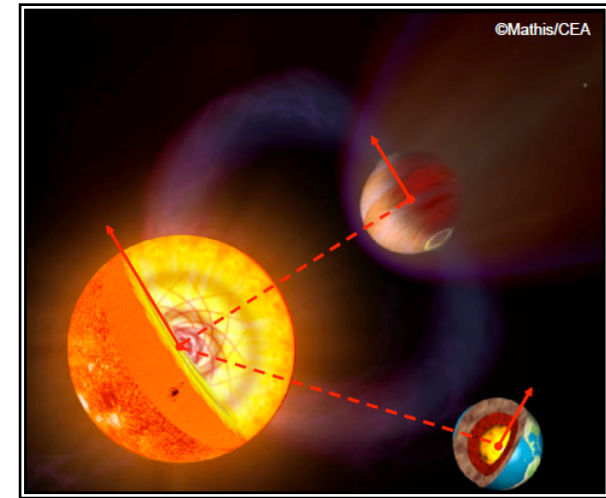


# ***Internal Waves Interaction with (differential) Rotation and Magnetic Field in Single Stars or in Stars Hosting a Companion***



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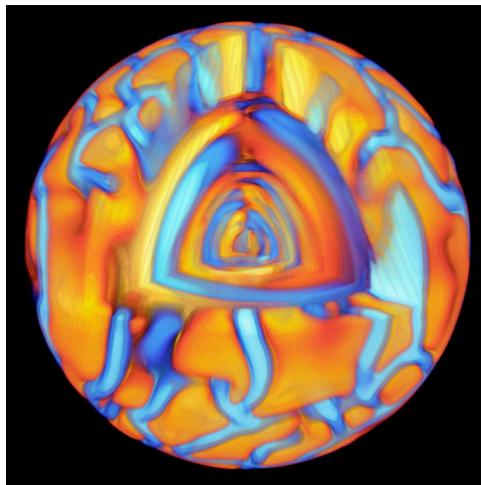


**The Impact of Asteroseismology across Stellar Astrophysics  
October 24 – 28 2011; KITP, UC Santa Barbara, USA**

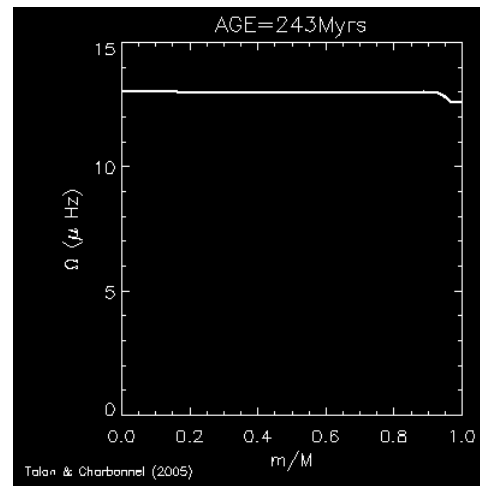
# Internal waves in stellar interiors

## Convective excitation

*Press 1981; Schatzman 1993; Zahn et al. 1997; Talon & Charbonnel 2005; Rogers et al. 2006-2008; Mathis et al. 2008; Mathis 2009; Brun, Miesch & Toomre 2011*



Brun, Miesch & Toomre 2011

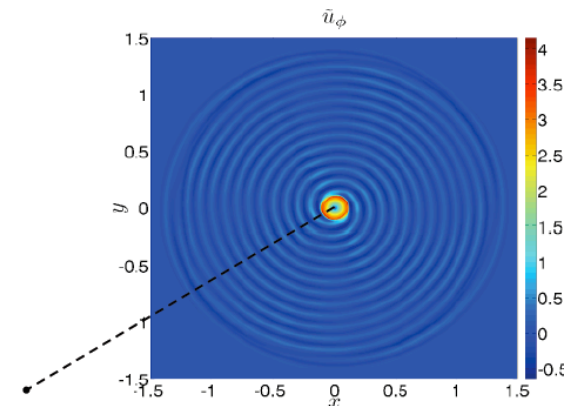
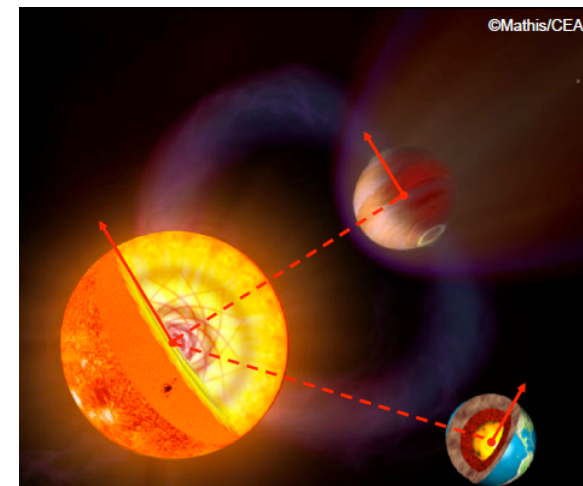


→ **Angular momentum transport**

## Tidal excitation

*Zahn 1975, Witte & Savonjie 1999-2001-2002, Barker & Ogilvie 2010, Barker 2011*

*Friday session*

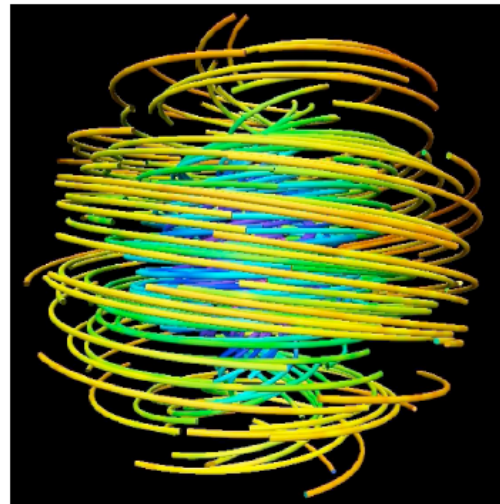
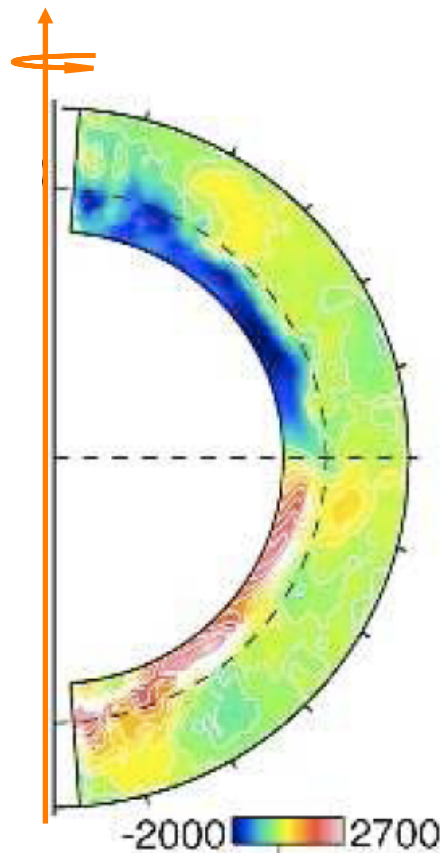


# Internal waves region of excitation and propagation

Complex magnetic fields

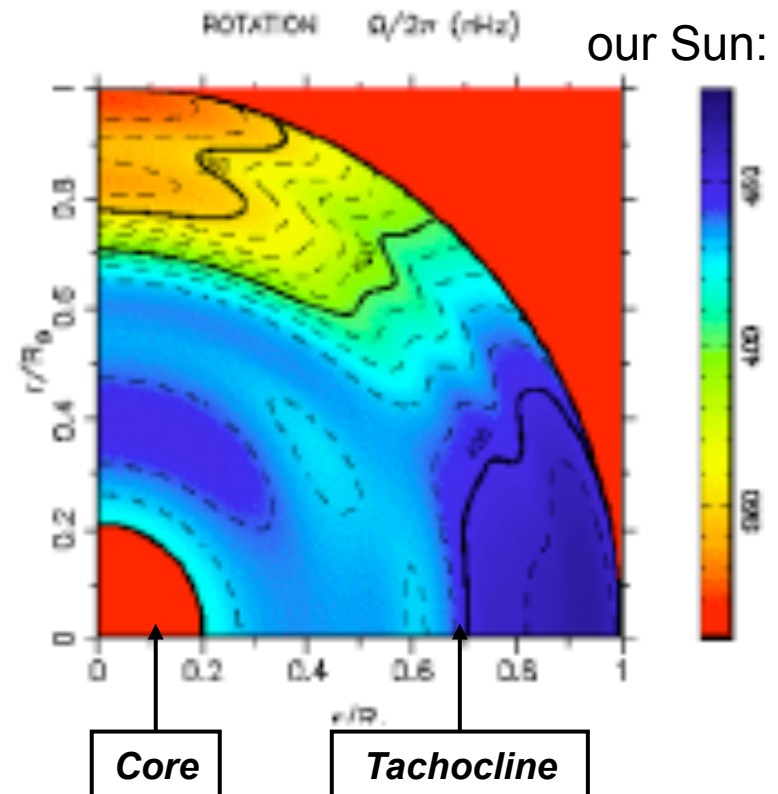
- *Dynamo Browning et al. 2006*

$\Omega$  - *Fossil Duez, Mathis & Braithwaite 2010*



Differential rotation

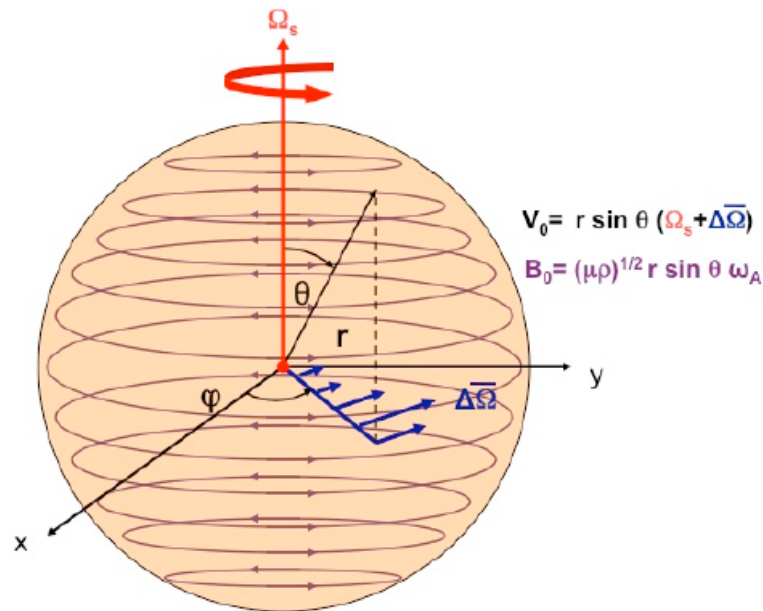
*Schou et al. 1998, Garcia et al. 2007, Eff-Darwich et al. 2008*



**A coherent picture of internal wave mechanisms**

→ **needs to take into account the (differential) rotation and magnetic fields**

# A first global Magneto-Gravito-Inertial waves set-up



- Velocities:

$$V(r, t) = V_0(r, t) + \underline{u(r, t)} \text{ with } V_0 = r \sin \theta \Omega(r, \theta) \hat{e}_\varphi$$

Wave's velocity field

$$\bar{\Omega}(r) = \underline{\Omega_s} + \underline{\Delta\bar{\Omega}(r)}, \text{ where } \Delta\bar{\Omega}(r) \ll \Omega_s$$

Uniform rotation:  
waves structure

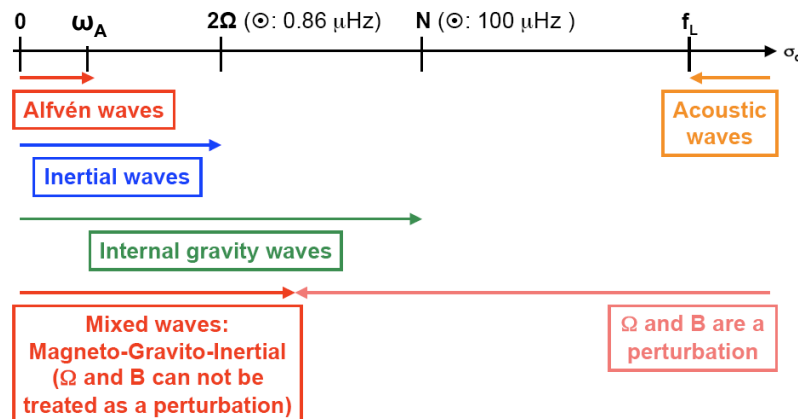
Differential rotation:  
thermal diffusion

-Magnetic fields:

$$B(r, t) = B_0^T(r, t) + \underline{b(r, t)} \text{ with } B_0^T = \sqrt{\mu\rho} r \sin \theta \omega_A \hat{e}_\varphi$$

Wave's magnetic field

Uniform Alfvén  
frequency



$$\sigma_s^2 \approx (\widehat{B} \cdot \widehat{k})^2 V_A^2 + (N \times \widehat{k})^2 + 4(\Omega \cdot \widehat{k})^2$$

- Local approach: Schatzman 1993; Kumar, Talon & Zahn 1999; Kim & McGregor 2003
- Equatorial modelling: Rogers & Mc Gregor 2010-2011

# The Magneto-Gravito-Inertial waves dynamics - I

Friedlander 1987-1989;  
Mathis & de Brye 2011

- Induction equation ( $q = \eta/K \ll 1$ )

$$\mathbf{b} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0^T) \longrightarrow \mathbf{b} = \sqrt{\mu \bar{\rho}} \omega_A \partial_\varphi \boldsymbol{\xi}$$

- Momentum equation ( $P_r = \nu/K \ll 1$ )

$$\left( \partial_t + \Omega_s \partial_\varphi \right) \left[ \left( \partial_t + \Omega_s \partial_\varphi \right) \boldsymbol{\xi} + 2 \Omega_s \widehat{\mathbf{e}}_z \times \boldsymbol{\xi} \right] =$$

$$-\frac{1}{\bar{\rho}} \nabla \Pi(r, t) - \nabla \widetilde{\Phi} + \frac{\bar{\rho}}{\bar{\rho}^2} \nabla \bar{P} + \frac{F_{\mathcal{L}}^{\text{Te}}(\boldsymbol{\xi})}{\bar{\rho}}$$

Wave's total pressure

$$\Pi = \bar{P} + \frac{\mathbf{B}_0^T \cdot \mathbf{b}}{\mu}$$

Wave's volumetric  
magnetic tension force

$$F_{\mathcal{L}}^{\text{Te}}(\boldsymbol{\xi}) = \frac{1}{\mu} \left[ (\mathbf{B}_0^T \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{B}_0^T \right]$$

$$= \bar{\rho} \omega_A^2 \left[ \partial_{\varphi^2} \boldsymbol{\xi} + 2 \widehat{\mathbf{e}}_z \times \partial_\varphi \boldsymbol{\xi} \right]$$

- Continuity equation: *anelastic approximation*
- Energy equation: *regime dominated par thermal diffusion*; i.e.  $P_r$  &  $q \ll 1$
- Poisson's equation: *the Cowling's approximation is assumed*

# The Magneto-Gravito-Inertial waves dynamics - II

Using an expansion in Fourier's series  $\exp(im\varphi)\exp(i\sigma t)$

$$u' = i\sigma_s \xi'$$

$$b' = im\sqrt{\mu\bar{\rho}}\omega_A \xi'$$

$$-\mathcal{A}\xi' + i\mathcal{B}\widehat{e}_z \times \xi' = -\nabla W' + \frac{\rho'}{\bar{\rho}^2} \nabla \bar{P}$$

Gravito-inertial waves like  $\rightarrow$  Poincaré equation

$$0 < \mathcal{A} = \sigma_M^2 = \sigma_s^2 - m^2\omega_A^2$$

Vertical trapping if  $\mathcal{A} < 0$

$$\mathcal{B} = 2(\Omega_s\sigma_s - m\omega_A^2)$$

*Braginsky & Roberts 1975;  
Friedlander 1987-1989; Mathis & de Brye 2011*

The strong stratification case: the MHD Traditional Approximation

In stellar radiation zones  $S_\Omega = \frac{N}{2\Omega_s}$  and  $S_B = \frac{N}{\omega_A} \ll 1 \rightarrow$  asymptotic expansion

# M.-G.-I. waves angular structure under MHD TA

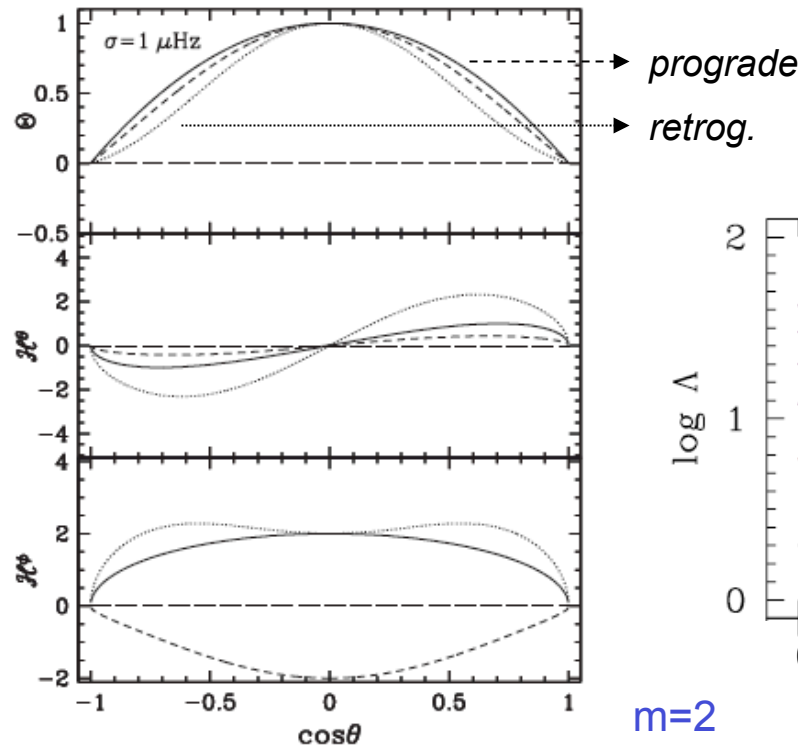
M.-G.-I. waves horizontal eigenfunctions:

Hough functions (eigenfunctions of the **Laplace Tidal Operator**; Laplace 1799, Hough 1898)

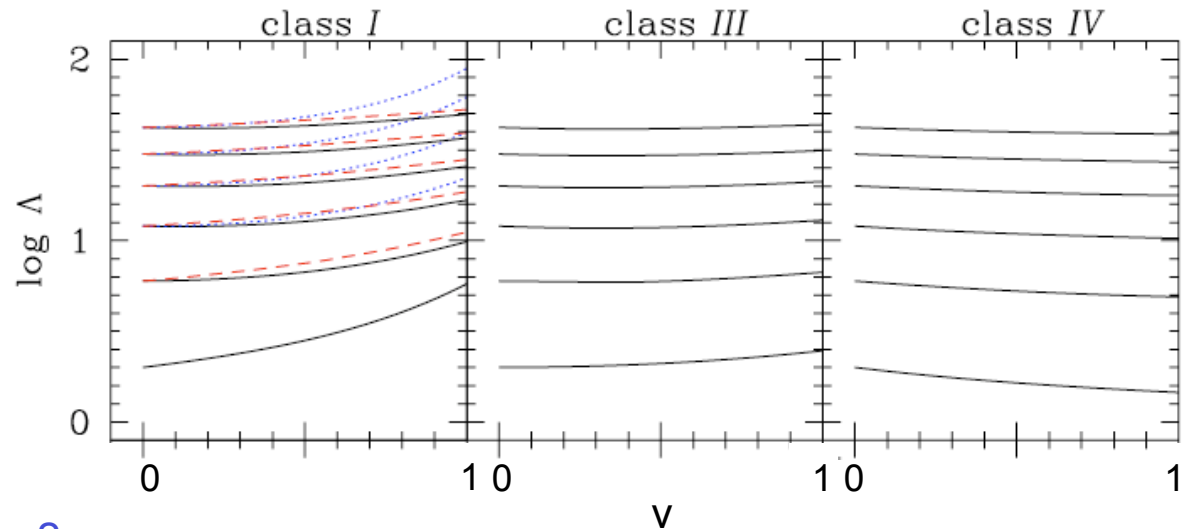
$$\mathcal{L}_{v_{M;m}} [\Theta_{k,m}(x; v_{M;m})] = -\Lambda_{k,m}(v_{M;m}) \Theta_{k,m}(x; v_{M;m})$$

$$\mathcal{L}_{v_{M;m}} \equiv \frac{d}{dx} \left( \frac{1-x^2}{1-v_{M;m}^2 x^2} \frac{d}{dx} \right) - \frac{1}{1-v_{M;m}^2 x^2} \left( \frac{m^2}{1-x^2} + m v_{M;m} \frac{1+v_{M;m}^2 x^2}{1-v_{M;m}^2 x^2} \right)$$

$$v_{M;m} = R_o^{-1} \frac{1 - m \Lambda_E}{1 - \frac{m^2}{2} R_o^{-1} \Lambda_E} \quad \left\{ \begin{array}{l} R_o = \frac{\sigma_s}{2\Omega_s} \\ \Lambda_E = \frac{\omega_A^2}{\Omega_s \sigma_s} \end{array} \right.$$



$m=2$





# The regular Magneto-Gravito-Inertial waves structure

Wave velocity and magnetic fields

$$\mathbf{u} = \sum_{j=\{r,\theta,\varphi\}} \left[ \sum_{\sigma,m,k} u_{j;k,m}(\mathbf{r}, t) \right] \widehat{\mathbf{e}}_j$$

$$u_{r;k,m} = -\mathcal{E}_{k,m}(r) \Theta_{k,m}(\cos \theta; \nu_{M;m}) \sin [\zeta_{k,m}(r, \varphi, t)] \\ \times \exp \left[ -\frac{\tau_{k,m}(r; \nu_{M;m}, \Delta \bar{\Omega})}{2} \right],$$

same form in the  $\theta$  &  $\varphi$  directions

$$\mathbf{b} = \sum_{j=\{r,\theta,\varphi\}} \left[ \sum_{\sigma,m,k} b_{j;k,m}(\mathbf{r}, t) \right] \widehat{\mathbf{e}}_j$$

$$b_{j;k,m} = \sqrt{\mu \bar{\rho}} \omega_A \frac{m}{\sigma_s} u_{j;k,m}.$$

- Wave propagation function

$$\zeta_{k,m}(r, \varphi, t) = \int_r^{r_c} k_{V;k,m}(r') dr' + m\varphi + \sigma_s t \quad k_{V;k,m} \equiv \left( \frac{N}{\sigma_M} \right) \frac{\Lambda_{k,m}^{1/2}(\nu_{M;m})}{r} \\ \equiv F_r^{-1} \left( 1 - \frac{m^2}{2} R_o^{-1} \Lambda_E \right)^{-1/2} \frac{\Lambda_{k,m}^{1/2}(\nu_{M;m})}{r}$$

- Wave damping

$$\underline{\tau_{k,m}(r; \nu_{M;m}, \Delta \bar{\Omega})} = \Lambda_{k,m}^{3/2}(\nu_{M;m}) \int_r^{r_c} K \frac{N_T^2 N}{\tilde{\sigma}_m \tilde{\sigma}_{M;m}^3} \frac{dr'}{r'^3} \quad \begin{cases} \tilde{\sigma}_m(r) = \sigma_s + m \Delta \bar{\Omega}(r) \\ \tilde{\sigma}_{M;m}(r) = \tilde{\sigma}_m - m^2 \omega_A^2 \end{cases}$$



# Magneto-Gravito-Inertial waves propagation

## Control parameters

- MHD local frequency:  $\mathcal{A} = \sigma_M^2 = \sigma_s^2 - m^2 \omega_A^2$

- MHD TA control parameter:  $v_{M;m} = R_o^{-1} \frac{1 - m\Lambda_E}{1 - \frac{m^2}{2} R_o^{-1} \Lambda_E}$

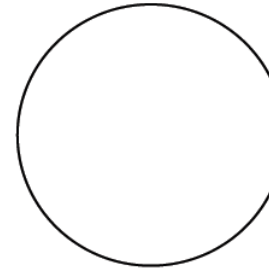
$m > 0$  - retrograde

$m < 0$  - prograde

$$R_o = \frac{\sigma_s}{2\Omega_s} \quad \Lambda_E = \frac{\omega_A^2}{\Omega_s \sigma_s}$$

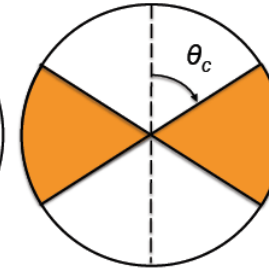
Vert.  
trapping

T ( $A \leq 0$ )



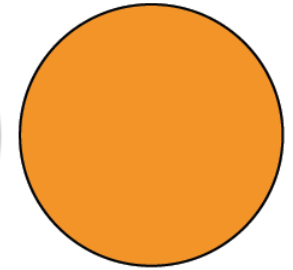
Equatorial trapped  
waves

H ( $A > 0; v_M \geq 1$ )

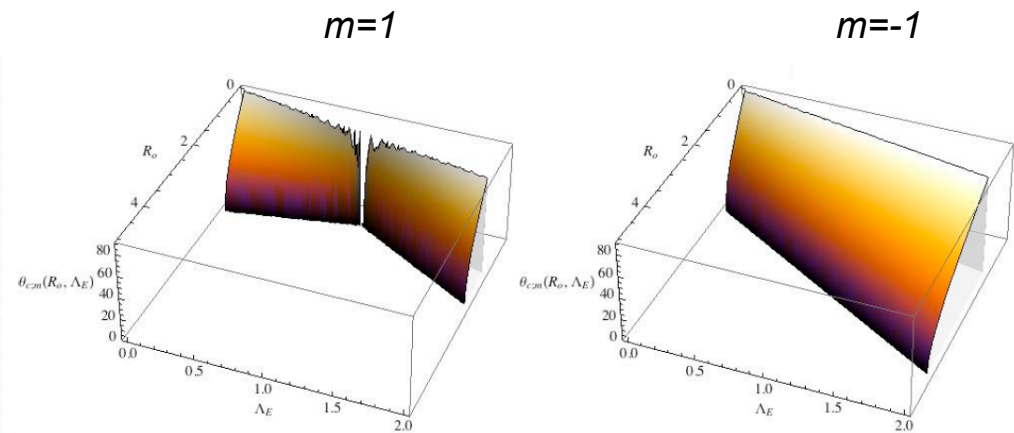
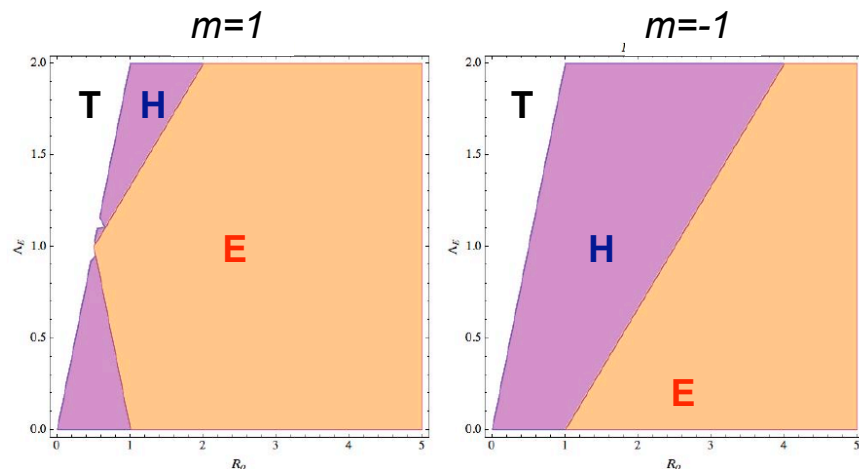


Regular waves  
& MHD T. A.

E ( $A > 0; v_M < 1$ )



$$\theta_{c;m}(v_{M;m}) = \arccos(|v_{M;m}|^{-1})$$



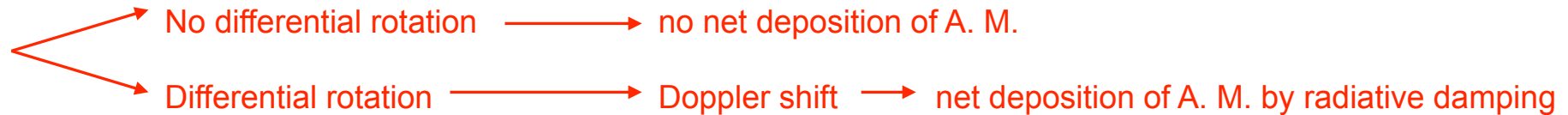
→ Net bias between pro & retrograde waves:  $\theta_c(\text{prograde}) > \theta_c(\text{retrograde})$

# Regular M.-G.-I. waves classification

- **Class I** (*gravity waves + Coriolis + Lorentz force*)
- **Class II** (*conservation of specific vorticity + curvature + Lorentz force, retrograde waves*; Rossby waves); the angular momentum flux is negative
- **Class III** (*mixed class I & class II*; Yanai waves): idem class I
- **Class IV** (*conservation of specific vorticity + stratification + Lorentz force, prograde waves*; Kelvin waves); the angular momentum flux is positive

# Transport of Angular Momentum by internal waves

If prograde and retrograde waves are equally excited:



$m > 0$  - retrograde (extraction)

$m < 0$  - prograde (deposit)

Example: dynamical evolution of a  $1M_{\odot}$  star with a magnetic braking ( $V_i = 50 \text{ Km.s}^{-1}$ );  $\langle \Omega \rangle_{\theta}$

High degree waves below the convection zone:

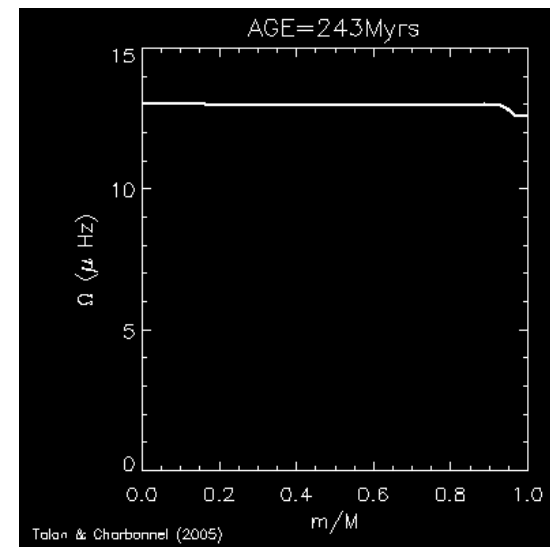
Shear Layer Oscillation (or not?)

Transport by low degree ( $l \leq 10$ ), low frequency waves ( $\nu < 5 \mu\text{Hz}$ )

Secular A. M. extraction driven by the wind (S.L.O. filtered out)

$\longrightarrow$  **nearly uniform rotation profile (cf. solar R. Z.)**

*Talon &  
Charbonnel 2005  
(see also Rogers et al. 2008  
& Belkacem et al. 2008)*



# Action of angular momentum

Definition:

$$\begin{aligned}\mathcal{L}_V^{\text{AM}}(r, \theta) &= \sum_{\sigma, k, m} \left\{ r^2 \mathcal{F}_{V; k, m}^{\text{AM}} \right\} = \sum_{\sigma, k, m} \left\{ -\frac{m}{\sigma_s} \left( r^2 \mathcal{F}_{V; k, m}^{\text{E}} \right) \right\} \\ &= r^2 \sum_{\sigma, k, m} \left\{ \underbrace{\mathcal{F}_{V; k, m}^{\text{Re}}(r, \theta)}_{\text{Lagrangian wave's Reynolds stresses}} + \underbrace{\mathcal{F}_{V; k, m}^{\text{Ma}}(r, \theta)}_{\text{Lagrangian wave's Maxwell stresses}} \right\} \quad \text{Energy flux at the borders with CZ}\end{aligned}$$

Grimshaw 1984  
Mathis & de Brye 2011

$$\left\{ \begin{aligned}\mathcal{F}_{V; k, m}^{\text{Re}} &= \bar{\rho} r \sin \theta \left\langle u_{r; k, m} \left( u_{\varphi; k, m} + \sigma_s R_0^{-1} \cos \theta \xi_{\theta; k, m} \right) \right\rangle_{\varphi} \\ \mathcal{F}_{V; k, m}^{\text{Ma}} &= \\ &- \bar{\rho} r \sin \theta m R_0^{-1} \Lambda_E \left\langle u_{r; k, m} \left( \frac{m}{2} u_{\varphi; k, m} + \sigma_s \cos \theta \xi_{\theta; k, m} \right) \right\rangle_{\varphi}\end{aligned}\right.$$

→ act against  
Reynolds stresses  
and scales as  
 $(\omega_A/\sigma_s)^2$

The case of solar type stars: energy flux < 0

- prograde waves ( $m < 0$ ) → angular momentum flux < 0: deposit
- ondes rétrogrades ( $m > 0$ ) → angular momentum flux > 0: extraction

# Angular momentum transport

Angular momentum transport:

$$\bar{\rho} \frac{d}{dt} (r^2 \bar{\Omega}) = -\frac{3}{2} \frac{1}{r^2} \partial_r \overline{\mathcal{L}_V^{\text{AM}}}$$

$$\begin{aligned} \overline{\mathcal{L}_V^{\text{AM}}}(r) &= \langle \mathcal{L}_V^{\text{AM}} \rangle_\theta \\ &= \sum_{\sigma, k, m} \overline{\mathcal{L}_{V; k, m}^{\text{AM}}}(r_c; \nu_{M; m}) \exp \left[ -\tau_{k, m}(r; \nu_{M; m}, \Delta \bar{\Omega}) \right] \end{aligned}$$

Excited  
spectrum

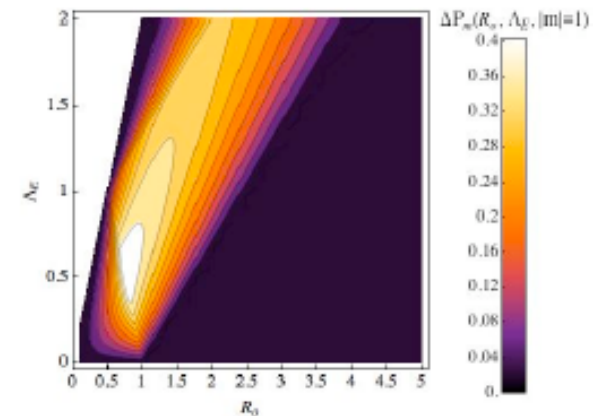
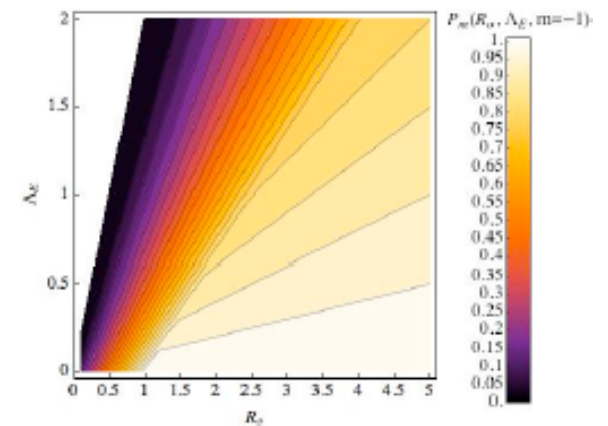
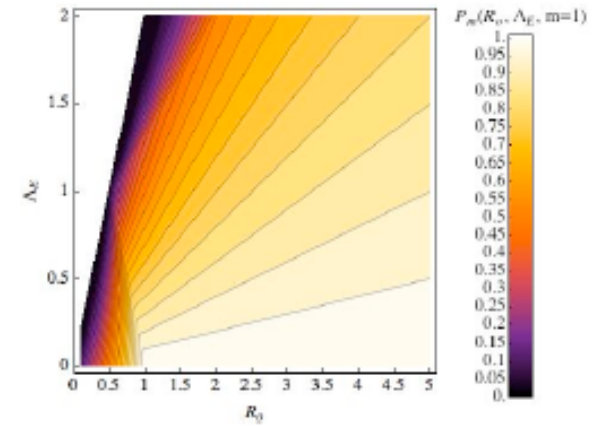
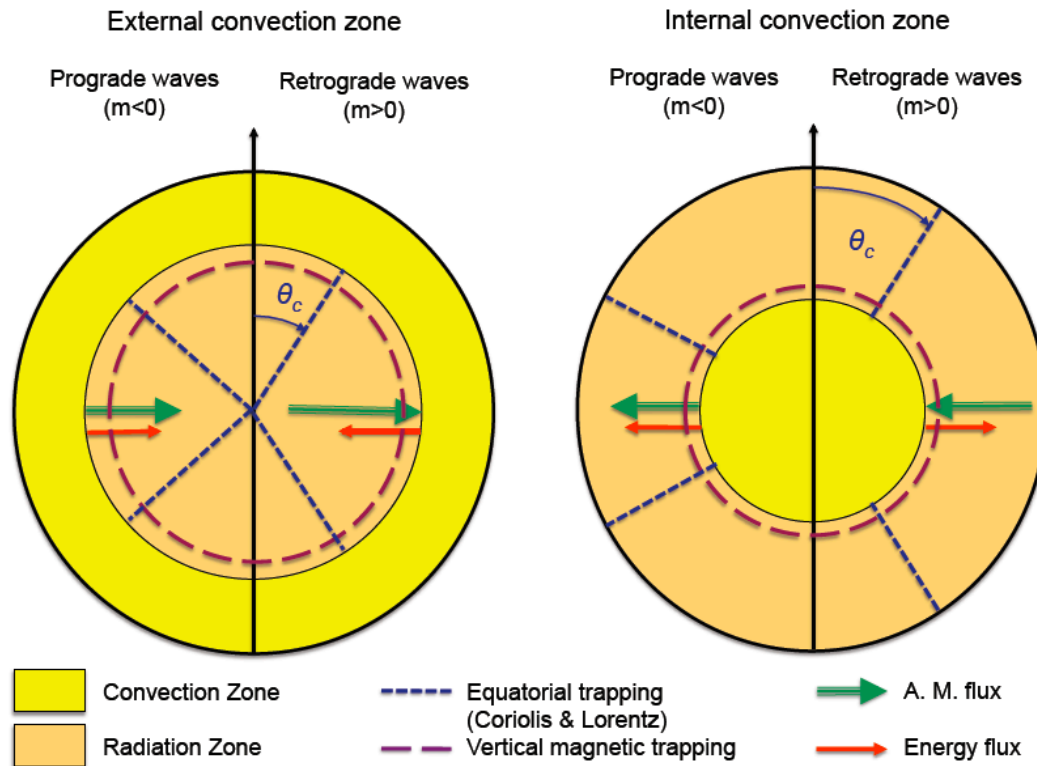
A.-M. flux at the borders with CZ      Radiative damping

$$\tau_{k, m}(r; \nu_{M; m}, \Delta \bar{\Omega}) = \Lambda_{k, m}^{3/2}(\nu_{M; m}) \int_r^{r_c} K \frac{N_T^2 N}{\tilde{\sigma}_m \tilde{\sigma}_{M; m}^3} \frac{dr'}{r'^3} \quad \begin{cases} \tilde{\sigma}_m(r) = \sigma_s + m \Delta \bar{\Omega}(r) \\ \tilde{\sigma}_{M; m}(r) = \tilde{\sigma}_m - m^2 \omega_A^2 \end{cases}$$

Radiative damping and Doppler effect:

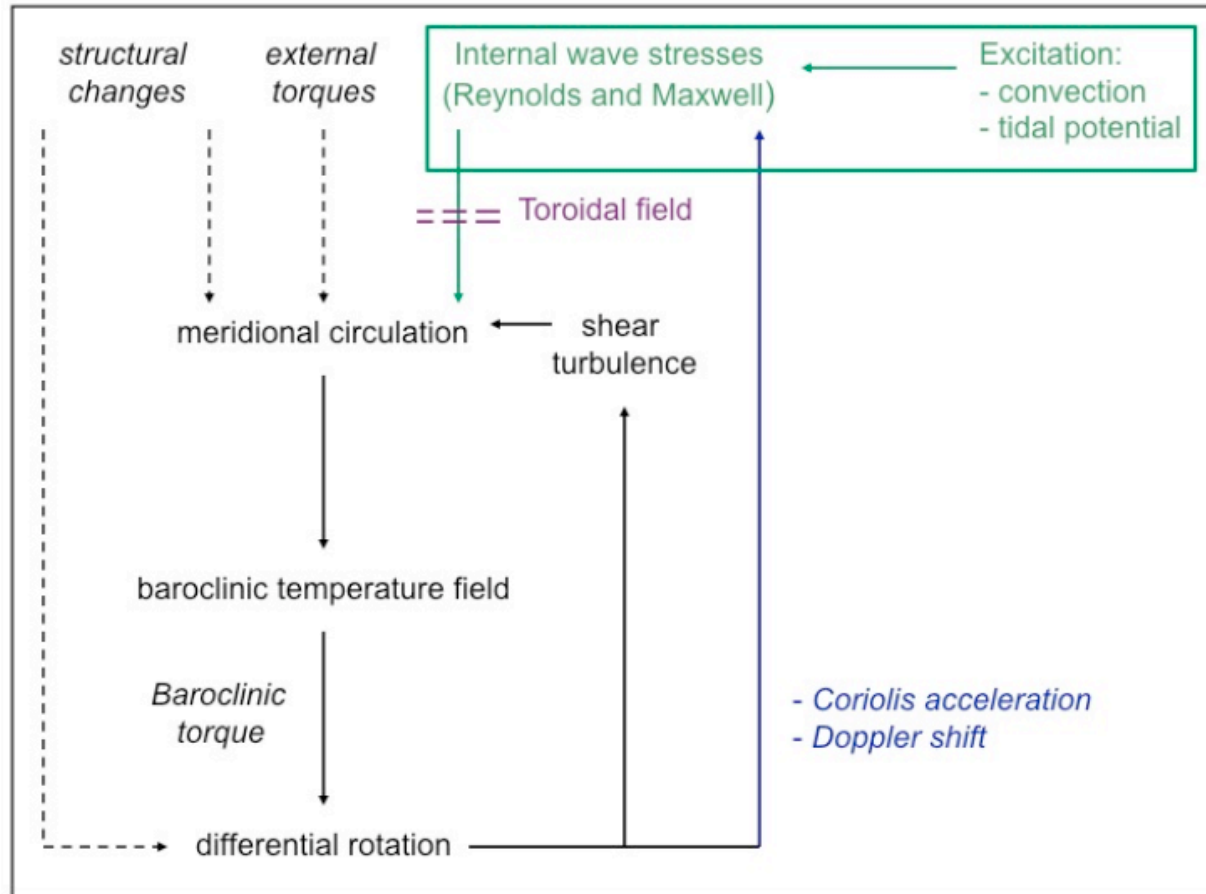
- Doppler effect:  $\sigma_m(\text{prograde}) < \sigma_m(\text{retrograde})$ : prograde waves damped before retrograde waves
- $\Lambda_{k, m}(\text{prograde}) < \Lambda_{k, m}(\text{retrograde})$ : reduces the bias between prograde and retrograde waves
- $\Lambda_{k, m} > \Lambda_{k, m}(\Omega \text{ \& } B_0 = 0)$ : waves are damped closer to their excitation region

# Excitation energy transmission

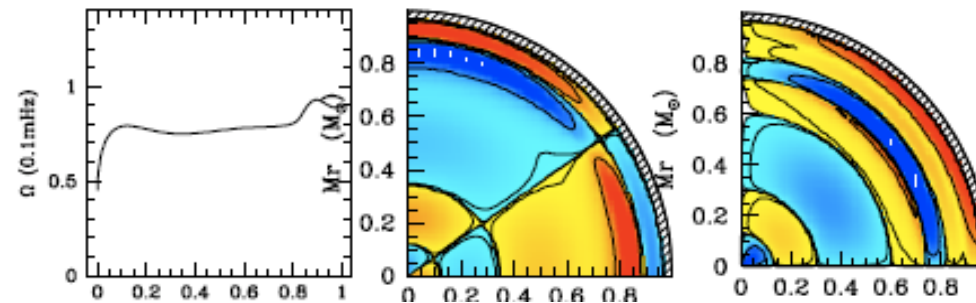


$$\begin{aligned}
 \mathcal{P}_m &= \underbrace{\left( \frac{\sigma_M}{\sigma_s} \right)^2}_{\text{vertical trapping}} \underbrace{\left[ \frac{1}{2\pi} \int_{\theta_{c,m}}^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\varphi \right]}_{\text{equatorial trapping}} \\
 &= \left( 1 - \frac{m^2}{2} R_o^{-1} \Lambda_E \right) [\cos \theta_{c,m} H_e(|v_{M,m}| - 1) + H_e(1 - |v_{M,m}|)]
 \end{aligned}$$

# Transport loop



$1M_{\odot}$ ,  $v_{ini} = 75\text{km/s}$ ,  $t = 6.815 \times 10^8 \text{ yr}$



Mathis & de Brye  
and Mathis et al. 2011



# Conclusion & prospects

## Conclusions

- **Global treatment** of low-frequency internal waves in a deep rotating and magnetised shell,
- The toroidal magnetic field induces a **vertical trapping which grows with  $(\omega_A/\sigma_s)^2$** ,
- The combined action of the Lorentz force and the Coriolis acceleration modifies waves angular structure which becomes different for prograde and retrograde waves: **the associated equatorial trapping is stronger for prograde waves that modifies the excitation energy transmission**,
- The **damping is enhanced as soon as rotation and magnetic field amplitude grow**.

## Prospects

- Combine with ASH numerical simulations of waves excitation and implementation in the hydrodynamical stellar evolution code STAREVOL,
- General differential rotation and toroidal magnetic fields ( $\Omega(r,\theta)$  &  $\omega_A(r,\theta)$ ),
- Poloidal geometry,
- Dynamo action.