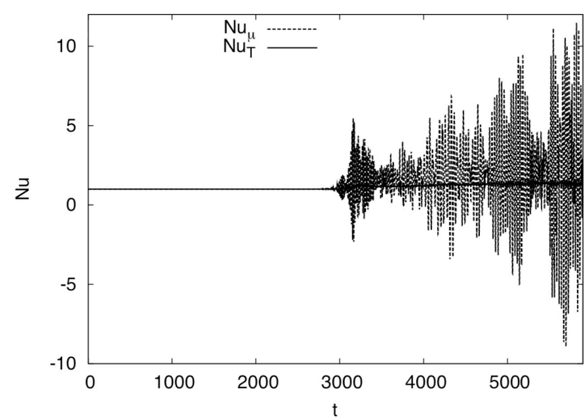
$$\mathrm{Pr} = 0.03$$
$$\tau = 0.03$$
$$R_0^{-1} = 1.5$$





$$\mathrm{Pr} = 0.03$$
$$\tau = 0.03$$
$$R_0^{-1} = 5$$





$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha \left| T_{0z} - T_{0z}^{ad} \right|}$$

Outstanding questions

Three fundamental questions:

- Can we predict when staircases form/don't form?
- Can we understand what controls transport through a staircase as well as the dynamics of mergers?
- Can we understand what controls transport in the absence of staircases?

Double-diffusive convection

- What we expect naively (overturning convection)
- Double-diffusive convection
- Layered convection vs. non-layered convection
  - Numerical model and sample results
  - Criterion for layer formation
  - Transport through a staircase
- A new model for double-diffusive convection

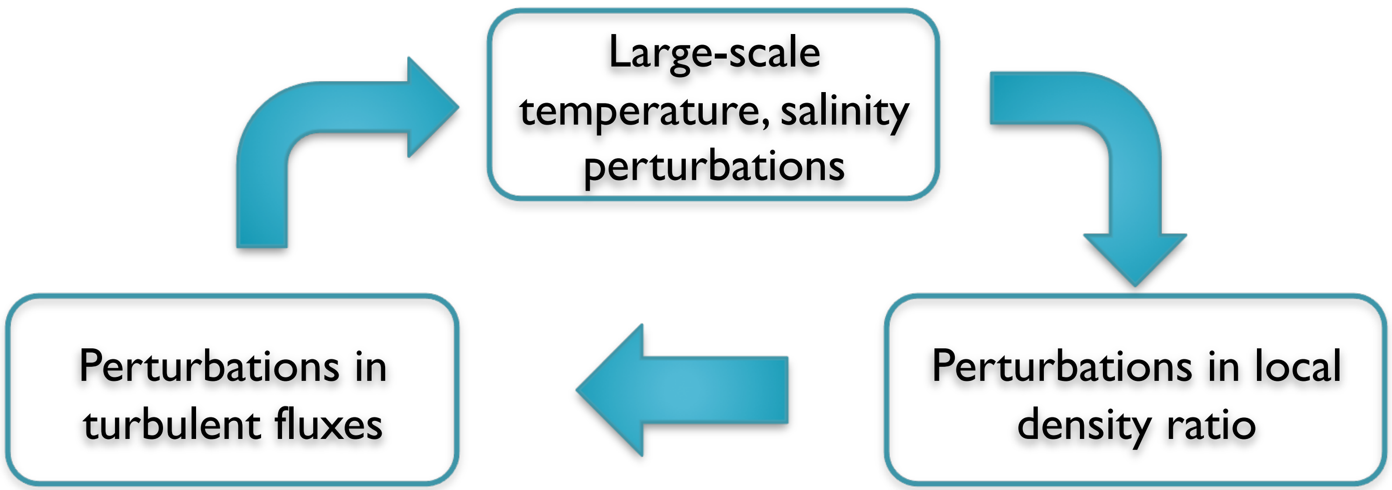
Criterion for staircase formation

- Emergence of large-scale structures in double-diffusive convection can be understood using "mean-field" theory
  - Long tradition of this approach for fingering (thermohaline) convection in the ocean: Stern & Turner, 1969; Walsh & Ruddick, 1995; Stern et al. 2001; Radko 2003. …

- Mean-field theory (Radko 2003, Rosenblum et al. 2011)
  - Assume that emerging staircase scale >> basic instability scale
  - Spatially average governing equations over small scales
  - Use empirically motivated closure to model turbulent transport by the small-scales
  - Study the resulting evolution of the large-scale fields

Criterion for staircase formation

- Physical interpretation of mean-field instability: positive feedback between large-scale temperature/composition perturbation and induced fluxes.

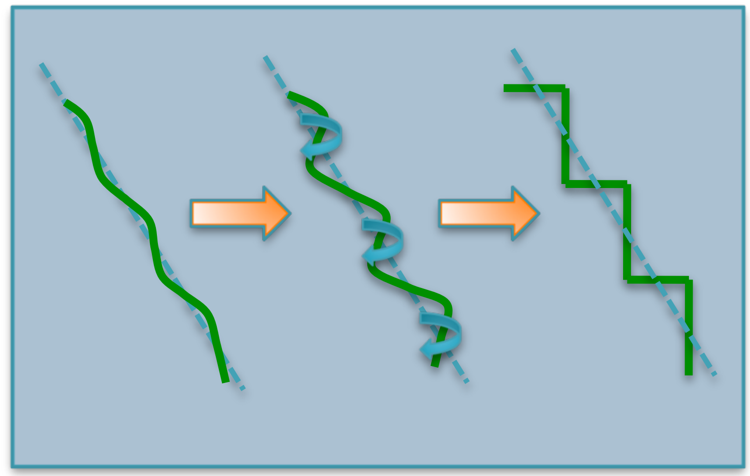


- Different feedback loops can lead to different "mean-field" instabilities, e.g. layering instability, large-scale gravity wave excitation, intrusive instability.

Criterion for staircase formation

Layering instability:

- Modes of instability are horizontally invariant, vertically sinusoidal perturbations in temperature/composition/density.
- The mode overturns into a staircase when amplitude is large enough.
- A necessary condition for the layering instability is that flux ratio γ_{tot}^{-1} should be a decreasing function of density ratio R_0^{-1} : Radko's γ -instability.



$$\gamma_{tot}^{-1} = \frac{\text{Total buoy. flux from composition}}{\text{Total buoy. flux from heat}}$$

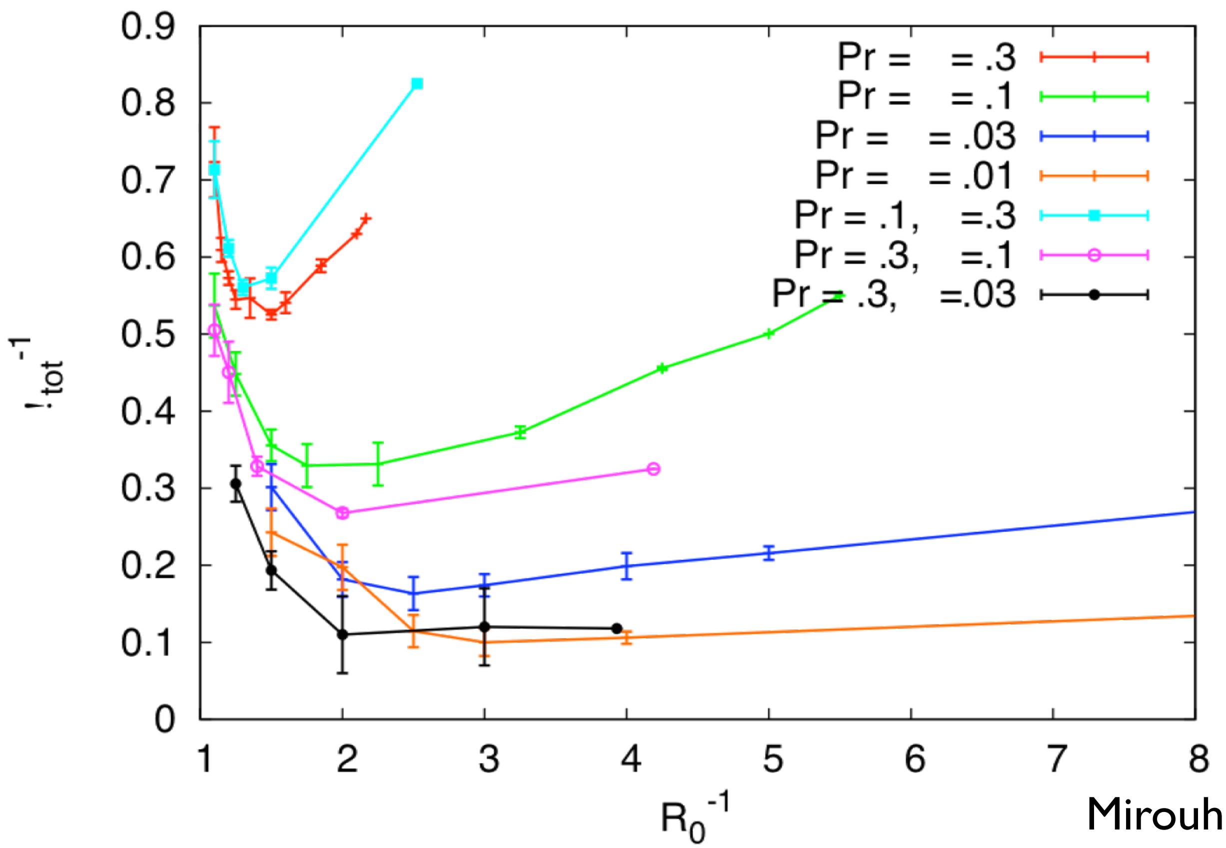
$$= \frac{-\kappa_\mu \mu_z + < w\mu >}{-\kappa_T T_z + < wT >}$$

$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha \left| T_{0z} - T_{0z}^{ad} \right|}$$

Criterion for staircase formation

- Since the layering instability occurs only when γ_{tot}^{-1} is a decreasing function of R_ρ^{-1}, knowing when staircases are expected boils down to measuring the function $\gamma_{tot}^{-1}(R_\rho^{-1})$



Mirouh et al. in prep.

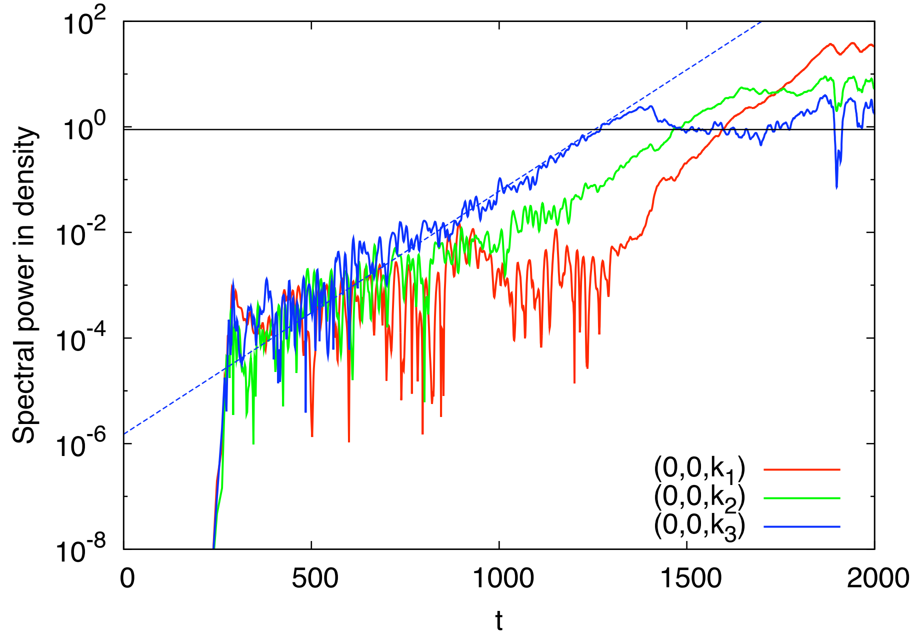
$$\gamma_{tot}^{-1} = \frac{-\kappa_\mu \mu_z + < w\mu >}{-\kappa_T T_z + < wT >}$$

$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha |T_{0z} - T_{0z}^{ad}|}$$

Criterion for staircase formation

- We find that layers indeed form in regions with $\gamma_{tot}^{-1}(R_\rho^{-1})$ decreasing. Furthermore, the layer growth rate matches theory very well.

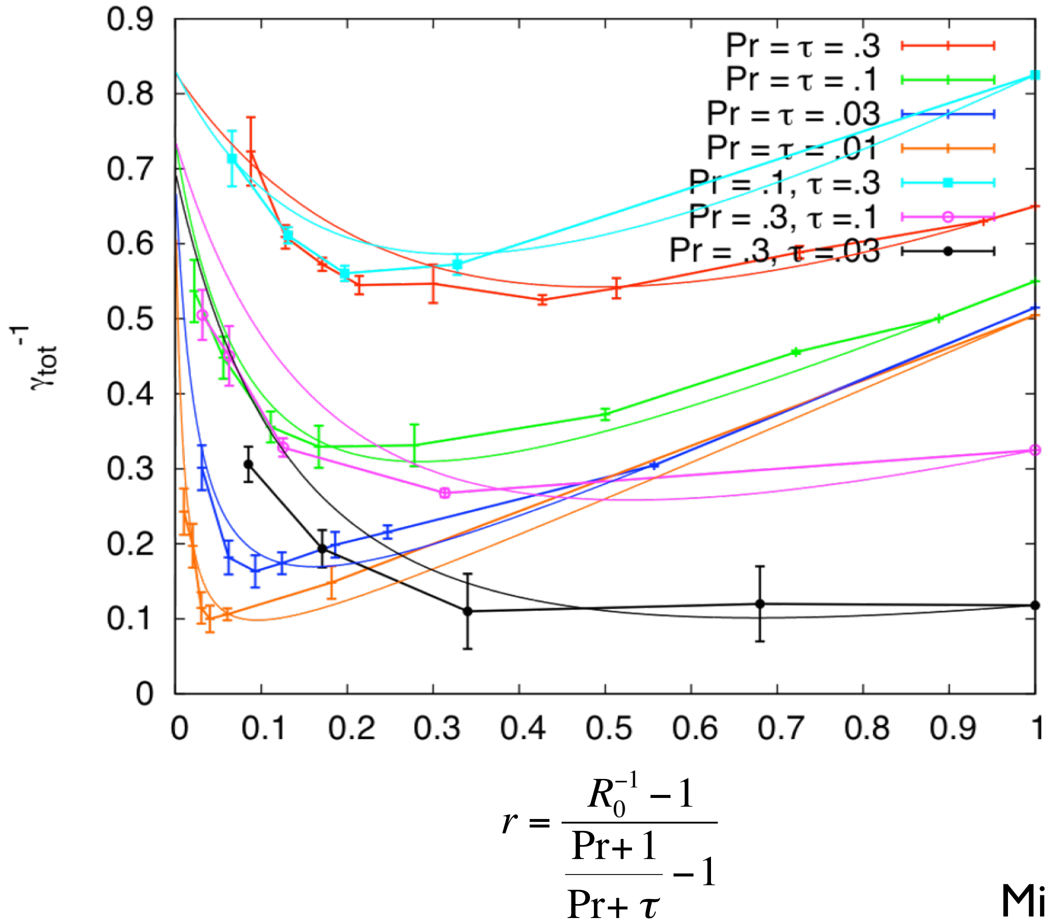


- **Problem:** how do we apply this idea to stellar interiors, which are in a parameter regime inaccessible to simulations?

Mirouh et al. in prep.

Criterion for staircase formation

- Solution: find a model to predict $\gamma_{tot}^{-1}(R_\rho^{-1})$ from linear theory (!).



$$\gamma_{tot}^{-1} = \frac{-\kappa_\mu \mu_z + < w\mu >}{-\kappa_T T_z + < wT >}$$

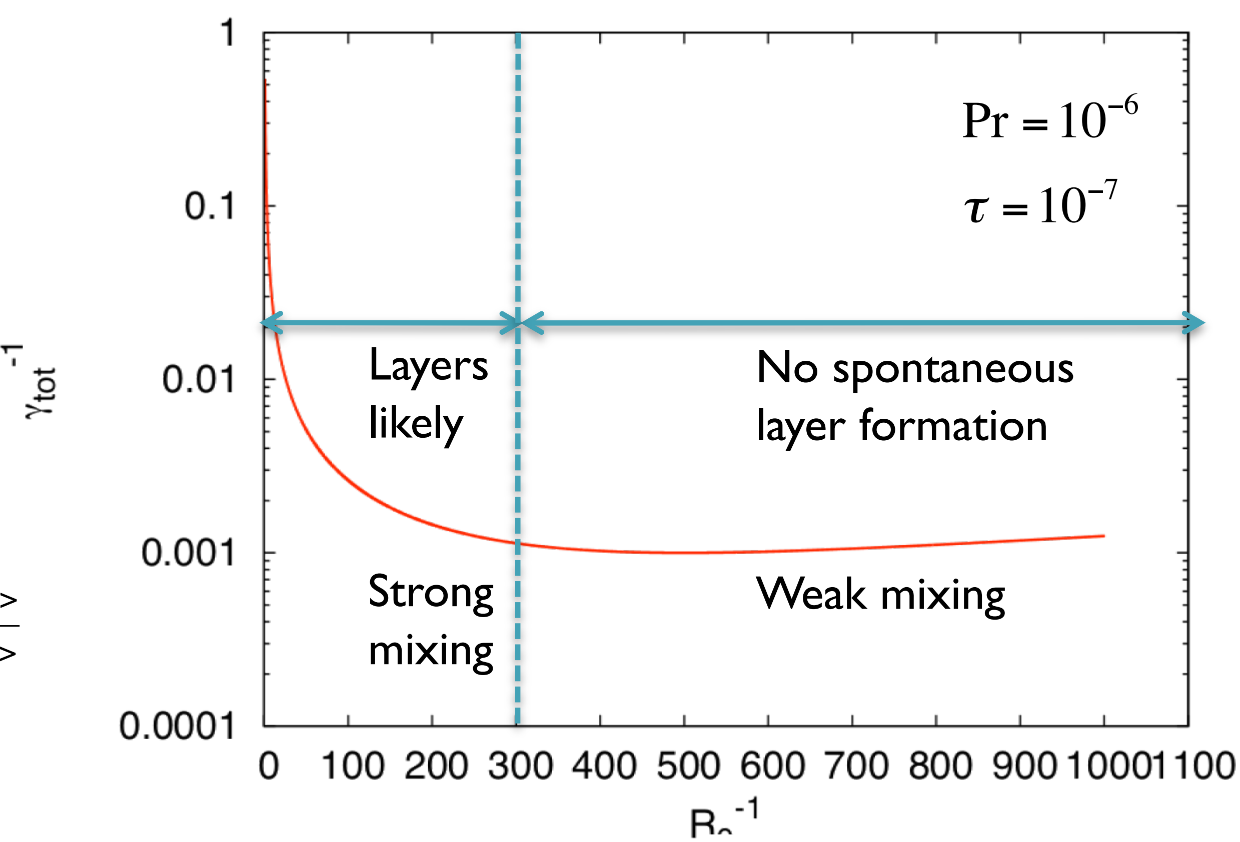
$$Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha |T_{0z} - T_{0z}^{ad}|}$$

Mirouh et al. in prep.

Criterion for staircase formation

- Predictions for stellar interiors:



$$\gamma_{tot}^{-1} = \frac{-\kappa_\mu \mu_z + \langle w\mu \rangle}{-\kappa_T T_z + \langle wT \rangle}$$

$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

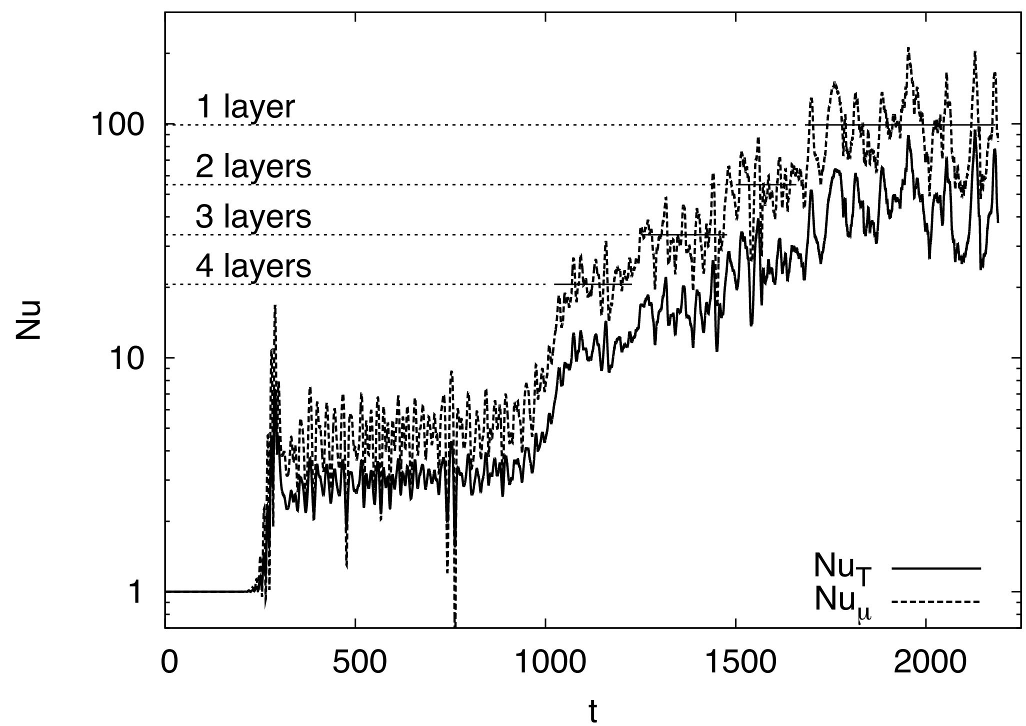
$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha \left| T_{0z} - T_{0z}^{ad} \right|}$$

Double-diffusive convection

- What we expect naively (overturning convection)
- Double-diffusive convection
- Layered convection vs. non-layered convection
  - Numerical model and sample results
  - Criterion for layer formation
  - Transport through a staircase
- A new model for double-diffusive convection

Staircase transport

- Staircase formation and each subsequent merger increases turbulent transport for both heat and composition.



$$Nu_T = \frac{\text{Total heat flux}}{\text{Diffusive heat flux}}$$

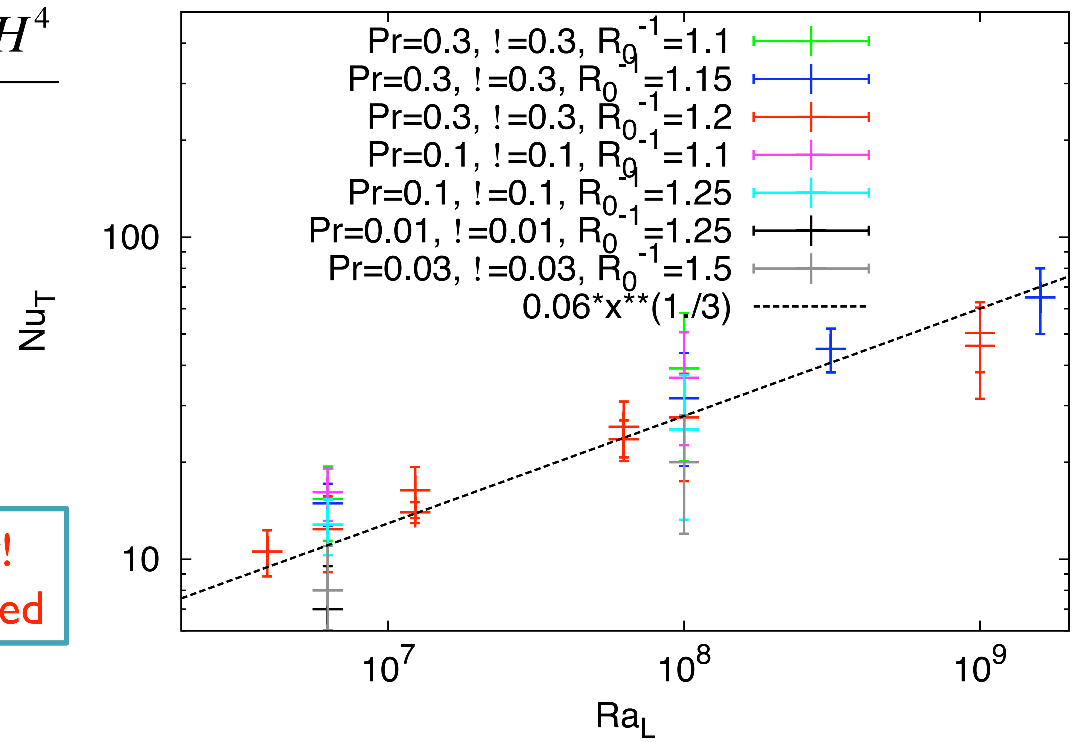
Rosenblum et al. 2011.

Staircase transport

- The heat transport properties in the layered convection case is "well" explained with Rayleigh-Benard scaling laws.
- The mixing rate depends mostly on layer height!

$$Ra_L = \frac{\alpha g \left| T_{0z} - T_{0z}^{ad} \right| H^4}{\nu \kappa_T}$$

$$Nu_T = 0.06 Ra_L^{1/3}$$



$$Nu_T = \frac{\text{Total heat flux}}{\text{Diffusive heat flux}}$$

$$\gamma_{tot}^{-1} = \frac{-\kappa_\mu \mu_z + \langle w\mu \rangle}{-\kappa_T T_z + \langle wT \rangle}$$

$$\Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha \left| T_{0z} - T_{0z}^{ad} \right|}$$

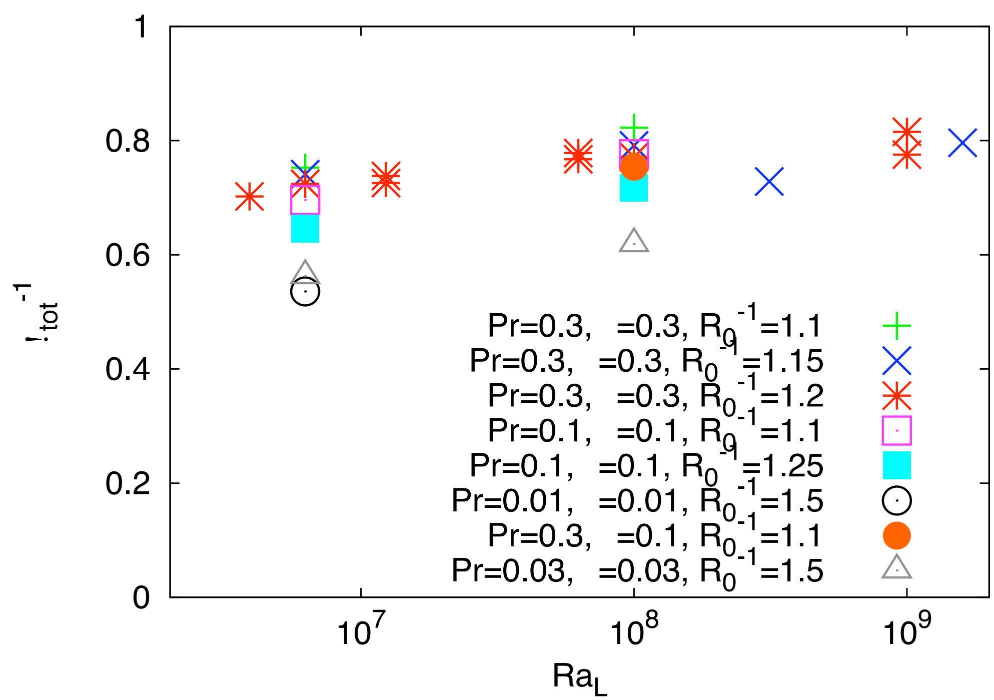
Preliminary!
To be verified

Rosenblum et al 2011, + new results in prep.

Staircase transport

- The compositional transport properties in the layered convection case seems to be "well" explained assuming that a more-or-less constant order-unity flux ratio

$$\gamma_{tot}^{-1} \approx 0.6 - 0.8$$



Rosenblum et al 2011, + new results in prep.

$$Nu_T = \frac{\text{Total heat flux}}{\text{Diffusive heat flux}}$$

$$\gamma_{tot}^{-1} = \frac{-\kappa_\mu \mu_z + \langle w\mu \rangle}{-\kappa_T T_z + \langle wT \rangle}$$

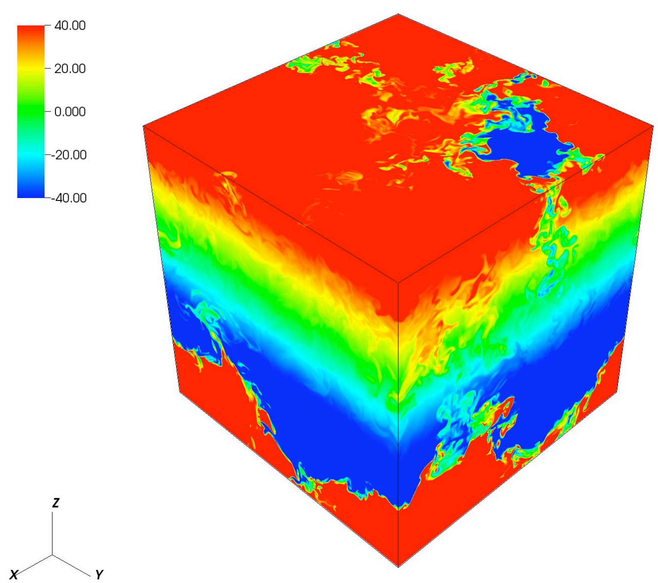
$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

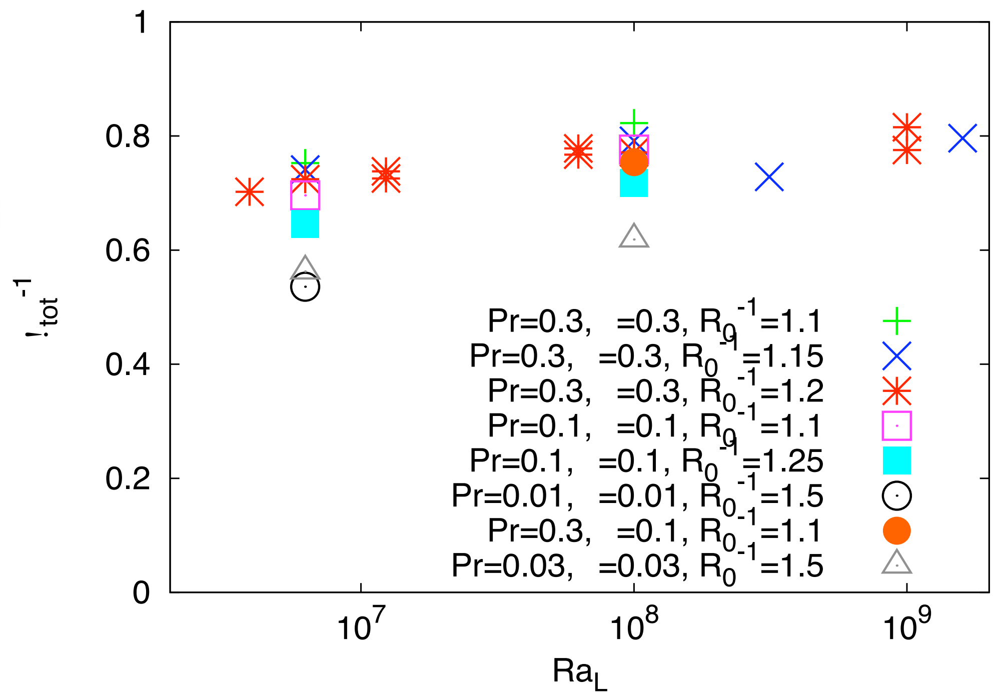
$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha \left| T_{0z} - T_{0z}^{ad} \right|}$$

Preliminary!
To be verified

Staircase transport

- This is in contrast with previous predictions for double-diffusive convection, which assumed that $\gamma_{tot}^{-1} \propto \tau^{-1/2}$
- This difference stems from the very turbulent nature of transport across layers in the low Prandtl number vs diffusive transport at higher Prandtl number.

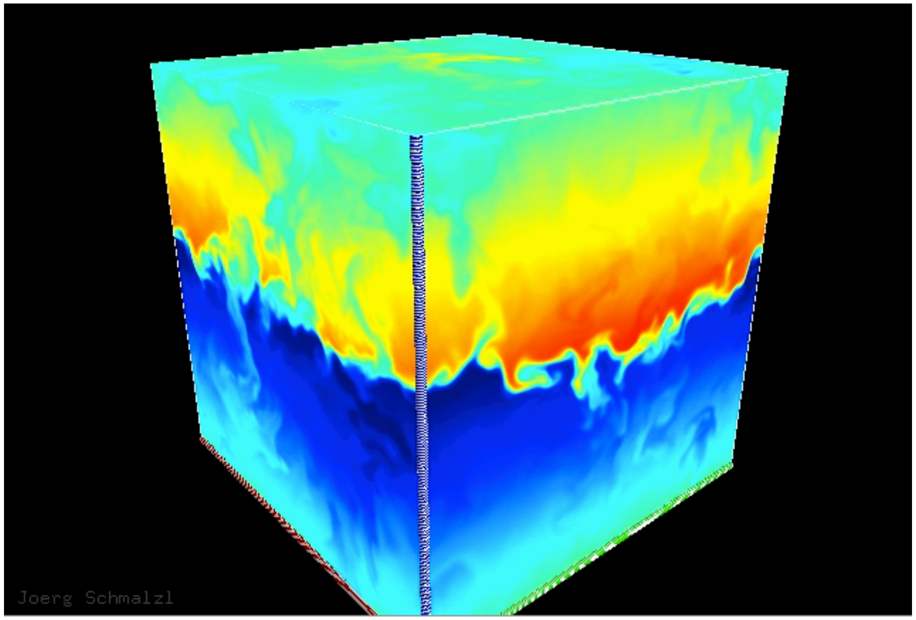




Rosenblum et al 2011, + new results in prep.

Staircase transport

- In all simulations performed to date (except one), layers merge until only one remains in the box. However, what if the box was larger?
- **OPEN QUESTION: what is the thickness of layers in real stellar/planetary interiors?**



Rosenblum et al 2011, + new results in prep.

Double-diffusive convection

- What we expect naively (overturning convection)
- Double-diffusive convection
- Layered convection vs. non-layered convection
  - Numerical model and sample results
  - Criterion for layer formation
  - Transport through a staircase
- A new model for double-diffusive convection

A new model for double-diffusive convection (summary of results so far)

- Semi-convection occurs for $1 < R_0^{-1} = \frac{\nabla_\mu}{\nabla - \nabla_{ad}} < \frac{\Pr + 1}{\Pr + \tau} = \frac{\nu + \kappa_T}{\nu + \kappa_\mu}$
- 2 possible outcomes: homogeneous or layered semi-convection
- Criterion for layer formation depends on $\gamma_{tot}^{-1}\left(R_0^{-1}\right)$ where

$$\gamma_{tot}^{-1} = \frac{\text{Total buoy. flux from composition}}{\text{Total buoy. flux from heat}}$$

- Numerical experiments reveal that

$$1 < R_0^{-1} < R_L^{-1} : \quad \text{efficient layered semi-convection}$$
$$R_0^{-1} > R_L^{-1} : \quad \text{very inefficient homogeneous semi-convection}$$

- Critical density ratio for spontaneous layer formation can be estimated semi-analytically.

$$\gamma_{tot}^{-1} = \frac{-\kappa_\mu \mu_z + \langle w\mu \rangle}{-\kappa_T T_z + \langle wT \rangle}$$

$$\Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_\mu}{\kappa_T}$$

$$R_0^{-1} = \frac{\beta \mu_{0z}}{\alpha \left| T_{0z} - T_{0z}^{ad} \right|}$$

A new model for double-diffusive convection (summary of results so far)

- Transport in layered convection depends on layer height with:
  - Heat transport scaling as Rayleigh Benard convection

$$Nu_T = \frac{\text{Total heat flux}}{\text{Radiative heat flux}} \propto \left(\frac{\alpha g \left| T_{0z} - T_{0z}^{ad} \right| H^4}{\kappa_T \nu} \right)^{1/3}$$

  - Compositional buoy. transport of same order as heat buoy. Transport

$$\gamma_{tot}^{-1} = \frac{\text{Total buoy. flux from composition}}{\text{Total buoy. flux from heat}} \approx 0.6 - 0.8$$

- What controls layer height remains to be determined…

Double-diffusive convection