Degeneracy and the mode classification of adiabatic oscillations of stars

Masao Takata

Department of Astronomy, University of Tokyo

October 26, 2011

Introduction: mode classification and identification

- mode classification:
  providing the definition of the radial order n of each eigenmode.

- mode identification:
  determining n, ℓ, and m of observed modes.

The problem of mode identification (of n) is meaningless if we do not have a proper answer to the problem of mode classification.

Current understanding of mode classification

The case of adiabatic oscillations of spherical stars:

radial modes ($\ell = 0$)	*solved*	*1
dipole modes ($\ell = 1$)	*solved*	*2
higher-degree modes ($\ell \geq 2$)	*unsolved*	*3

*1 Sturm–Liouville problem

*2 Momentum conservation helps (Takata 2006b)

*3 The problem has been solved only under the Cowling approximation (neglect of the perturbation to the gravitational field) (Scuflaire 1974; Osaki 1975)

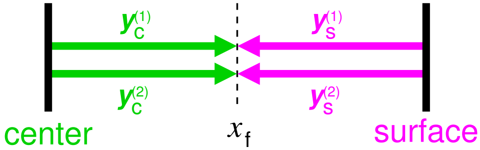
Tools of mode classification (1): basic equations

Equations of linear adiabatic stellar oscillations:

- a fourth-order system of ODEs (for each ℓ)
- squared frequency (ω^2) is a parameter
- boundary conditions:
  two at each of the center and the surface

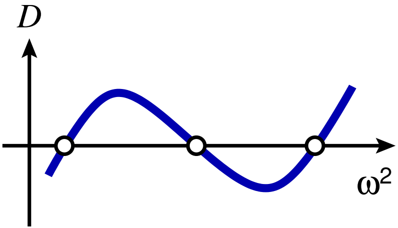
Tools of mode classification (2): discriminant of eigenmodes

Solution by the shooting method:



The condition for an eigenfrequency:

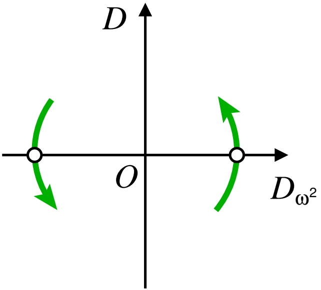
$$D\left(\omega^2\right) = \det\left(y_c^{(1)}(x_f) \quad y_c^{(2)}(x_f) \quad y_s^{(1)}(x_f) \quad y_s^{(2)}(x_f)\right) = 0$$



A key idea of mode classification: (D_{ω^2}, D) diagram

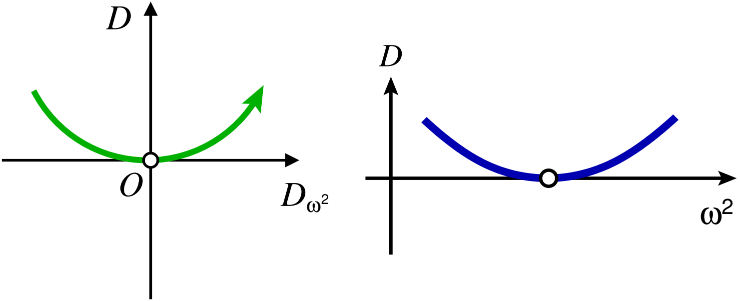
cf. Woodhouse (1988)

- Plot a point on the (D_{ω^2}, D) plane for each ω^2.
- An eigenmode is found every time the point comes on the abscissa axis.
- If ω^2 increases, the point always crosses the abscissa axis in the counterclockwise direction.
- Each eigenmode has a specific value of the polar angle $(n\pi)$.



The degeneracy problem (1)

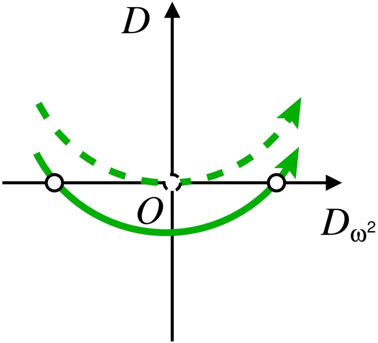
- A possible problem happens when $(D_{\omega^2}, D) = (0, 0)$.
- This corresponds to the case of degeneracy, which means, in this context, that two eigenmodes with the same ℓ have the same eigenfrequency.



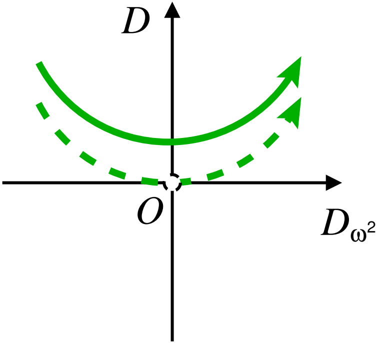
The degeneracy problem (2)

How do the degenerate modes react to *continuous change in the equilibrium structure*?

case 1: splitting

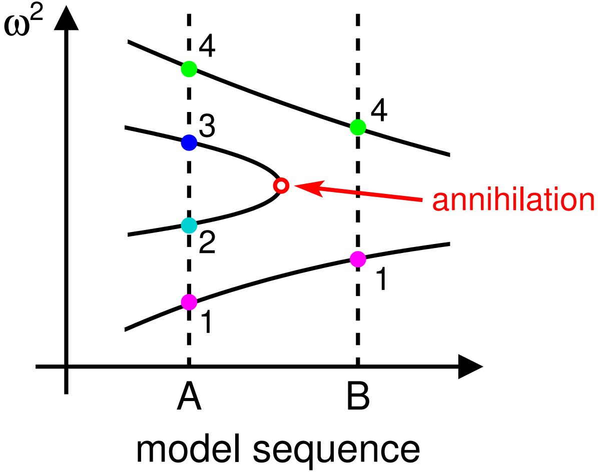


case 2: *annihilation*!!



The degeneracy problem (3)

Annihilation of the degenerate modes causes a serious problem to the mode classification.

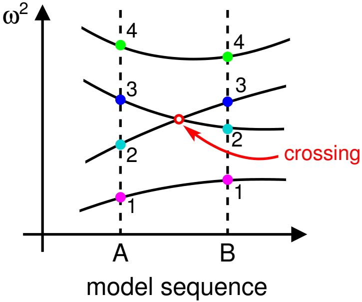


The degeneracy problem (4): resolution

Mathematical theories rescue us.

- The problem of adiabatic stellar oscillations can be formulated as an eigenvalue problem of a *self-adjoint* operator (e.g. Dyson & Schutz 1979).
- Eigenvalues of a self-adjoint operator are never lost by any perturbation, as far as the perturbed operator remains self-adjoint (Kato 1976).

Every degenerate eigenvalue must correspond to crossing (or merging) of two eigenmodes, which does not complicate the mode classification.



A scheme of mode classification (1)

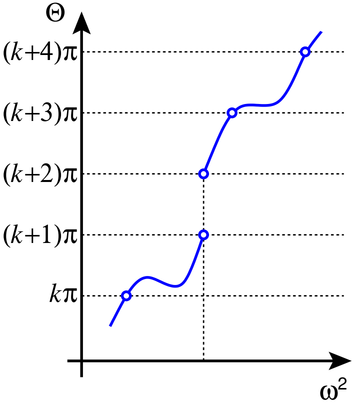
How can we know the polar angle on the (D_{ω^2}, D) plane?

- The sign of D can be calculated from the four solutions, $\boldsymbol{y}_{\mathbf{c}}^{(1)}$, $\boldsymbol{y}_{\mathbf{c}}^{(2)}$, $\boldsymbol{y}_{\mathbf{s}}^{(1)}$, and $\boldsymbol{y}_{\mathbf{s}}^{(2)}$.
- The sign of D_{ω^2} when $D = 0$ can also be evaluated from the four solutions.
- The angle Θ is introduced by

$$\tan\Theta = \frac{\text{[Expression of the sign of } D\text{]}}{\text{[Expression of the sign of } D_{\omega^2} \text{ when } D = 0\text{]}} .$$

A scheme of mode classification (2)

- The origin of Θ is adjusted so that the radial fundamental mode has $\Theta = \pi$.
- Each eigenmode is characterized by $\Theta = n\pi$, where n is interpreted as the radial order.
- Each pair of degenerate modes, if exist, is allocated two consecutive radial orders.



A scheme of mode classification (3)

Mode classification based on the radial order n

	$\ell = 0$	$\ell = 1$	$\ell \geq 2$
$n \geq 1$	$\mathbf{p}_n$ mode		
$n = 0$	—	$\mathbf{g}_1$ mode	**f** mode
$n \leq -1$	—	$\mathbf{g}_{\lvert n \rvert+1}$ mode	$\mathbf{g}_{\lvert n \rvert}$ mode

Summary

- A scheme has been proposed to classify eigenmodes of adiabatic oscillations of spherically symmetric stars.
- Degenerate modes do not complicate the mode classification. (They simply have two consecutive radial orders.)
- The scheme accurately lets us know which modes are found in a given frequency interval by computing Θ only at the boundaries of the interval.

Discussions (1)

Q. Does the radial order thus defined have any relation to the number of nodes of the eigenfunctions?

A. So far, the answer is no. We cannot compute the radial order only from the eigenvalue and the associated eigenfunctions. In other words, the information contained in the structure of the eigenfunctions is not sufficient to fix the radial order.

Discussions (2)

Q. What is the physical meaning of the radial order?

A. As we know in the case of mixed modes, p (g) modes are not purely composed of acoustic (gravity) waves. We probably need other indexes to represent to what extent a given eigenmode has a mixed character. Even in that case, the radial order proposed here would probably serve as the primary index to characterize the eigenmode.

Discussions (3)

Q. Any relation to or impact on observations?

A.
1. Mode identification of observed modes is meaningful only if we can classify the eigenmodes theoretically.
2. Better theoretical insights into the problem must be valuable when we interpret observational data.

Discussions (4)

Q. Rotation? Nonadiabatic case?

A. Those are all too advanced problems at this stage. We have just finished the simplest case, which has persisted for several decades!

References

- Dyson, J., & Schutz, B. F., 1979, Proc. R. Soc. Lond. A, 368, 389
- Kato, T., 1976, Perturbation Theory for Linear Operators, 2nd edn. (New York: Springer-Verlag)
- Osaki, Y., 1975, PASJ, 27, 237
- Scuflaire, R., 1974, A&A, 36, 107
- Takata, M., 2006a, PASJ, 58, 759
- Takata, M., 2006b, PASJ, 58, 893
- Woodhouse, J. H., 1988, in Seismological algorithms: computational methods and computer programs, ed. D. J. Doornbos (London: Academic Press Limited), 323