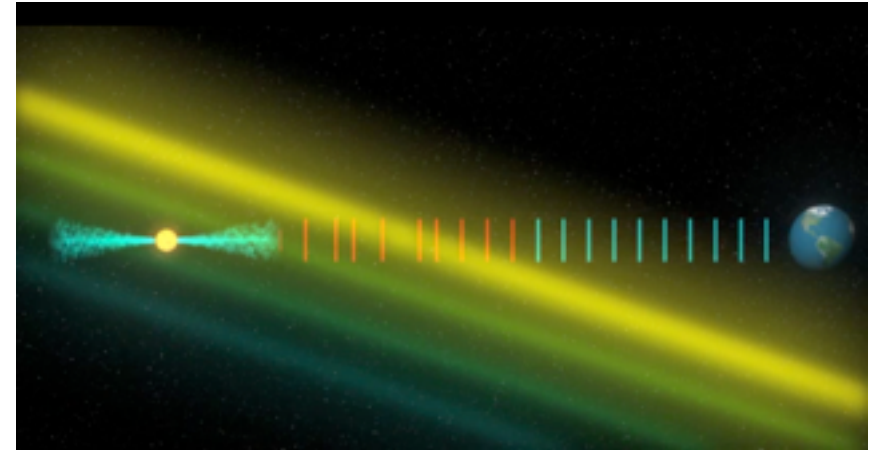
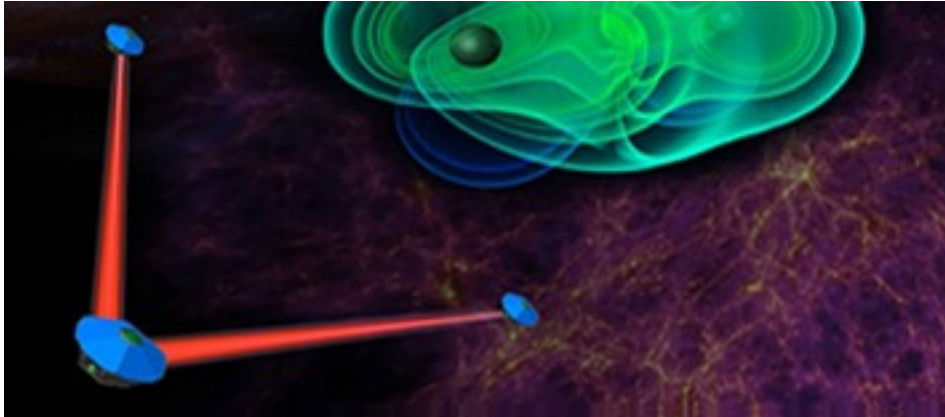


Gravitational waves from massive black hole binaries



Using gravitational-wave observations of black hole mergers to probe the growth of black holes from the early universe

Basic basics

Gravitational radiation *necessary* in any relativistic theory of gravity: Need a mechanism to causally communicate changes in the gravitational field.

In GR, tidal fields (“curvature”) play role similar to electric and magnetic fields in E & M ... radiation takes form of tidal gravitational field propagating from source.

Leading radiation *quadrupolar*: monopole violates conservation of energy; dipole violates conservation of momentum.

$$h = \frac{2G}{c^4} \frac{1}{r} \frac{d^2 Q}{dt^2}$$
$$\approx \frac{2G}{c^4} \frac{1}{r} \times mv^2$$

How to measure

The GW h is an oscillation in spacetime, has an impact on propagation of light:

Behavior of light in spacetime with wave:

$$ds^2 = -c^2 dt^2 + [1 + h(t, x)] dx^2 = 0$$

Solve for the speed of light in this coordinate system:

$$\frac{dx}{dt} = \frac{c}{\sqrt{1 + h(t, x)}}$$

How to measure

The GW h is an oscillation in spacetime, has an impact on propagation of light:

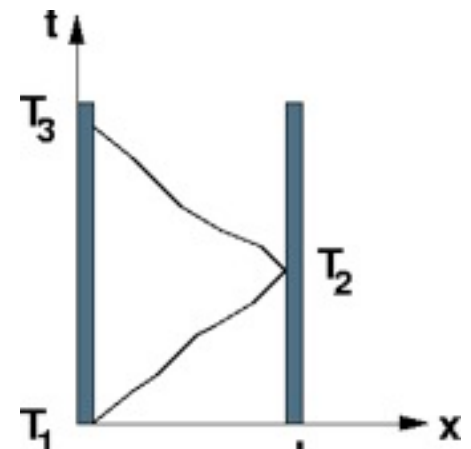
Behavior of light in spacetime with wave:

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Solve for the speed of light in this coordinate system:

$$\frac{dx}{dt} = \frac{c}{\sqrt{1 + h(t, x)}}$$

Now imagine that we have mirrors which fall freely in this spacetime. Bounce light between mirrors, record time between bounces.




How to measure

The GW h is an oscillation in spacetime, has an impact on propagation of light:

Behavior of light in spacetime with wave:

$$ds^2 = -c^2 dt^2 + [1 + h(t, x)] dx^2 = 0$$

Time interval between bounces:

$$\Delta T = \int \frac{dx}{dx/dt} \simeq \frac{1}{c} \int \left[1 - \frac{1}{2} h(t, x) \right] dx$$


Gravitational wave enters as an oscillation in interval from bounce to bounce (Bondi 1957).

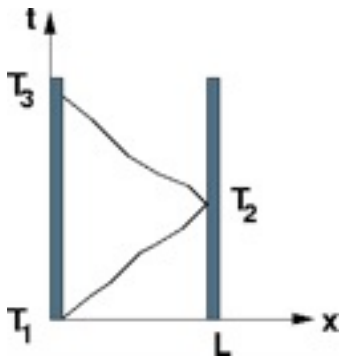
How to measure

The GW h is an oscillation in spacetime, has an impact on propagation of light:

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$$ds^2 = -c^2 dt^2 + [1 + h(t, x)] dx^2 = 0$$

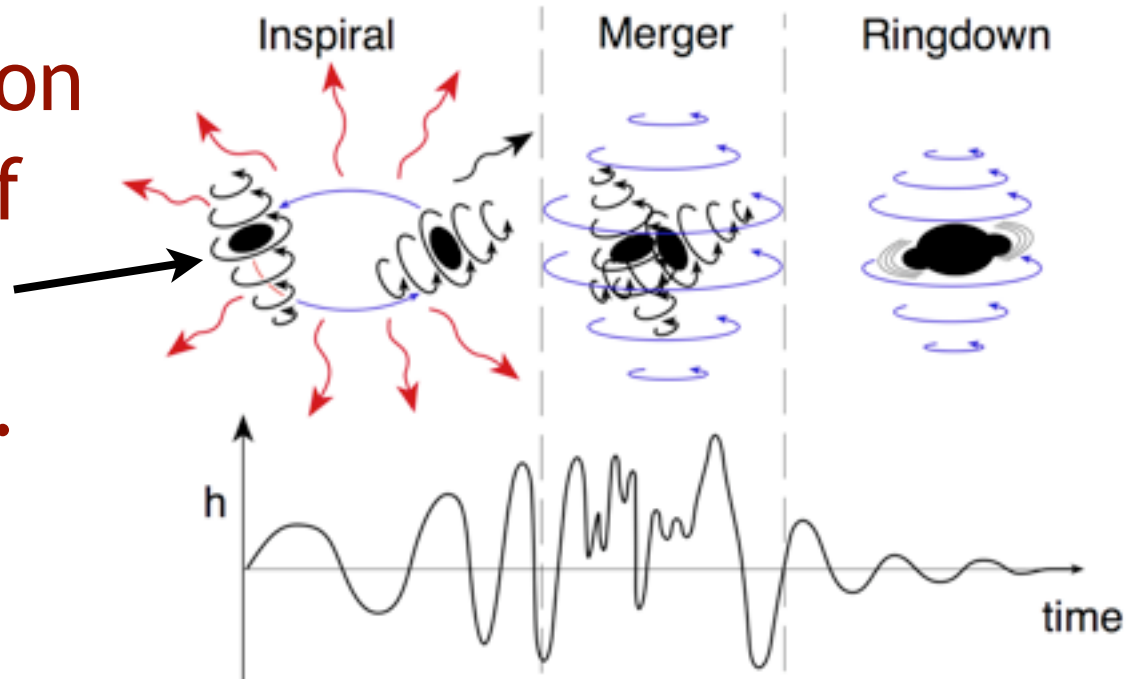
Two big ingredients needed to measure h :
Good inertial reference frame to define free fall, and good clock to measure time interval.



$$\Delta T = \int \frac{dx}{dx/dt} \simeq \frac{1}{c} \int \left[1 - \frac{1}{2} h(t, x) \right] dx$$

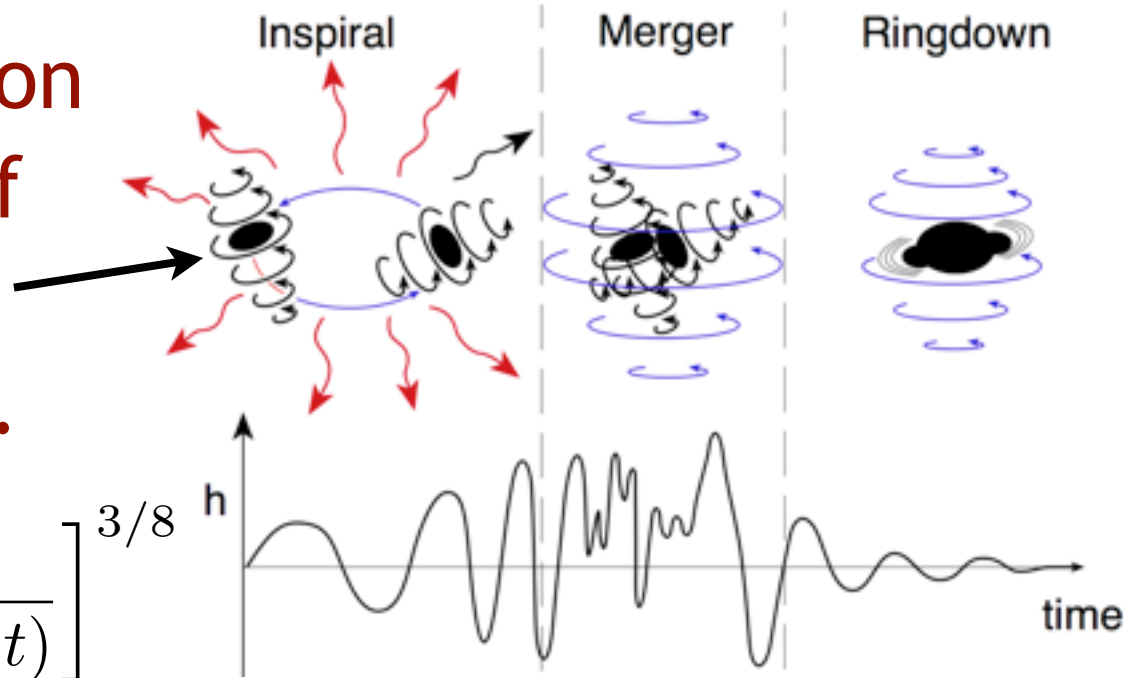
Gross features of BBH GWs

Inspiral: Slow evolution driven by GW loss of orbital energy and angular momentum.



Gross features of BBH GWs

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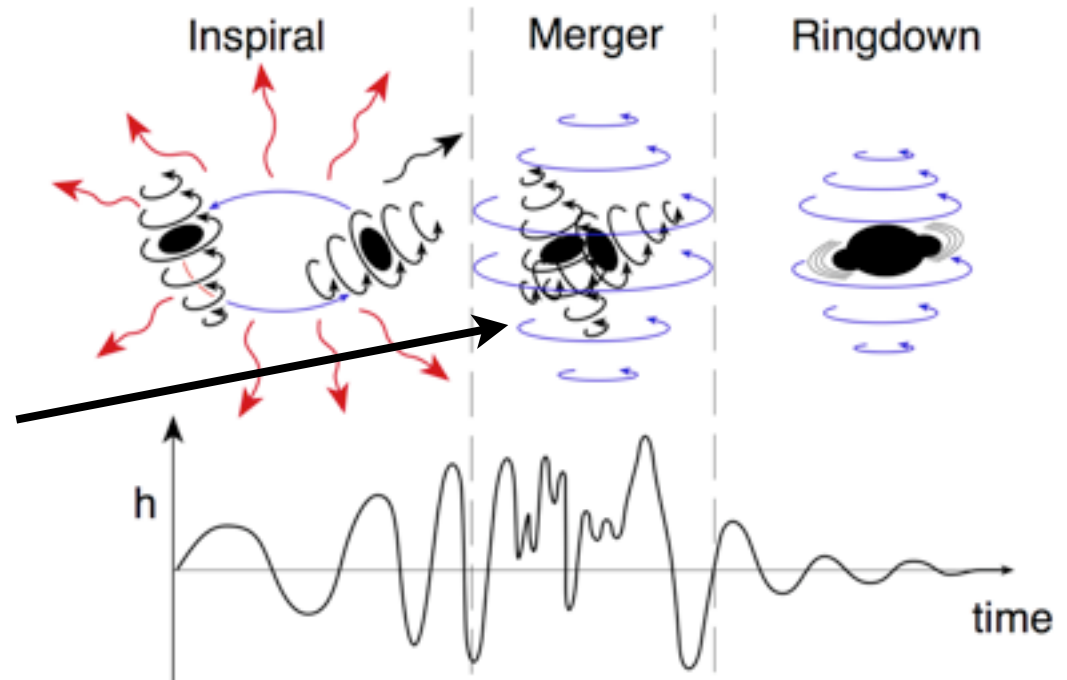


$$f(t) = \frac{1}{\pi} \left[\frac{5c^5}{256(G\mathcal{M})^{5/3}(t_c - t)} \right]^{3/8}$$

Leading solution for rate of change of wave frequency as system evolves ... more careful calculation shows that inspiral encodes a lot of information about members' masses and spins.

Gross features of BBH GWs

Merger: Extremely violent dynamics of spacetime: Two black holes smash together, leaving one behind.



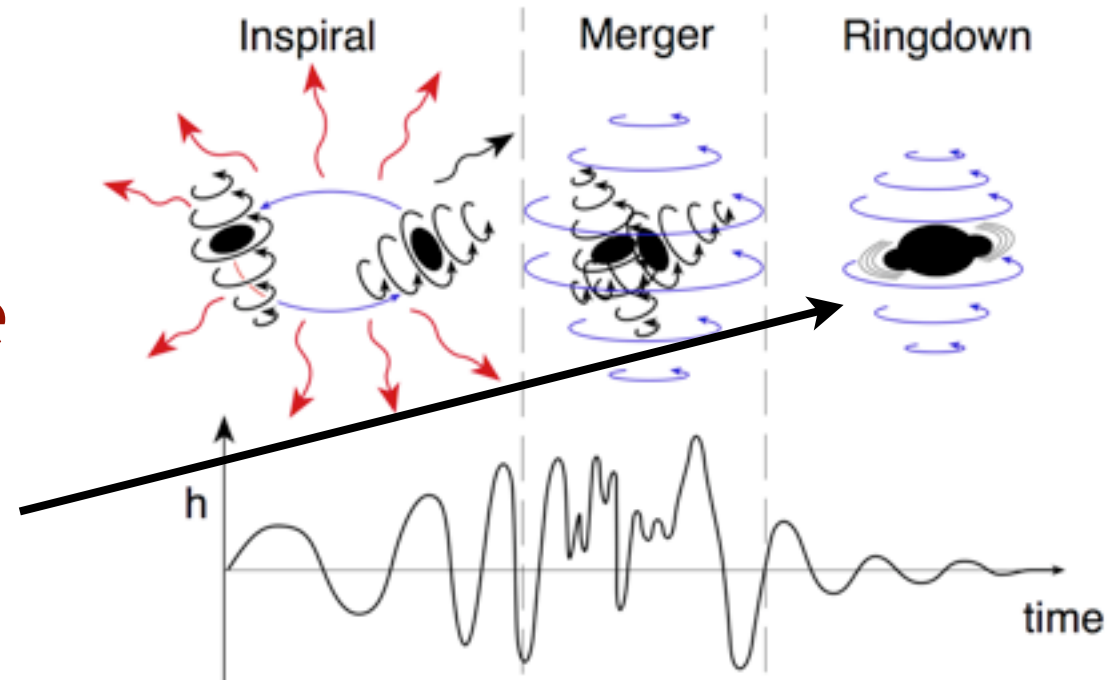
Transition from inspiral to merger happens at

$$f_{\text{merge}} \simeq \frac{c^3}{GM_{\text{tot}}} \frac{[2 - 6]^{-3/2}}{\pi} = (0.02 - 0.004) \text{ Hz} \left(\frac{10^6 M_{\odot}}{M_{\text{tot}}} \right)$$

Late inspiral/merger modeled numerically; a lot of binary mass (up to ~10%) comes out in GWs.

Gross features of BBH GWs

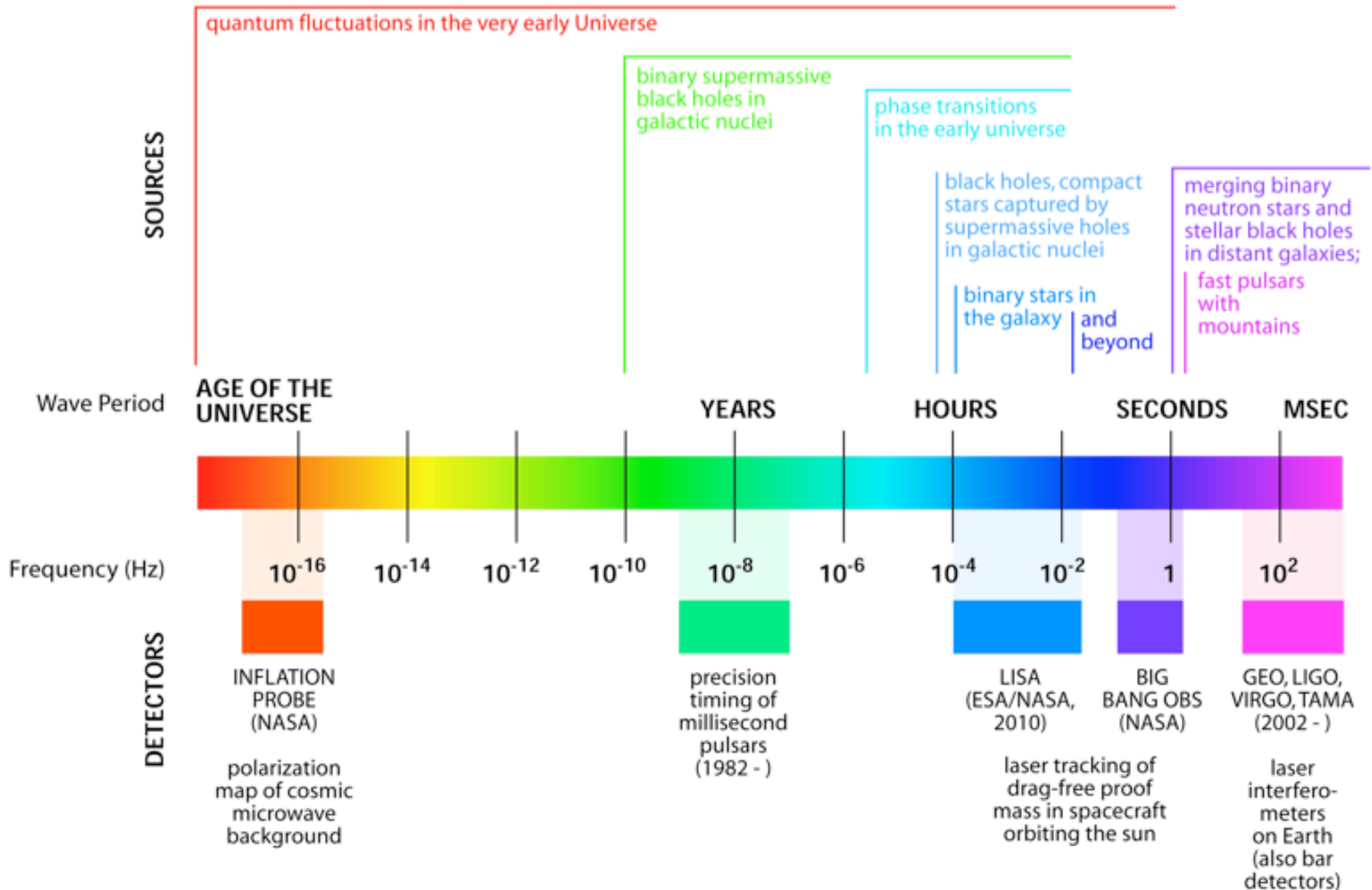
Ringdown: Last wiggles of the merger, enforce the black hole “No Hair” theorems. Kerr solution at end.



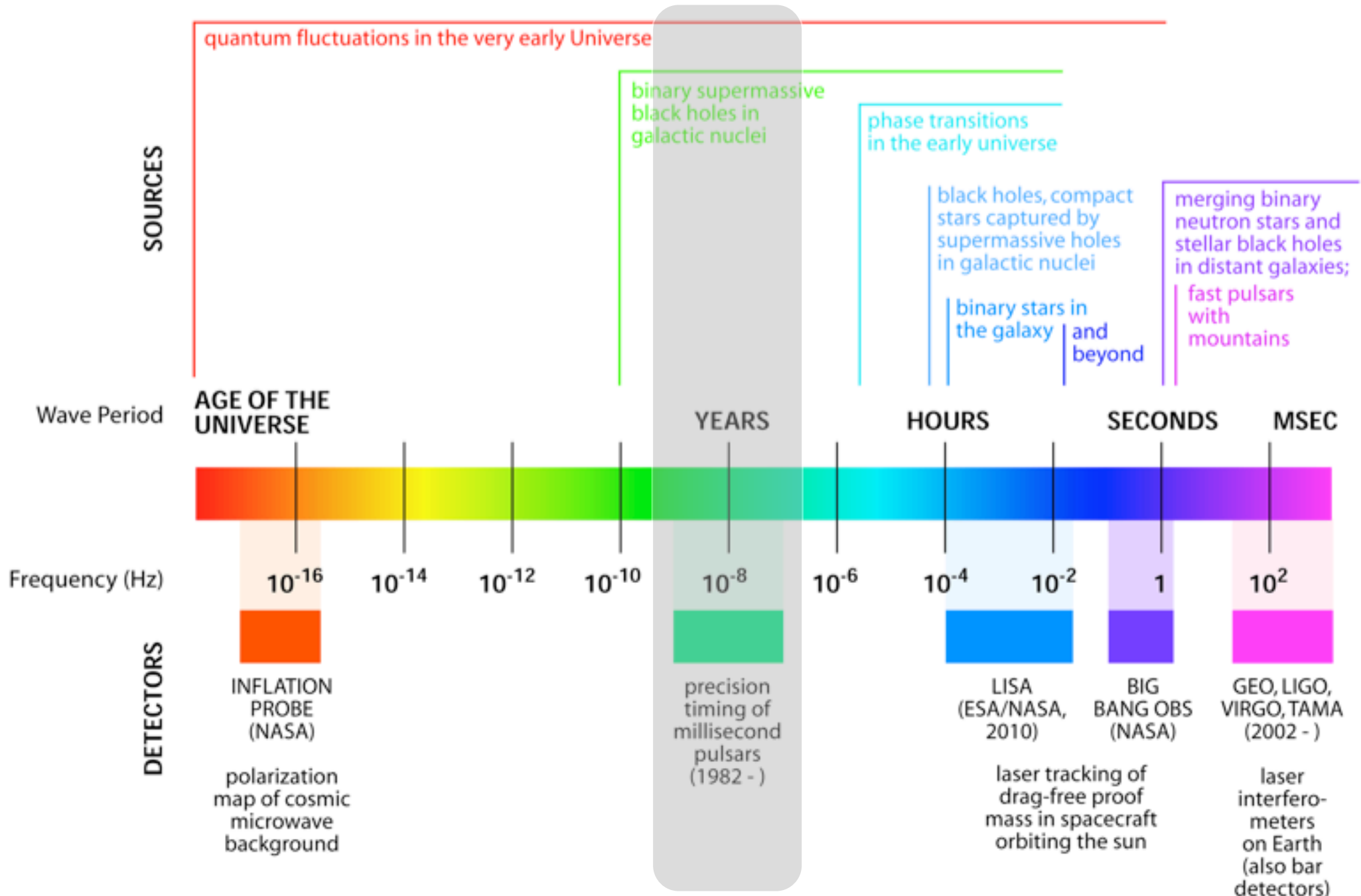
Simply described using black hole perturbation theory. Expect mix of modes; mode frequency and damping time set by final mass and spin.

$$f_{22} \approx \frac{c^3}{2\pi GM} \left[1 - 0.63(1 - a/M)^{0.3} \right] \quad \tau_{22} \approx \frac{2}{\pi f_{22}} (1 - a/M)^{-0.45}$$

THE GRAVITATIONAL WAVE SPECTRUM



THE GRAVITATIONAL WAVE SPECTRUM



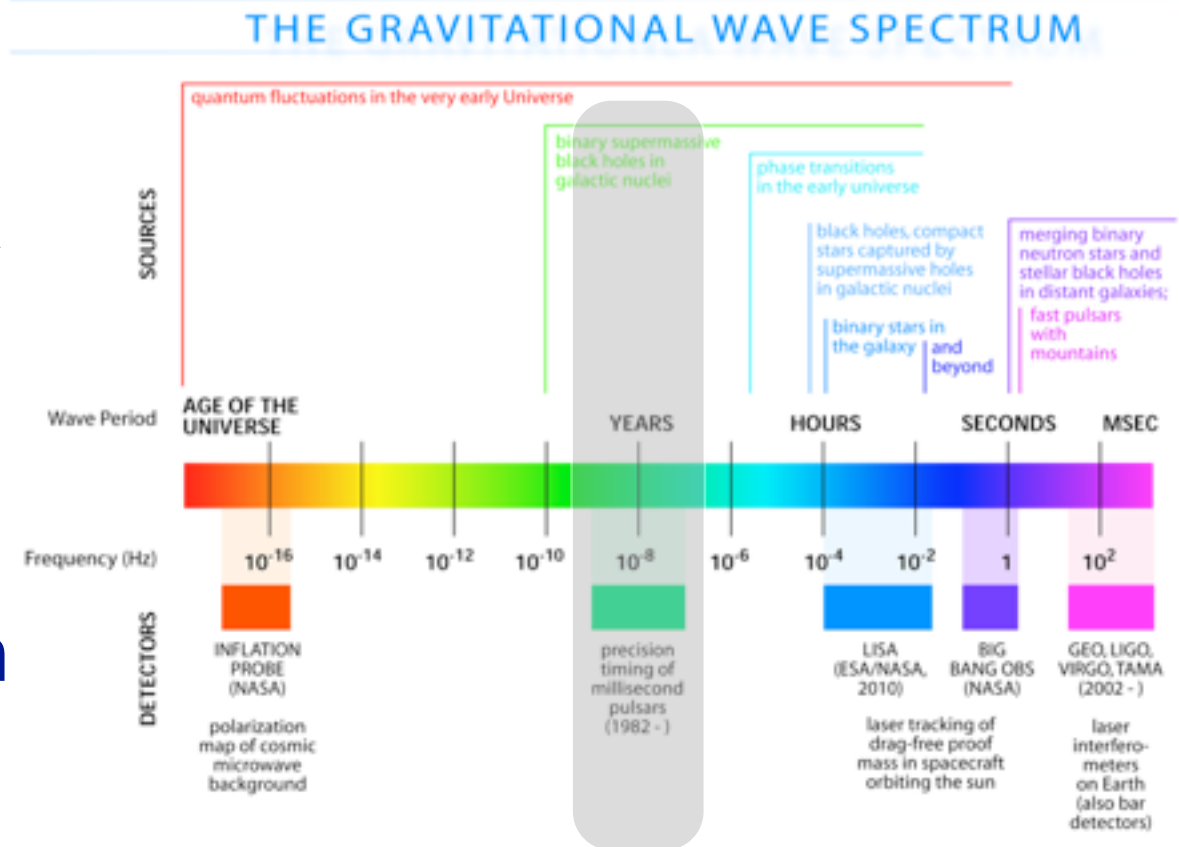
Very low frequency

Frequencies: (years)⁻¹

to (months)⁻¹

Measure in this band by
precision timing of
millisecond pulsars:

Pulsars are clocks, GWs
cause coherent variation
in pulse arrival times.



Build a network of pulsars,
time them well, look for
pulse variations with a
particular angular
distribution on the sky.

Pulsar timing movie courtesy Penn State Gravitational
Wave Astronomy Group, <http://gwastro.org>

Massive BHs: Birth, Growth, and Impact, KITP, 7 August 2013

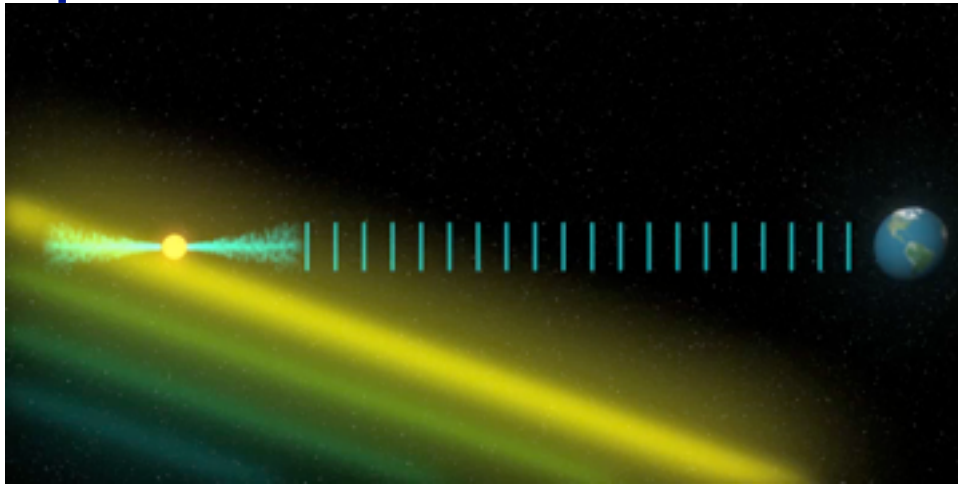
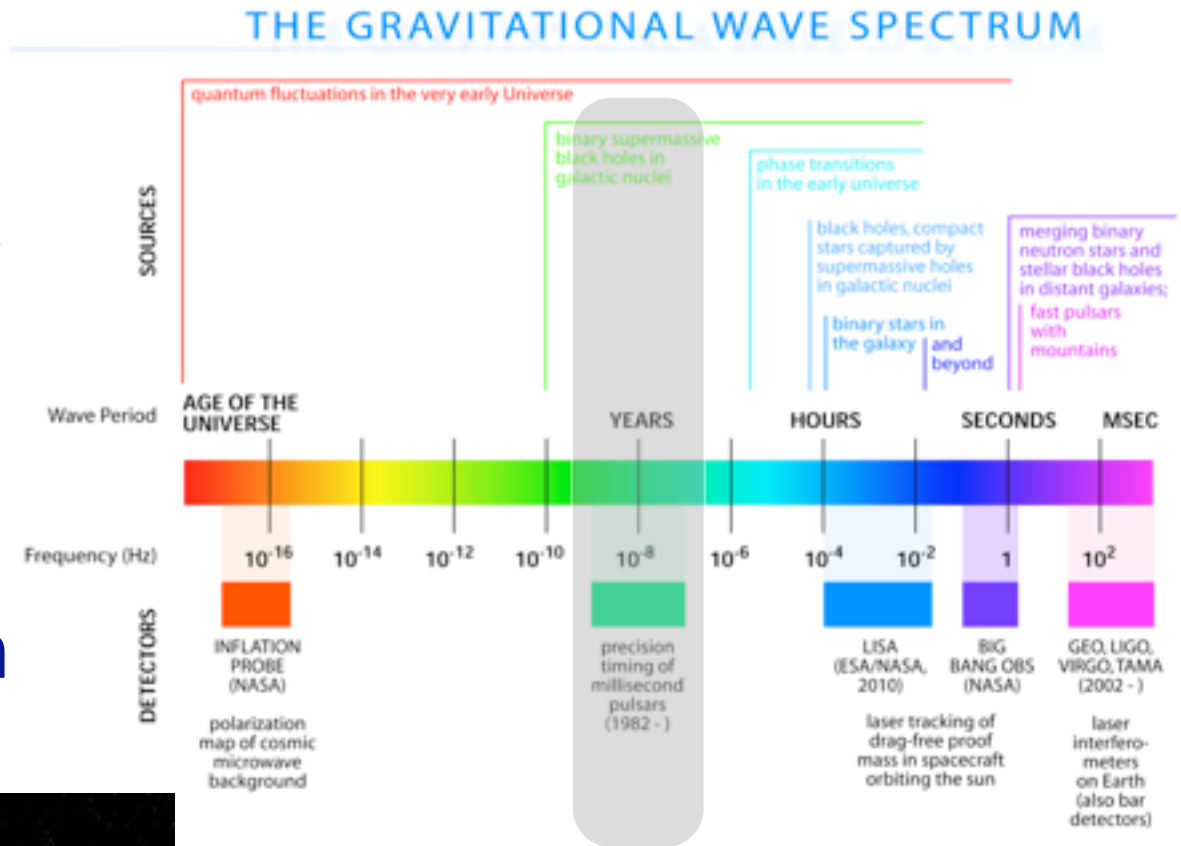
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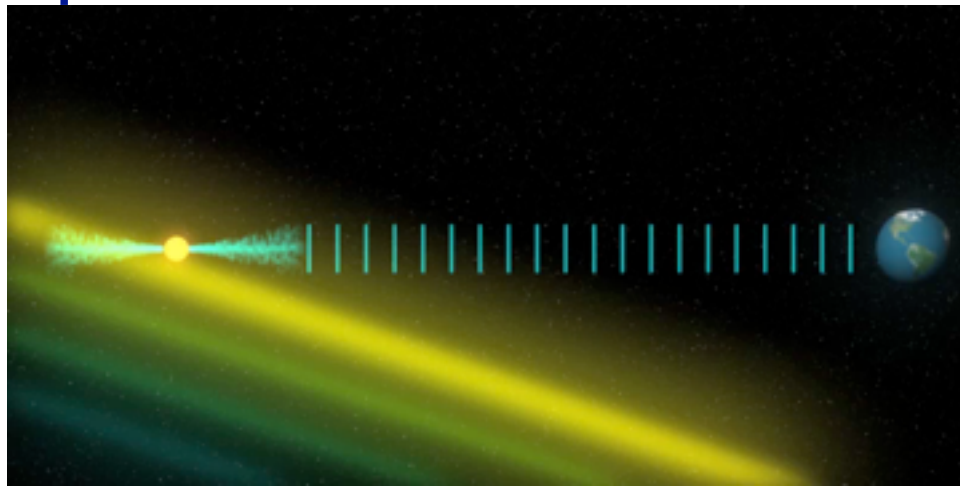
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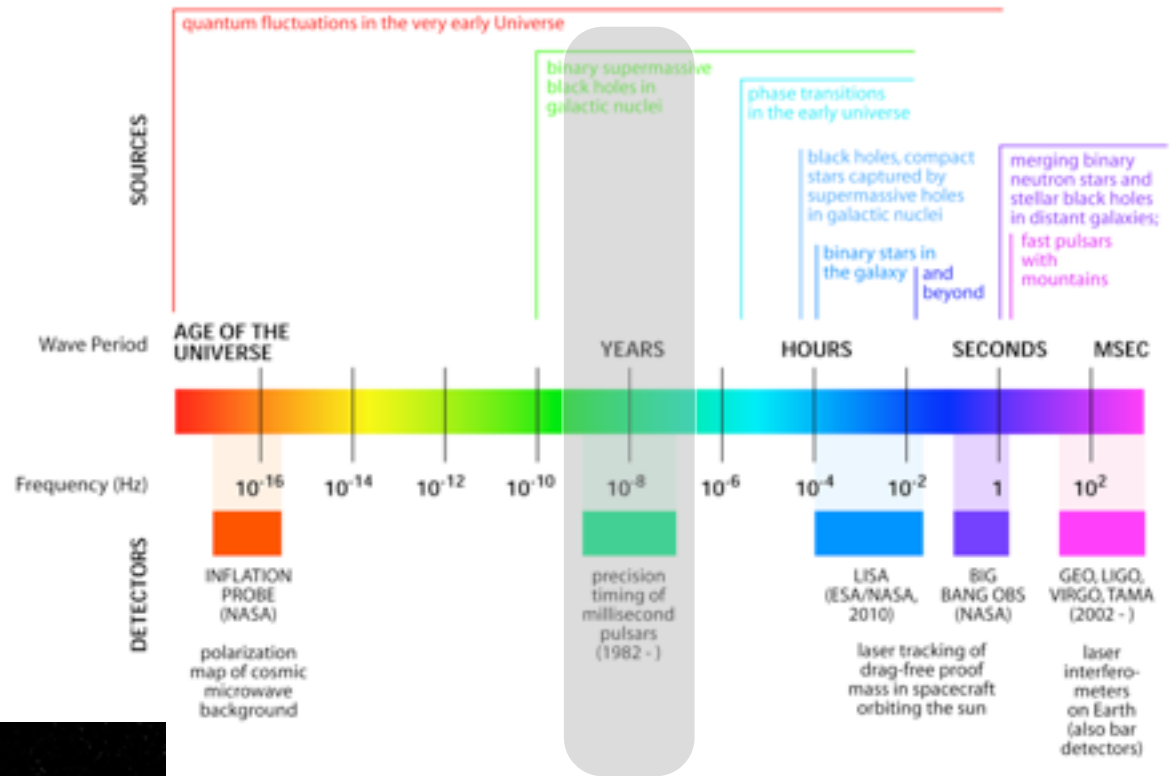
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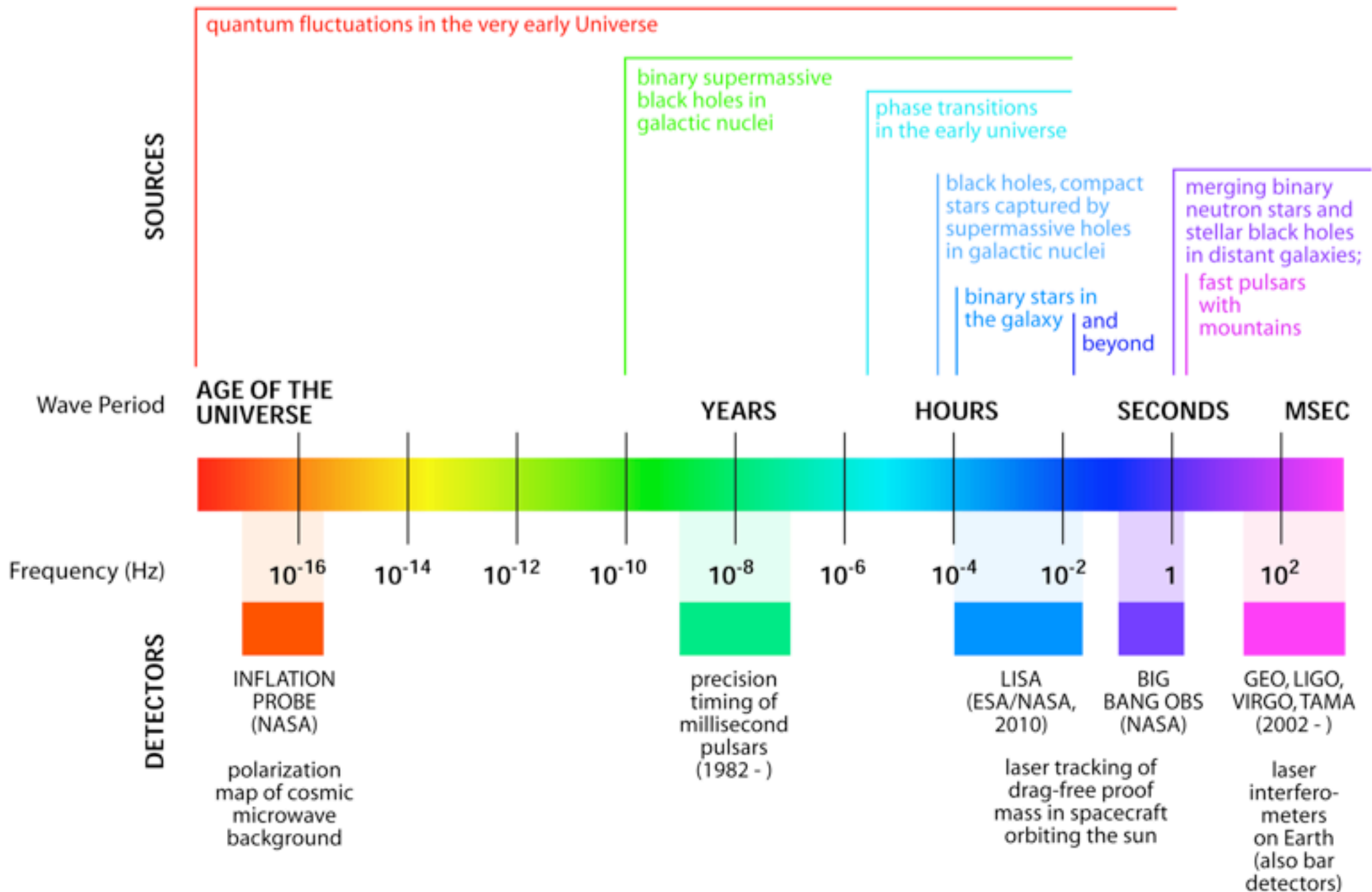
Pulsar timing movie courtesy Penn State Gravitational
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THE GRAVITATIONAL WAVE SPECTRUM

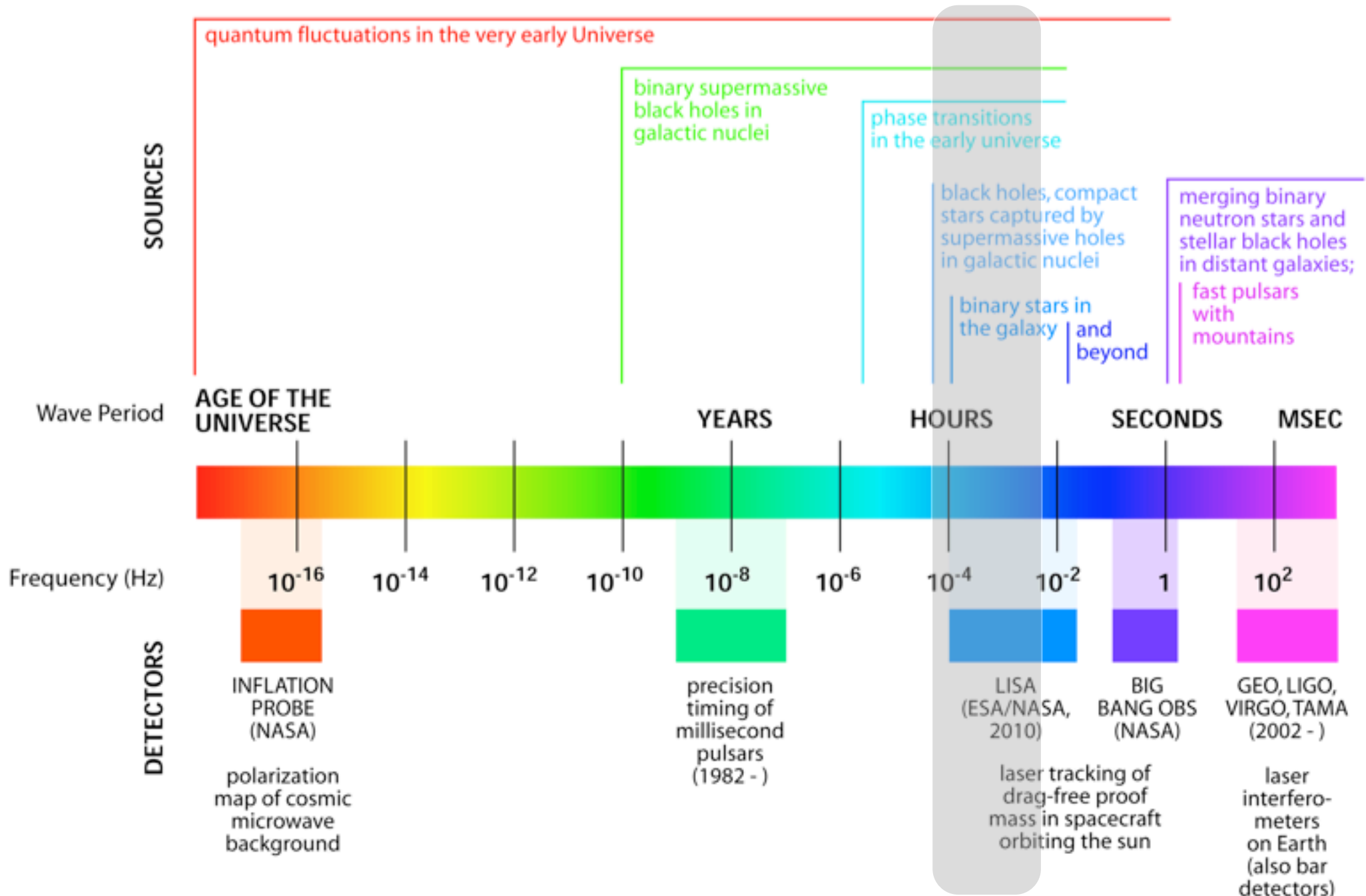


Network now has almost 40
pulsars, setting impressive
limits on backgrounds ...
getting close to predictions.

THE GRAVITATIONAL WAVE SPECTRUM



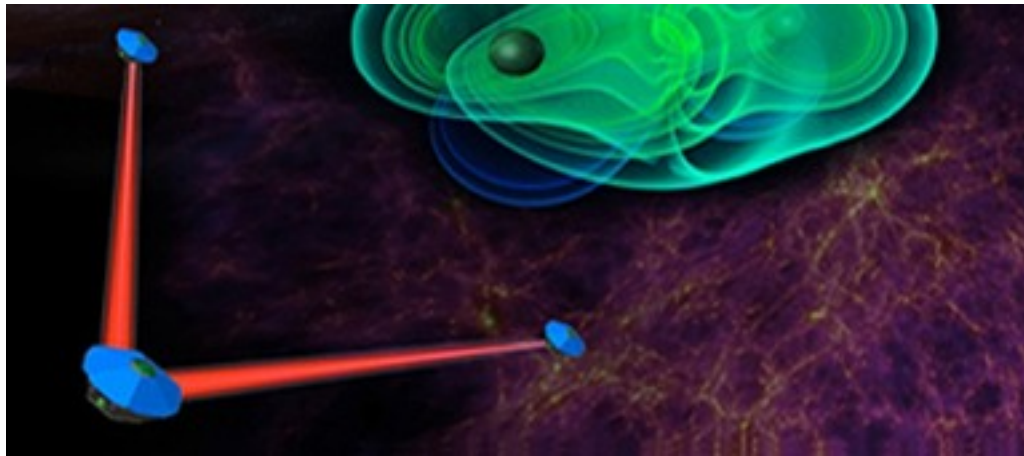
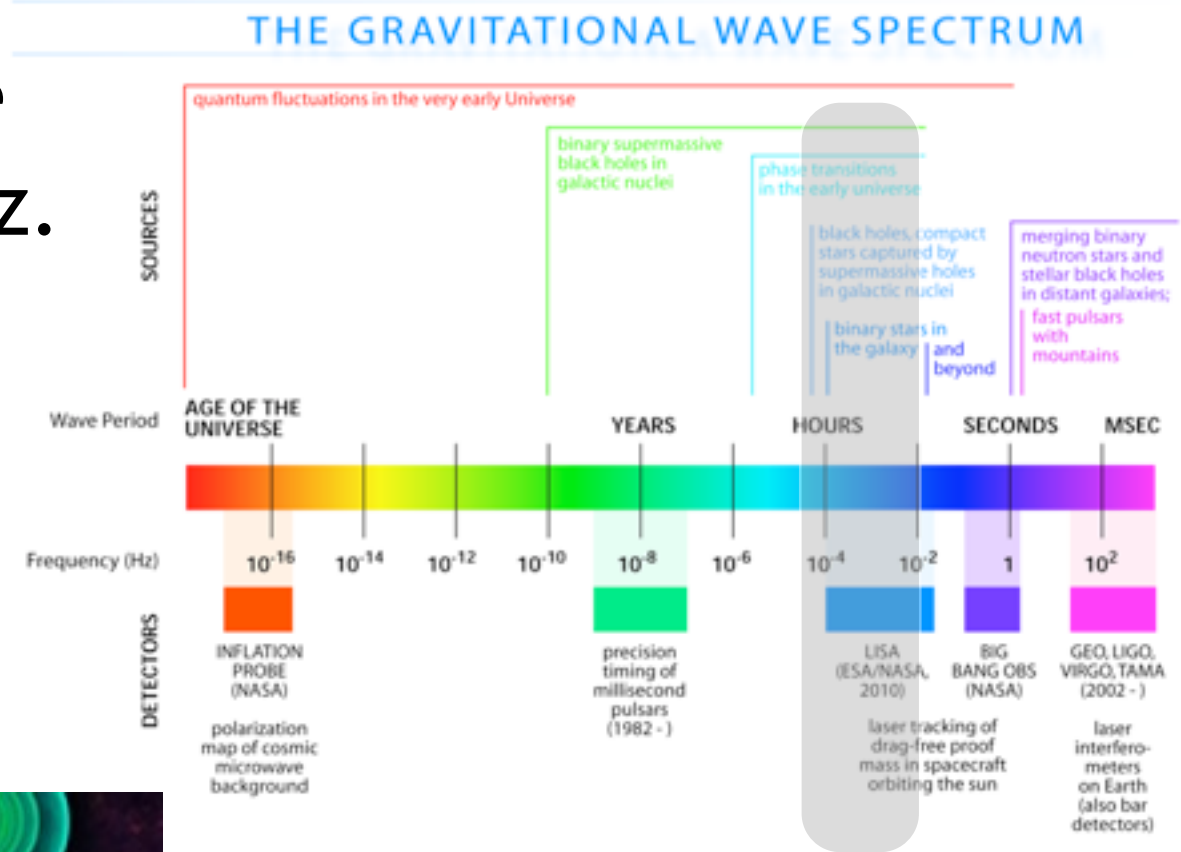
THE GRAVITATIONAL WAVE SPECTRUM



Low frequency

Frequencies: Inverse hours up to about 1 Hz.

Laser interferometry best tool to measure these waves ... Earth too noisy, detector must go into space.

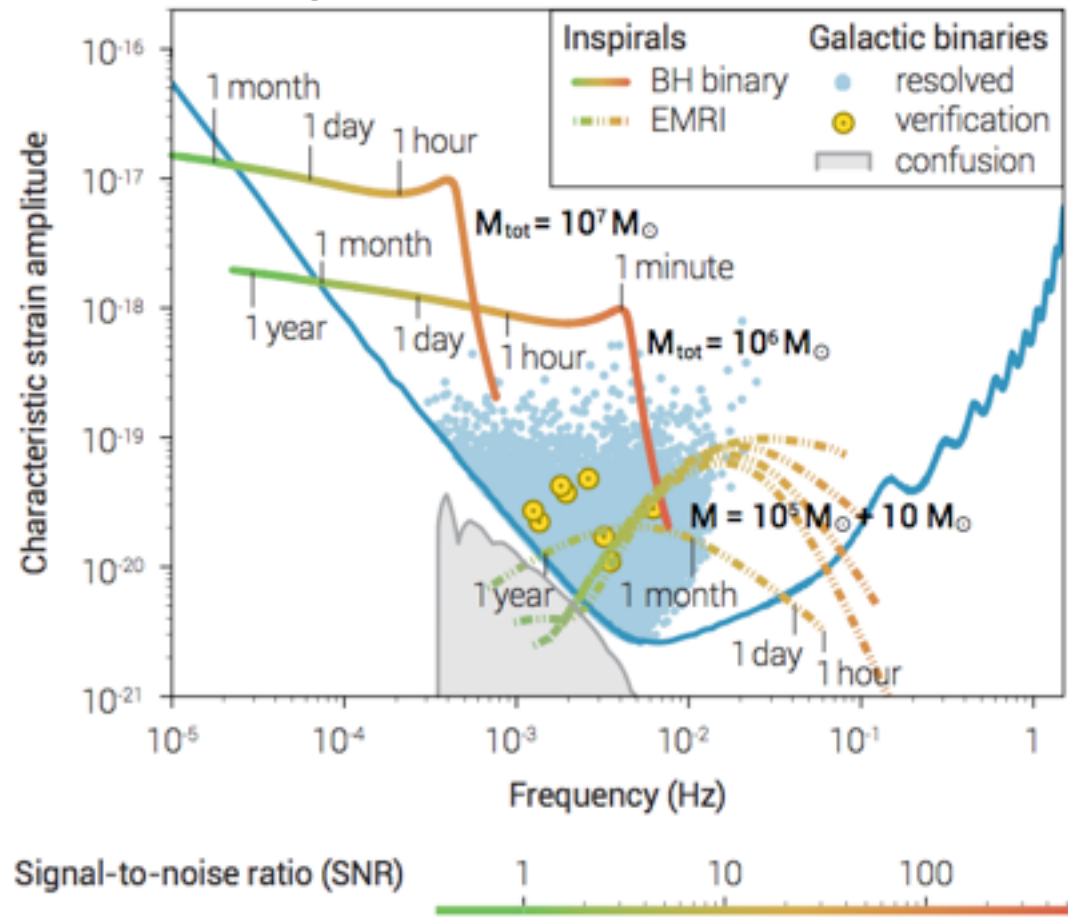


Space antenna like eLISA designed to measure waves in this band.

eLISA sensitivity

Sensitivity is very good for black holes for late inspiral (last few $10^2 - 10^3$ orbits) and merger/ringdown for redshifted masses $(1+z)M_{\text{tot}} \sim 10^5 - 10^7$

Fig 13 of Danzmann et al, arxiv:1305.5720

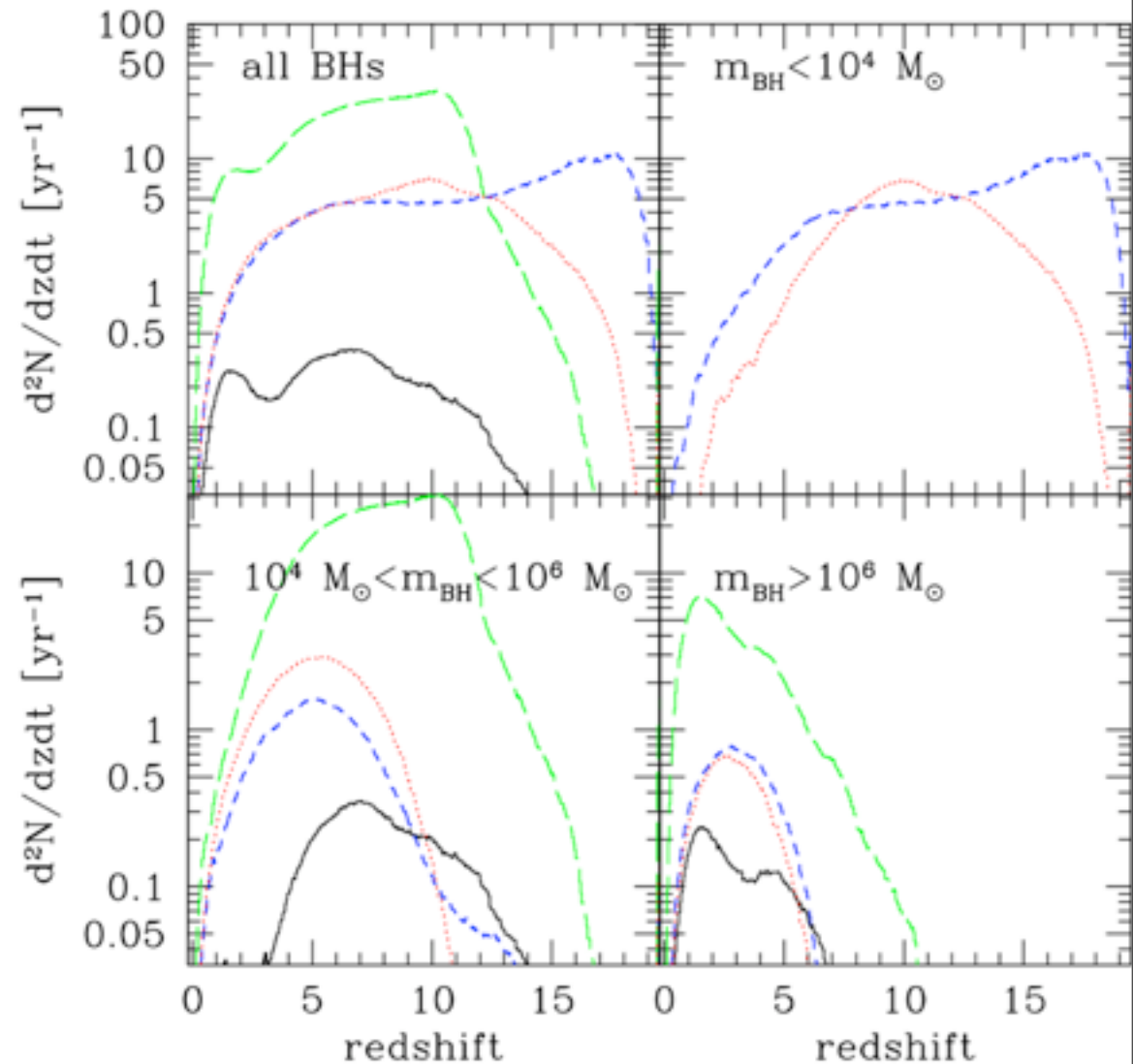


Perfectly suited to going after early cosmological seeds in binaries.

Gravitational waves and MBBH

Wide range of MBBH binaries accessible to an instrument like eLISA with decent events rates ...

... provided that we can nail down events happening at $z \sim 5 - 15$.



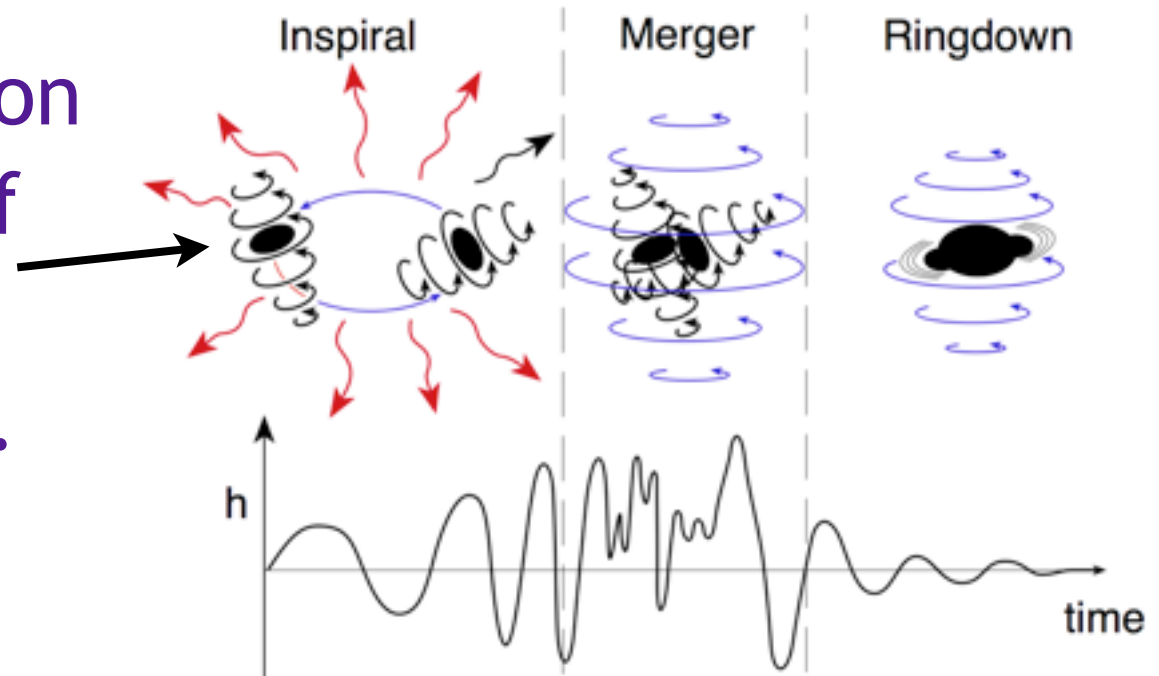
Sesana, Volonteri, & Haardt 2007 MNRAS 377, 1711

Inspiral waves

Inspiral: Slow evolution driven by GW loss of orbital energy and angular momentum.

Rather well understood.

Waveform described by 17 parameters in general.



2 masses

6 spin components

2 position angles

2 orientation angles

1 distance

1 initial semi-major axis

1 initial orbit anomaly

1 initial eccentricity

1 initial periapsis longitude

Inspiral waveform

$$h_+ = \frac{[GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} (1 + \cos^2 \iota) \cos \left[2\pi \int f(t) dt \right]$$

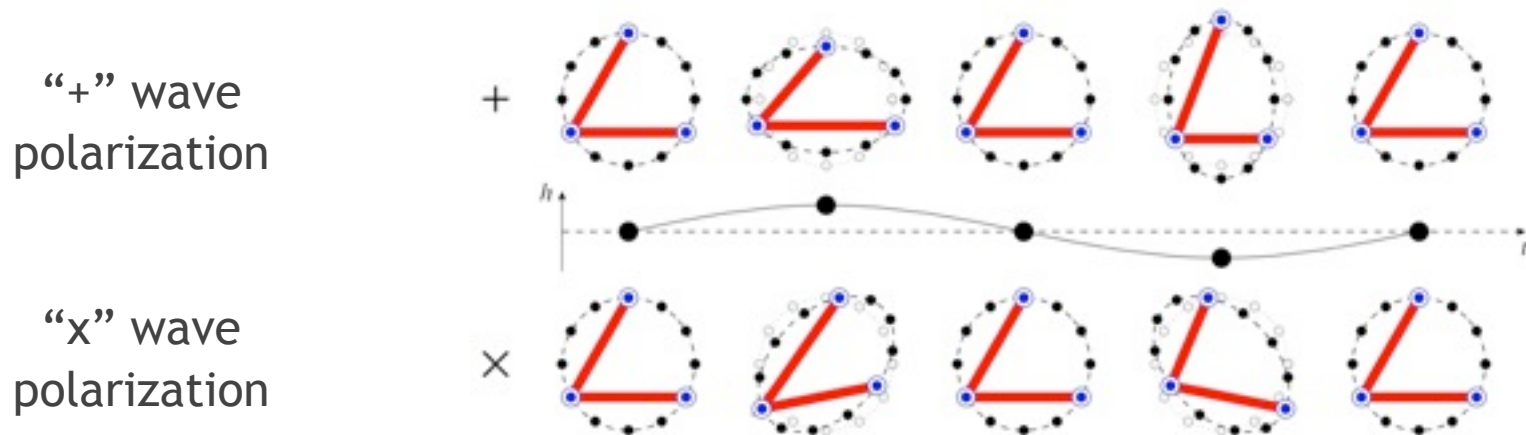
$$h_\times = \frac{2 [GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \cos \iota \sin \left[2\pi \int f(t) dt \right]$$

Inspiral waveform

$$h_+ = \frac{[GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} (1 + \cos^2 \iota) \cos \left[2\pi \int f(t) dt \right]$$

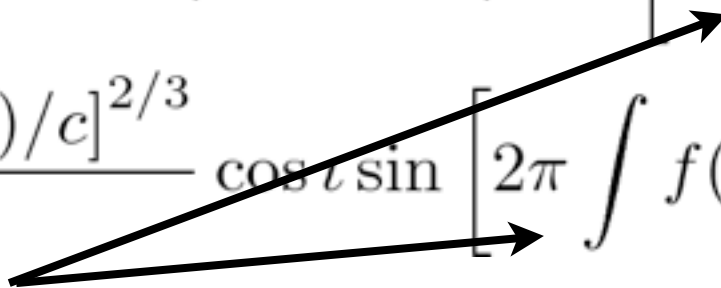
$$h_\times = \frac{2 [GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \cos \iota \sin \left[2\pi \int f(t) dt \right]$$

Gravitational waves have two polarizations, named for their tidal action upon a set of test masses:



With two arms (as in the baseline eLISA design), can only measure one polarization at a time.

Pieces of inspiral waveform

$$h_+ = \frac{[GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} (1 + \cos^2 \iota) \cos \left[2\pi \int f(t) dt \right]$$
$$h_\times = \frac{2 [GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \cos \iota \sin \left[2\pi \int f(t) dt \right]$$


1. Phase. Depends on how rapidly the orbit evolves. Rate is controlled by binary's masses and spins.
Measure the phase, measure masses and spins.

Phase: Comes from integrating up the (relativistic analog of) Kepler's law

To get that, need relativistic equations of motion. Post-Newtonian expansion of general relativity gives us a good form for inspiral:

$$a_1^i = -\frac{Gm_2 n_{12}^i}{r_{12}^2}$$

Lowest order piece:
Newtonian gravity

Phase: Comes from integrating up the (relativistic analog of) Kepler's law

To get that, need relativistic equations of motion. Post-Newtonian expansion of general relativity gives us a good form for inspiral:

$$a_1^i = -\frac{Gm_2 n_{12}^i}{r_{12}^2} + \frac{1}{c^2} \left\{ \left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2} \left(\frac{3}{2} (n_{12} v_2)^2 - v_1^2 + 4(v_1 v_2) - 2v_2^2 \right) \right] n_{12}^i + \frac{Gm_2}{r_{12}^2} (4(n_{12} v_1) - 3(n_{12} v_2)) v_{12}^i \right\}$$

Lowest order piece:
Newtonian gravity

Post-Newton gives corrections in v/c .

... and more corrections ...

... and more corrections ...

$$\begin{aligned}
 & + \frac{1}{c^4} \left\{ \left[-\frac{57G^3 m_1^2 m_2}{4r_{12}^4} - \frac{69G^3 m_1 m_2^2}{2r_{12}^4} - \frac{9G^3 m_2^3}{r_{12}^4} \right. \right. \\
 & \quad + \frac{Gm_2}{r_{12}^2} \left(-\frac{15}{8}(n_{12}v_2)^4 + \frac{3}{2}(n_{12}v_2)^2 v_1^2 - 6(n_{12}v_2)^2 (v_1 v_2) - 2(v_1 v_2)^2 + \frac{9}{2}(n_{12}v_2)^2 v_2^2 \right. \\
 & \quad \quad \left. \left. + 4(v_1 v_2)v_2^2 - 2v_2^4 \right) \right. \\
 & \quad + \frac{G^2 m_1 m_2}{r_{12}^3} \left(\frac{39}{2}(n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) + \frac{17}{2}(n_{12}v_2)^2 - \frac{15}{4}v_1^2 - \frac{5}{2}(v_1 v_2) + \frac{5}{4}v_2^2 \right) \\
 & \quad + \frac{G^2 m_2^2}{r_{12}^3} \left(2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2 - 8(v_1 v_2) + 4v_2^2 \right) \left. \right] n_{12}^i \\
 & \quad + \left[\frac{G^2 m_2^2}{r_{12}^3} (-2(n_{12}v_1) - 2(n_{12}v_2)) + \frac{G^2 m_1 m_2}{r_{12}^3} \left(-\frac{63}{4}(n_{12}v_1) + \frac{55}{4}(n_{12}v_2) \right) \right. \\
 & \quad + \frac{Gm_2}{r_{12}^2} \left(-6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2}(n_{12}v_2)^3 + (n_{12}v_2)v_1^2 - 4(n_{12}v_1)(v_1 v_2) \right. \\
 & \quad \quad \left. \left. + 4(n_{12}v_2)(v_1 v_2) + 4(n_{12}v_1)v_2^2 - 5(n_{12}v_2)v_2^2 \right) \right] v_{12}^i \left. \right\} \\
 & + \frac{1}{c^5} \left\{ \left[\frac{208G^3 m_1 m_2^2}{15r_{12}^4} (n_{12}v_{12}) - \frac{24G^3 m_1^2 m_2}{5r_{12}^4} (n_{12}v_{12}) + \frac{12G^2 m_1 m_2}{5r_{12}^3} (n_{12}v_{12})v_{12}^2 \right] n_{12}^i \right. \\
 & \quad \left. + \left[\frac{8G^3 m_1^2 m_2}{5r_{12}^4} - \frac{32G^3 m_1 m_2^2}{5r_{12}^4} - \frac{4G^2 m_1 m_2}{5r_{12}^3} v_{12}^2 \right] v_{12}^i \right\}
 \end{aligned}$$

... and a few more.

... and a few more.

$$\begin{aligned}
 & + \frac{1}{c^6} \left\{ \left[\frac{Gm_2}{r_{12}^3} \left(\frac{35}{16} (n_{12}v_2)^6 - \frac{15}{8} (n_{12}v_2)^4 v_2^2 + \frac{15}{2} (n_{12}v_2)^4 (v_1v_2) + 3(n_{12}v_2)^2 (v_1v_2)^2 \right. \right. \right. \\
 & \quad \left. \left. - \frac{15}{2} (n_{12}v_2)^4 v_2^2 + \frac{3}{2} (n_{12}v_2)^2 v_1^2 v_2^2 - 12(n_{12}v_2)^2 (v_1v_2)v_2^2 - 2(v_1v_2)^2 v_2^2 \right. \right. \\
 & \quad \left. \left. + \frac{15}{2} (n_{12}v_2)^2 v_2^2 + 4(v_1v_2)v_2^4 - 2v_2^6 \right) \right. \\
 & \quad + \frac{G^2 m_1 m_2}{r_{12}^4} \left(-\frac{171}{8} (n_{12}v_1)^4 + \frac{171}{2} (n_{12}v_1)^3 (n_{12}v_2) - \frac{723}{4} (n_{12}v_1)^3 (n_{12}v_2)^2 \right. \\
 & \quad \left. + \frac{383}{2} (n_{12}v_1)(n_{12}v_2)^3 - \frac{455}{8} (n_{12}v_2)^4 + \frac{229}{4} (n_{12}v_1)^2 v_1^2 \right. \\
 & \quad \left. - \frac{205}{2} (n_{12}v_1)(n_{12}v_2)v_1^2 + \frac{191}{4} (n_{12}v_2)^2 v_1^2 - \frac{91}{8} v_1^4 - \frac{229}{2} (n_{12}v_1)^2 (v_1v_2) \right. \\
 & \quad \left. + 244(n_{12}v_1)(n_{12}v_2)(v_1v_2) - \frac{225}{2} (n_{12}v_2)^2 (v_1v_2) + \frac{91}{2} v_1^2 (v_1v_2) \right. \\
 & \quad \left. - \frac{177}{4} (v_1v_2)^2 + \frac{229}{4} (n_{12}v_1)^2 v_2^2 - \frac{283}{2} (n_{12}v_1)(n_{12}v_2)v_2^2 \right. \\
 & \quad \left. + \frac{259}{4} (n_{12}v_2)^2 v_2^2 - \frac{91}{4} v_2^4 v_2^2 + 43(v_1v_2)v_2^2 - \frac{81}{8} v_2^4 \right) \\
 & \quad + \frac{G^2 m_1^2}{r_{12}^4} \left(-6(n_{12}v_1)^2 (n_{12}v_2)^2 + 12(n_{12}v_1)(n_{12}v_2)^2 + 6(n_{12}v_2)^4 \right. \\
 & \quad \left. + 4(n_{12}v_1)(n_{12}v_2)(v_1v_2) + 12(n_{12}v_2)^2 (v_1v_2) + 4(v_1v_2)^2 \right. \\
 & \quad \left. - 4(n_{12}v_1)(n_{12}v_2)v_2^2 - 12(n_{12}v_2)^2 v_2^2 - 8(v_1v_2)v_2^2 + 4v_2^4 \right) \\
 & \quad + \frac{G^2 m_1^2}{r_{12}^4} \left(-(n_{12}v_1)^2 + 2(n_{12}v_1)(n_{12}v_2) + \frac{43}{2} (n_{12}v_2)^2 + 18(v_1v_2) - 9v_2^2 \right) \\
 & \quad + \frac{G^2 m_1 m_2}{r_{12}^4} \left(\frac{115}{8} (n_{12}v_1)^2 - \frac{375}{4} (n_{12}v_1)(n_{12}v_2) + \frac{1113}{8} (n_{12}v_2)^2 - \frac{615}{64} (n_{12}v_{12})^2 \pi^2 \right. \\
 & \quad \left. + 18v_1^2 + \frac{123}{64} \pi^2 v_{12}^2 + 33(v_1v_2) - \frac{33}{2} v_2^2 \right) \\
 & \quad + \frac{G^2 m_1^2 m_2}{r_{12}^4} \left(-\frac{45887}{168} (n_{12}v_1)^2 + \frac{24925}{42} (n_{12}v_1)(n_{12}v_2) - \frac{10469}{42} (n_{12}v_2)^2 + \frac{48197}{840} v_1^2 \right. \\
 & \quad \left. - \frac{36227}{420} (v_1v_2) + \frac{36227}{840} v_2^2 + 110(n_{12}v_{12})^2 \ln\left(\frac{r_{12}}{r_1}\right) - 22v_{12}^2 \ln\left(\frac{r_{12}}{r_1}\right) \right) \\
 & \quad + \frac{16G^4 m_1^4}{r_{12}^4} + \frac{G^4 m_1^2 m_2^2}{r_{12}^4} \left(175 - \frac{41}{16} \pi^2 \right) + \frac{G^4 m_1^2 m_2}{r_{12}^4} \left(-\frac{3187}{1200} + \frac{44}{3} \ln\left(\frac{r_{12}}{r_1}\right) \right) \\
 & \quad + \frac{G^4 m_1 m_2^3}{r_{12}^4} \left(\frac{110741}{630} - \frac{41}{16} \pi^2 - \frac{44}{3} \ln\left(\frac{r_{12}}{r_1}\right) \right) \Big] n_{12}^4 \\
 & \quad + \left[\frac{Gm_2}{r_{12}} \left(\frac{15}{2} (n_{12}v_1)(n_{12}v_2)^4 - \frac{45}{8} (n_{12}v_2)^5 - \frac{3}{2} (n_{12}v_2)^3 v_1^2 + 6(n_{12}v_1)(n_{12}v_2)^2 (v_1v_2) \right. \right. \\
 & \quad \left. \left. - 6(n_{12}v_2)^2 (v_1v_2) - 2(n_{12}v_2)(v_1v_2)^2 - 12(n_{12}v_1)(n_{12}v_2)^2 v_2^2 + 12(n_{12}v_2)^3 v_2^2 \right. \right. \\
 & \quad \left. \left. + (n_{12}v_1)v_2^2 v_2^2 - 4(n_{12}v_1)(v_1v_2)v_2^2 + 8(n_{12}v_2)(v_1v_2)v_2^2 + 4(n_{12}v_1)v_2^4 \right. \right. \\
 & \quad \left. \left. - 7(n_{12}v_2)v_2^4 \right) \right. \\
 & \quad + \frac{G^2 m_2^2}{r_{12}^4} \left(-2(n_{12}v_2)^2 (n_{12}v_2) + 8(n_{12}v_1)(n_{12}v_2)^2 + 2(n_{12}v_2)^3 + 2(n_{12}v_1)(v_1v_2) \right. \\
 & \quad \left. + 4(n_{12}v_2)(v_1v_2) - 2(n_{12}v_1)v_2^2 - 4(n_{12}v_2)v_2^2 \right) \\
 & \quad + \frac{G^2 m_1 m_2}{r_{12}^4} \left(-\frac{243}{4} (n_{12}v_1)^3 + \frac{565}{4} (n_{12}v_1)^2 (n_{12}v_2) - \frac{209}{4} (n_{12}v_1)(n_{12}v_2)^2 \right. \\
 & \quad \left. - \frac{95}{12} (n_{12}v_2)^3 + \frac{207}{8} (n_{12}v_2)v_1^2 - \frac{137}{8} (n_{12}v_2)v_1^2 - 36(n_{12}v_1)(v_1v_2) \right. \\
 & \quad \left. + \frac{27}{4} (n_{12}v_2)(v_1v_2) + \frac{81}{8} (n_{12}v_1)v_2^2 + \frac{83}{8} (n_{12}v_2)v_2^2 \right) \\
 & \quad + \frac{G^2 m_1^2}{r_{12}^4} (4(n_{12}v_1) + 5(n_{12}v_2)) \\
 & \quad + \frac{G^2 m_1 m_2}{r_{12}^4} \left(-\frac{307}{8} (n_{12}v_1) + \frac{479}{8} (n_{12}v_2) + \frac{123}{32} (n_{12}v_{12}) \pi^2 \right) \\
 & \quad + \frac{G^2 m_1^2 m_2}{r_{12}^4} \left(\frac{31397}{420} (n_{12}v_1) - \frac{36227}{420} (n_{12}v_2) - 44(n_{12}v_{12}) \ln\left(\frac{r_{12}}{r_1}\right) \right) \Big] v_{12}^2 \Big\} \\
 & + \frac{1}{c^2} \left\{ \left[\frac{G^4 m_1^2 m_2}{r_{12}^4} \left(\frac{3992}{105} (n_{12}v_1) - \frac{4328}{105} (n_{12}v_2) \right) \right. \right. \\
 & \quad + \frac{G^4 m_1^2 m_2}{r_{12}^4} \left(-\frac{13576}{105} (n_{12}v_1) + \frac{2872}{21} (n_{12}v_2) - \frac{3172}{21} \frac{G^4 m_1 m_2}{r_{12}^4} (n_{12}v_{12}) \right. \\
 & \quad \left. + \frac{G^4 m_1^2 m_2}{r_{12}^4} \left(48(n_{12}v_1)^3 - \frac{696}{5} (n_{12}v_1)^2 (n_{12}v_2) + \frac{744}{5} (n_{12}v_1)(n_{12}v_2)^2 - \frac{288}{5} (n_{12}v_2)^3 \right. \right. \\
 & \quad \left. \left. - \frac{488}{105} (n_{12}v_1)v_1^2 + \frac{5056}{105} (n_{12}v_2)v_1^2 + \frac{2056}{21} (n_{12}v_1)(v_1v_2) \right. \right. \\
 & \quad \left. \left. - \frac{2224}{21} (n_{12}v_2)(v_1v_2) - \frac{1028}{21} (n_{12}v_1)v_2^2 + \frac{5812}{105} (n_{12}v_2)v_2^2 \right) \right. \\
 & \quad \left. + \frac{G^4 m_1 m_2}{r_{12}^4} \left(-\frac{582}{5} (n_{12}v_1)^3 + \frac{1746}{5} (n_{12}v_1)^2 (n_{12}v_2) - \frac{1954}{5} (n_{12}v_1)(n_{12}v_2)^2 \right. \right. \\
 & \quad \left. \left. + 158(n_{12}v_2)^3 + \frac{3568}{105} (n_{12}v_{12})v_1^2 - \frac{2864}{35} (n_{12}v_1)(v_1v_2) \right. \right. \\
 & \quad \left. \left. + \frac{10048}{105} (n_{12}v_2)(v_1v_2) + \frac{1432}{35} (n_{12}v_1)v_2^2 - \frac{5752}{105} (n_{12}v_2)v_2^2 \right) \right. \\
 & \quad \left. + \frac{G^2 m_1 m_2}{r_{12}^4} \left(-56(n_{12}v_{12})^4 + 60(n_{12}v_1)^3 v_{12}^2 - 180(n_{12}v_1)^2 (n_{12}v_2)v_{12}^2 \right. \right. \\
 & \quad \left. \left. + 174(n_{12}v_1)(n_{12}v_2)v_{12}^2 v_{12}^2 - 54(n_{12}v_2)^2 v_{12}^2 - \frac{205}{35} (n_{12}v_{12})v_1^4 \right. \right. \\
 & \quad \left. \left. + \frac{1068}{35} (n_{12}v_1)v_2^2 (v_1v_2) - \frac{984}{35} (n_{12}v_2)v_2^2 (v_1v_2) - \frac{1068}{35} (n_{12}v_1)(v_1v_2)^2 \right. \right. \\
 & \quad \left. \left. + \frac{180}{7} (n_{12}v_2)(v_1v_2)^2 - \frac{534}{35} (n_{12}v_1)v_1^2 v_2^2 + \frac{90}{7} (n_{12}v_2)v_1^2 v_2^2 \right. \right. \\
 & \quad \left. \left. + \frac{984}{35} (n_{12}v_1)(v_1v_2)v_2^2 - \frac{732}{35} (n_{12}v_2)(v_1v_2)v_2^2 - \frac{204}{35} (n_{12}v_1)v_2^4 \right. \right. \\
 & \quad \left. \left. + \frac{24}{7} (n_{12}v_2)v_2^4 \right) \right] n_{12}^2 \\
 & \quad + \left[-\frac{184}{21} \frac{G^4 m_1^2 m_2}{r_{12}^4} + \frac{6224}{105} \frac{G^4 m_1^2 m_2}{r_{12}^4} + \frac{6388}{105} \frac{G^4 m_1 m_2}{r_{12}^4} \right. \\
 & \quad \left. + \frac{G^4 m_1^2 m_2}{r_{12}^4} \left(\frac{52}{15} (n_{12}v_1)^2 - \frac{56}{15} (n_{12}v_1)(n_{12}v_2) - \frac{44}{15} (n_{12}v_2)^2 - \frac{132}{35} v_1^2 + \frac{152}{35} (v_1v_2) \right. \right. \\
 & \quad \left. \left. - \frac{48}{35} v_2^2 \right) \right. \\
 & \quad \left. + \frac{G^2 m_1 m_2}{r_{12}^4} \left(\frac{454}{15} (n_{12}v_1)^2 - \frac{372}{5} (n_{12}v_1)(n_{12}v_2) + \frac{854}{15} (n_{12}v_2)^2 - \frac{152}{21} v_1^2 \right. \right. \\
 & \quad \left. \left. + \frac{2864}{105} (v_1v_2) - \frac{1768}{105} v_2^2 \right) \right. \\
 & \quad \left. + \frac{G^2 m_1 m_2}{r_{12}^4} \left(60(n_{12}v_{12})^4 - \frac{348}{5} (n_{12}v_1)^2 v_{12}^2 + \frac{684}{5} (n_{12}v_1)(n_{12}v_2)v_{12}^2 \right. \right. \\
 & \quad \left. \left. - 66(n_{12}v_2)^2 v_{12}^2 + \frac{334}{35} v_1^2 - \frac{1336}{35} v_1^2 (v_1v_2) + \frac{1308}{35} (v_1v_2)^2 + \frac{654}{35} v_1^2 v_2^2 \right. \right. \\
 & \quad \left. \left. - \frac{1252}{35} (v_1v_2)v_2^2 + \frac{292}{35} v_2^4 \right) \right] v_{12}^2 \Big\} \\
 & + \mathcal{O}\left(\frac{1}{c^8}\right).
 \end{aligned}$$

[Blanchet 2006, Liv Rev Rel 9, 4, Eq. (168)]

Dynamical inclination

Relativistic effect: “Magnetic-type” coupling of mass currents to spacetime.

Creates new “forces”, modifying orbit acceleration; also causes spins of binary’s members to precess.

$$\begin{aligned}\frac{d\mathbf{S}_1}{dt} &= \frac{1}{r^3} \left[\left(2 + \frac{3}{2} \frac{m_2}{m_1} \right) \mu \sqrt{Mr} \hat{\mathbf{L}} \right] \times \mathbf{S}_1 + \frac{1}{r^3} \left[\frac{1}{2} \mathbf{S}_2 - \frac{3}{2} (\mathbf{S}_2 \cdot \hat{\mathbf{L}}) \hat{\mathbf{L}} \right] \times \mathbf{S}_1 \\ \frac{d\mathbf{S}_2}{dt} &= \frac{1}{r^3} \left[\left(2 + \frac{3}{2} \frac{m_1}{m_2} \right) \mu \sqrt{Mr} \hat{\mathbf{L}} \right] \times \mathbf{S}_2 + \frac{1}{r^3} \left[\frac{1}{2} \mathbf{S}_1 - \frac{3}{2} (\mathbf{S}_1 \cdot \hat{\mathbf{L}}) \hat{\mathbf{L}} \right] \times \mathbf{S}_2\end{aligned}$$

“Gravitomagnetic”
field due to
orbital motion

“Gravitomagnetic”
field due to other
body’s spin

Dynamical inclination

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Angular momentum is *globally* conserved:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 = \text{constant}$$

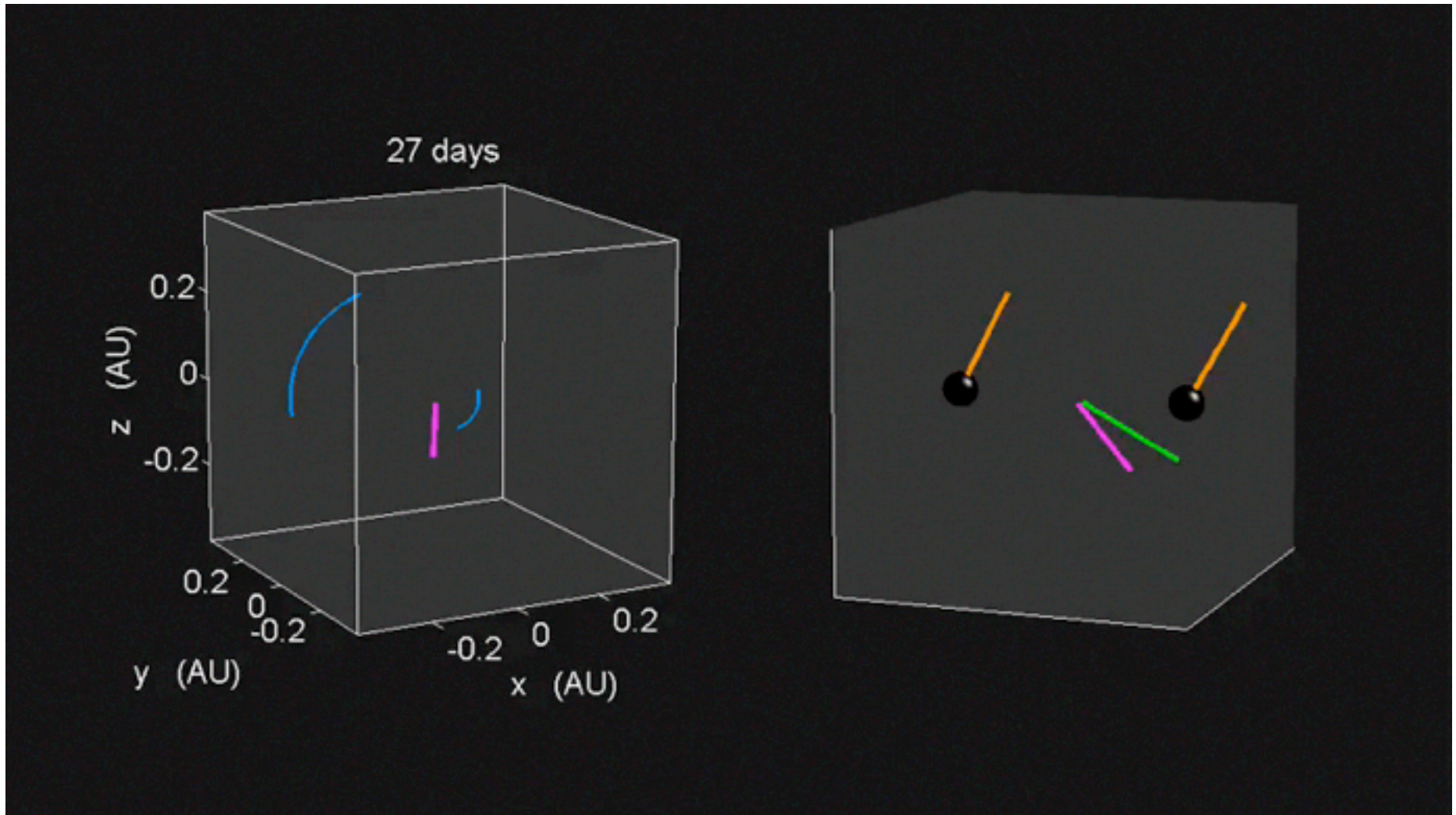
Means that the *orbital plane* precesses to compensate.
(Known as Lense-Thirring precession in weak-field.)

Dynamical inclination

Precession of angular momentum vectors in a binary black hole system.

(Animation credit: Peter Reinhardt)

Dynamical inclination



Precession of angular momentum vectors in a binary black hole system.

(Animation credit: Peter Reinhardt)

Pieces of inspiral waveform

$$h_+ = \frac{[G(1+z)\mathcal{M}/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \mathcal{F}(\text{“angles”}) \cos [\Phi(t)]$$

Integrate up motion and precession: $10^3 - 10^5$ radians of phase accumulate over measurement.

$$\begin{aligned} \phi(f) = & \phi_c - \frac{1}{16}(\pi\mathcal{M}f)^{-5/3} \left[1 + \frac{5}{3} \left(\frac{743}{336} + \frac{11}{4}\eta \right) (\pi Mf)^{2/3} - \frac{5}{2}(4\pi - \beta)(\pi Mf) \right. \\ & \left. + 5 \left(\frac{3058673}{1016064} + \frac{5429}{1008}\eta + \frac{617}{144}\eta^2 - \sigma \right) (\pi Mf)^{4/3} \right] \end{aligned}$$

$$\beta = \frac{1}{12} \sum_{i=1}^2 \left[113 \left(\frac{m_i}{M} \right)^2 + 75 \frac{\mu}{M} \right] \frac{\hat{\mathbf{L}} \cdot \mathbf{S}_i}{m_i^2}$$

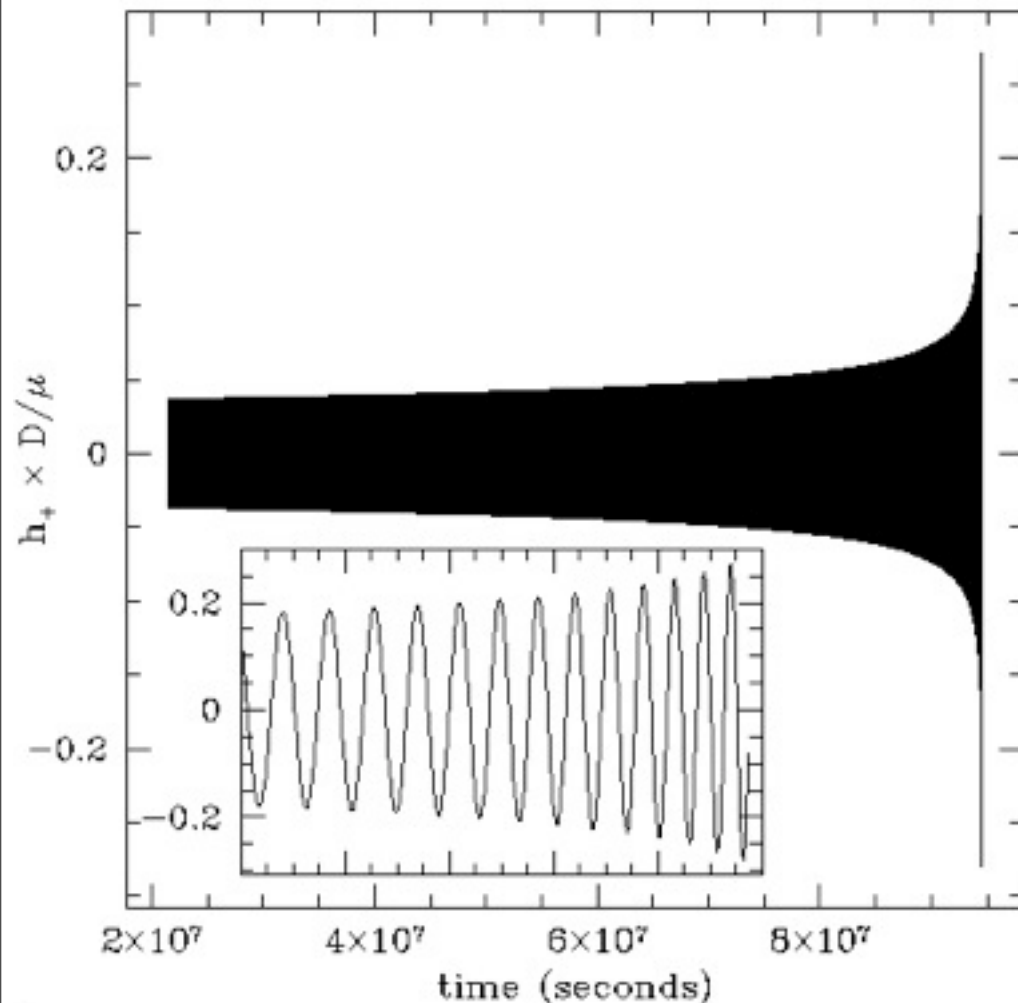
$$\sigma = \frac{\mu}{48M(m_1^2 m_2^2)} [721(\hat{\mathbf{L}} \cdot \mathbf{S}_1)(\hat{\mathbf{L}} \cdot \mathbf{S}_2) - 247(\mathbf{S}_1 \cdot \mathbf{S}_2)]$$

Key feature: The phase depends on – and thus encodes – masses & spins of the binary’s members.

Measure phase: Measure masses and spins.

Inspiral measurements

$$h_+ = \frac{[G(1+z)\mathcal{M}/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \mathcal{F}(\text{“angles”}) \cos [\Phi(t)]$$

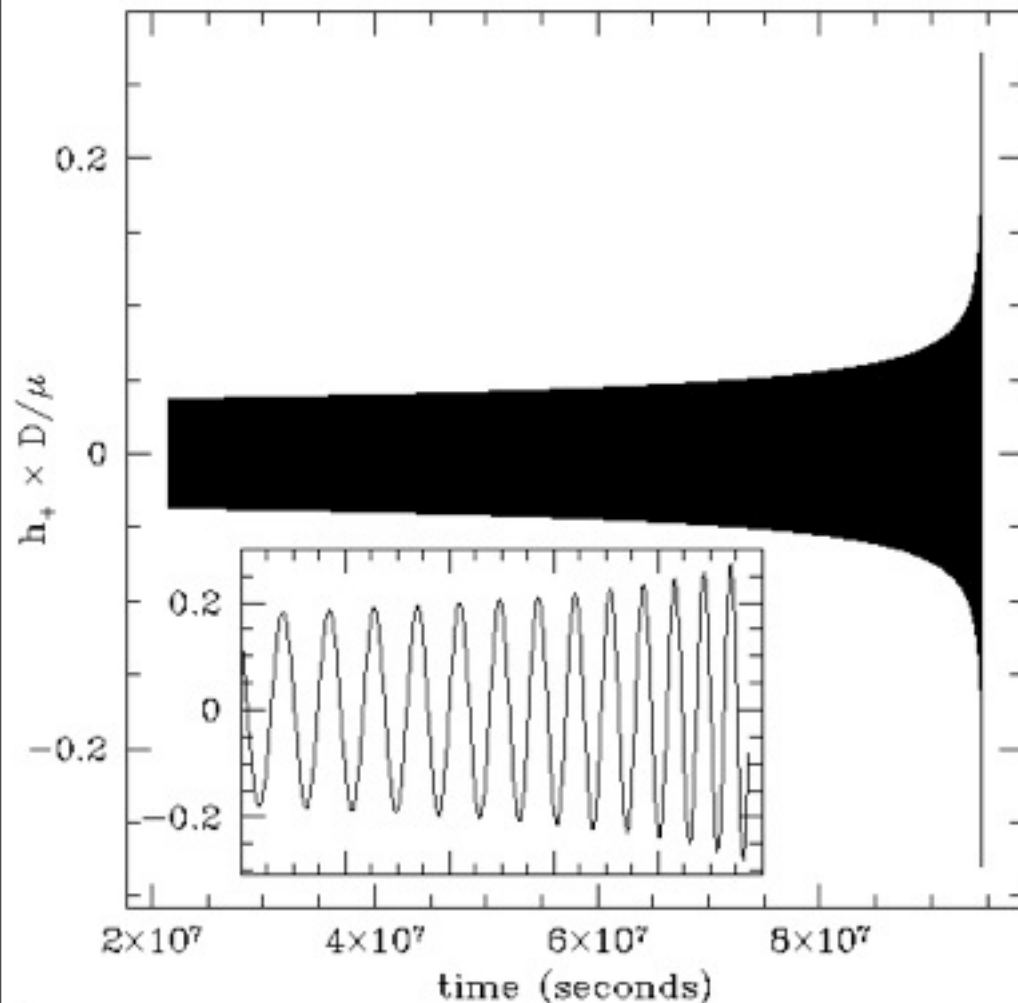


**Example waveform:
Both black holes non-
spinning.**

Smooth chirp from low
to high frequencies.

Inspiral measurements

$$h_+ = \frac{[G(1+z)\mathcal{M}/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \mathcal{F}(\text{“angles”}) \cos [\Phi(t)]$$

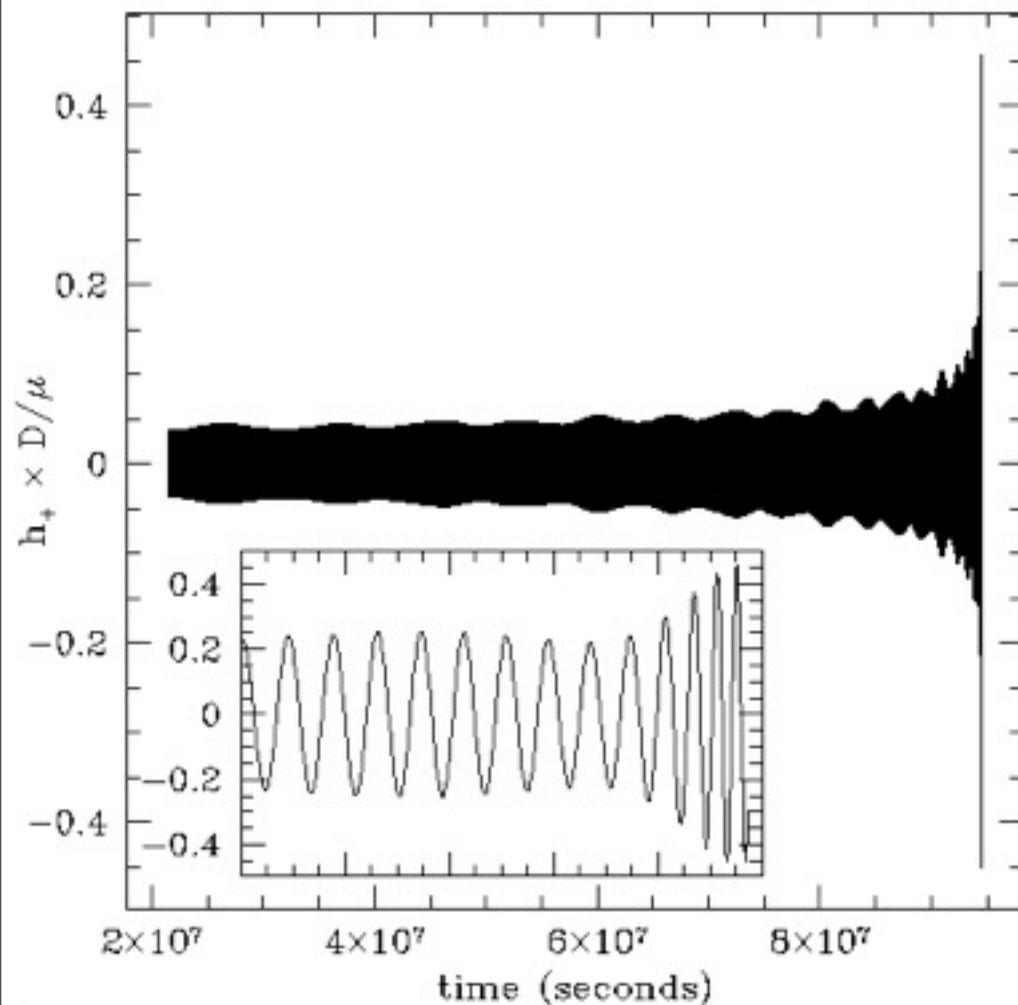


**Example waveform:
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Inspiral measurements

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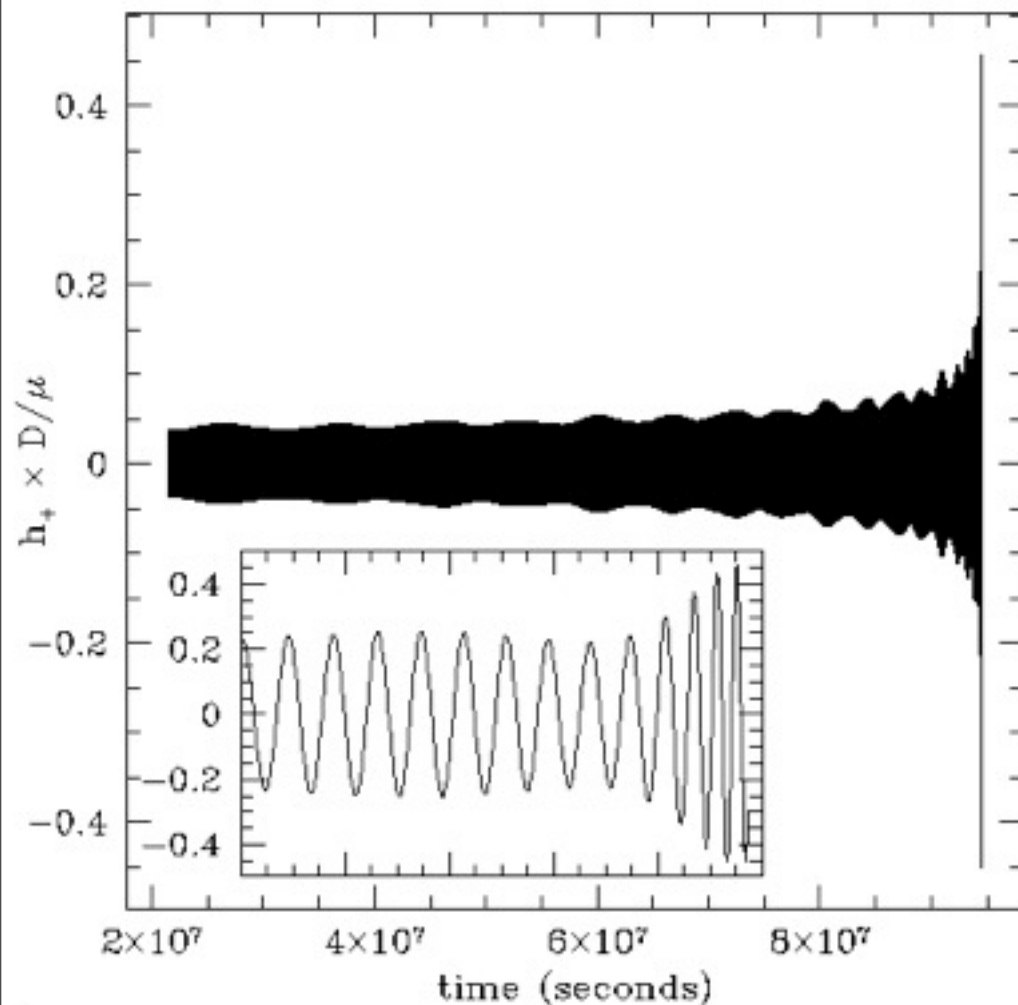


Spins cranked up!
Spin 1 = Spin 2 =
99% maximum

Strong frequency and
amplitude modulation
gives spin precision.

Inspiral measurements

$$h_+ = \frac{[G(1+z)\mathcal{M}/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \mathcal{F}(\text{“angles”}) \cos[\Phi(t)]$$



Spins cranked up!
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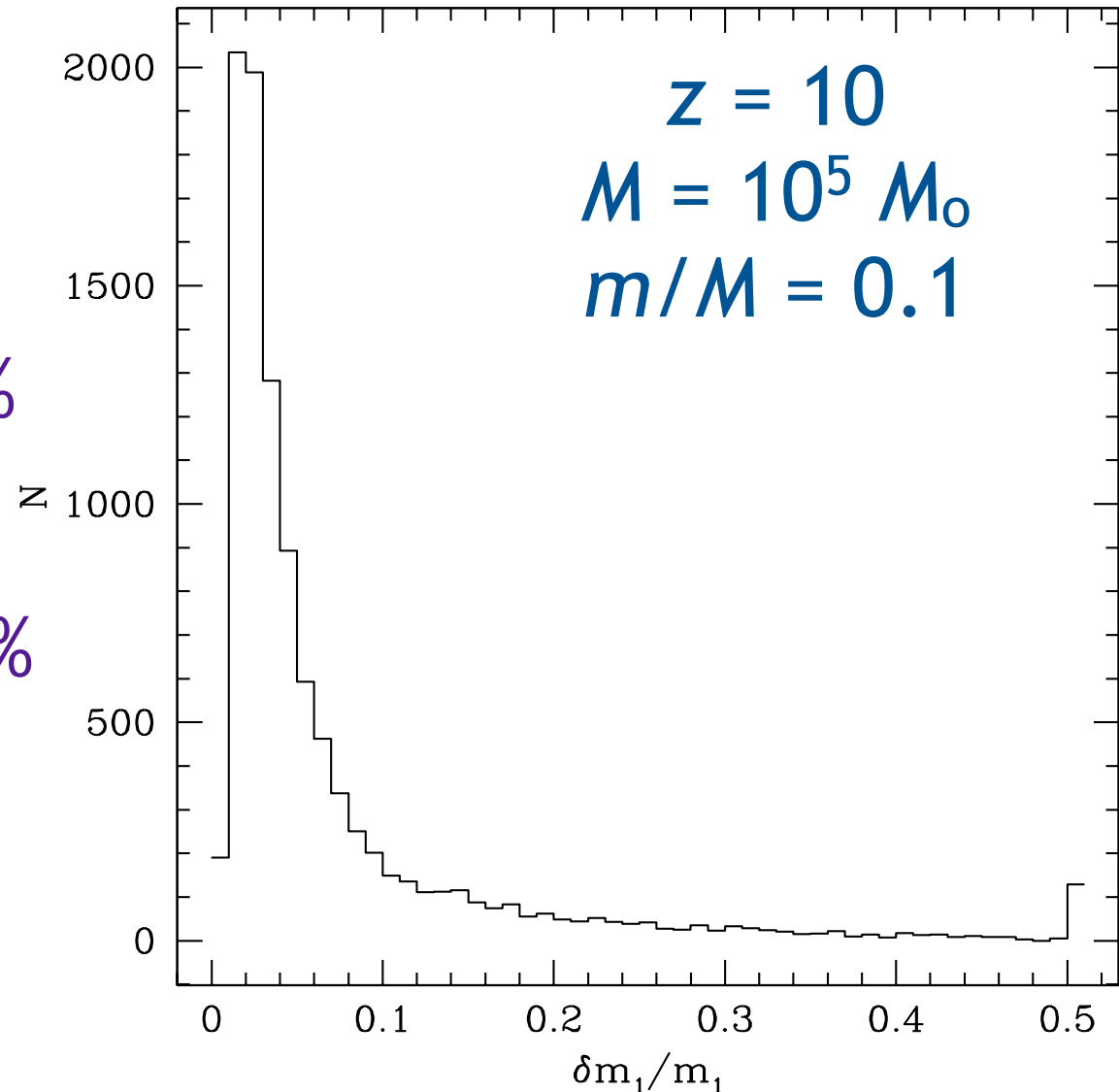
Strong frequency and
amplitude modulation
gives spin precision.

Highly precise masses, moderately precise spins

Example: distribution
of errors in mass
measurement

Median mass error 3.6%

90% of distribution
confined to $\delta m/m < 17\%$

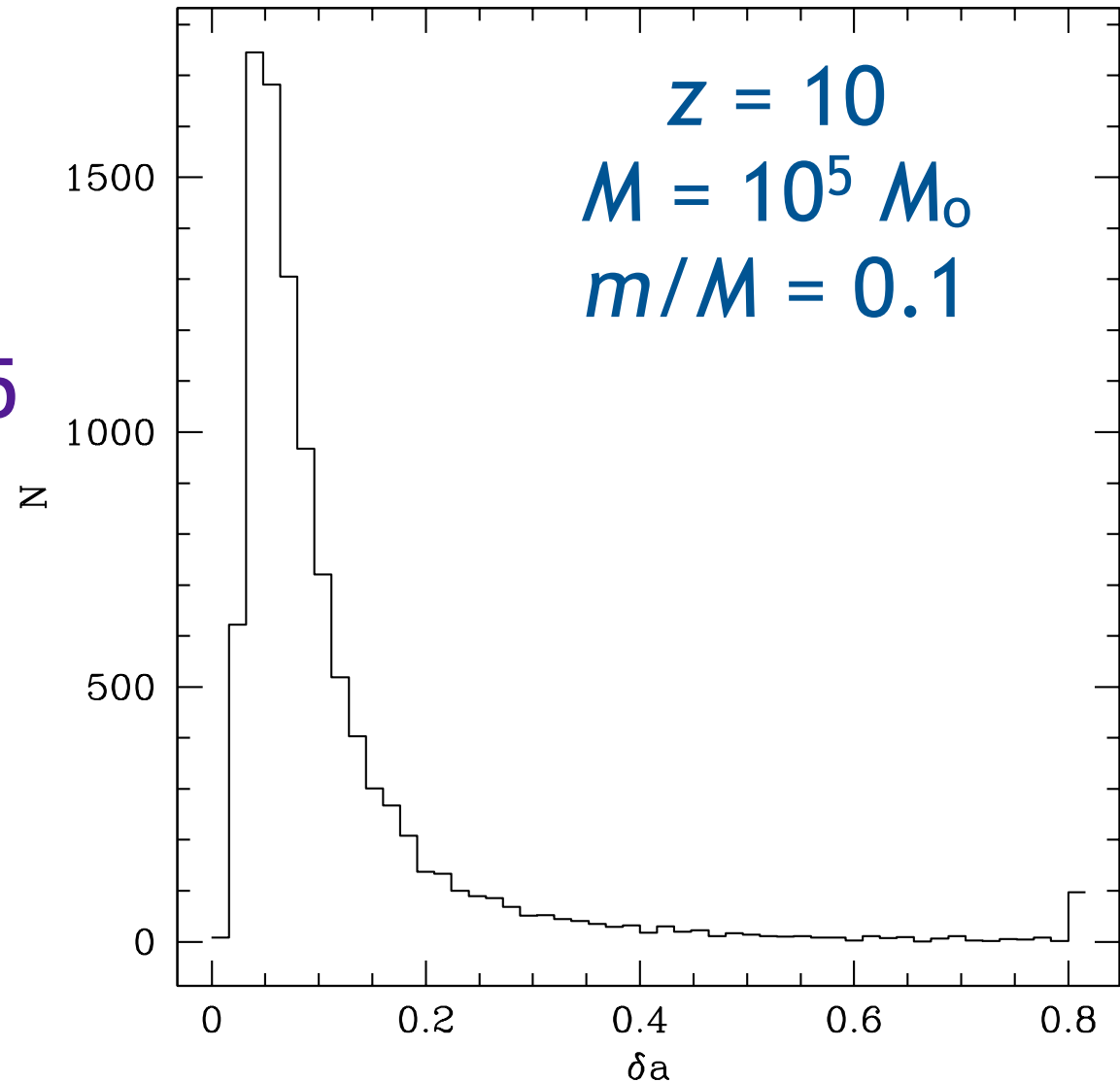


Highly precise masses, moderately precise spins

Example: distribution
of errors in spin
measurement

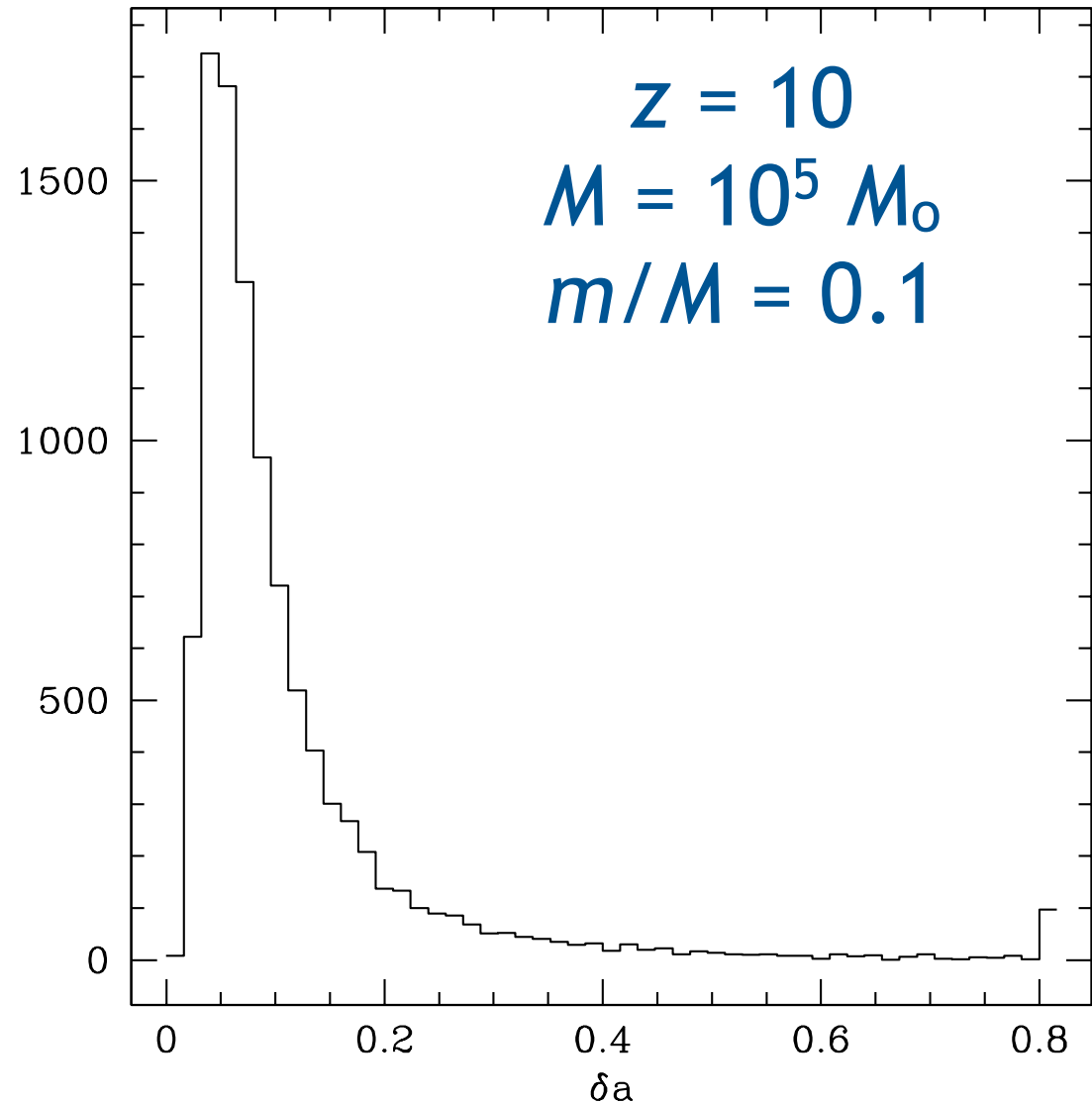
Median spin error 0.075

90% of distribution
confined to $\delta a < 0.22$



Highly precise masses, moderately precise spins

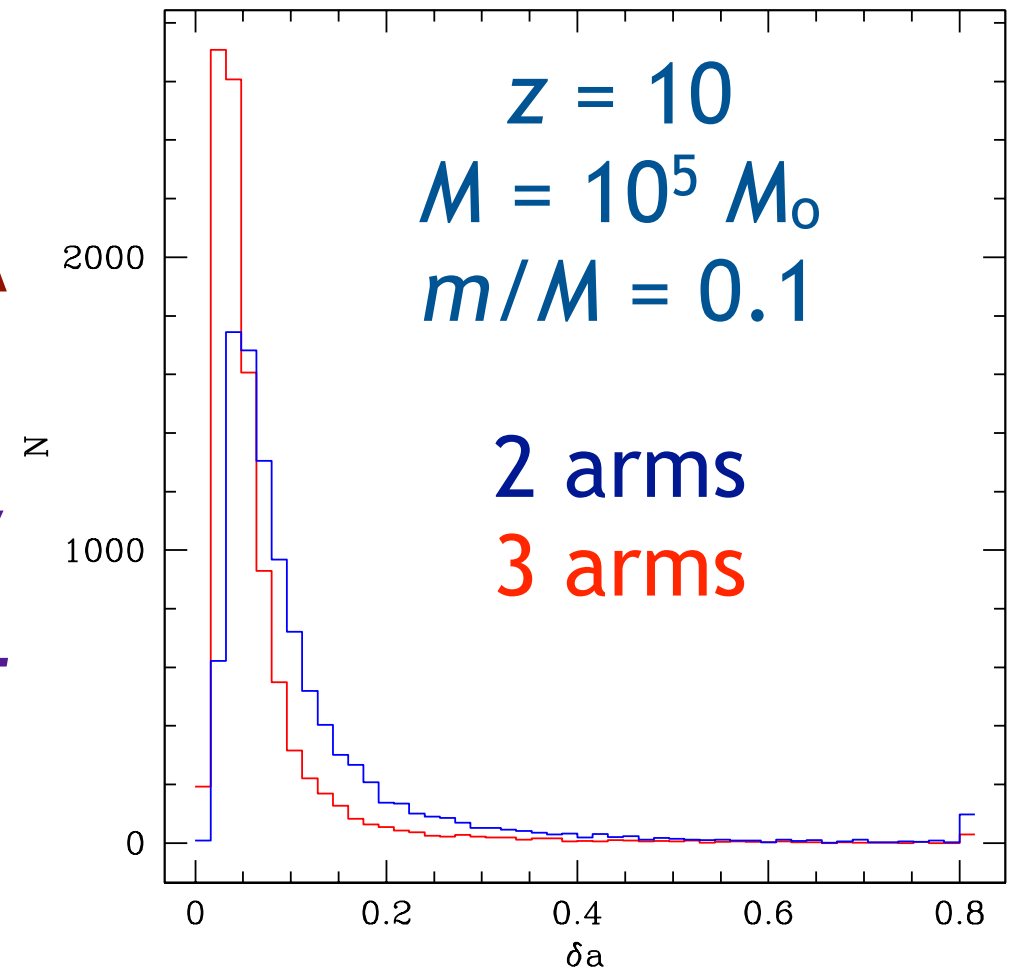
Similar measurement distributions for mass and spin found out to $z \sim 15$, and for masses of 10^4 - (a few) $10^5 M_{\odot}$.



Not much improvement on mass and spin if we imagine using a 3-armed variant of eLISA

Example: Precision of spin measurement, comparing eLISA with 2 arms to eLISA with 3 arms.

Improvement entirely due to boost of signal-to-noise ratio.



CAUTION: *Redshifted* masses are precisely measured

Consider nearby source: Phase encodes timescale for orbit change; tells us mass scale:

$$\int f(t) dt \rightarrow \tau_{\text{orbit}} \propto \mathcal{M}$$

Consider cosmological source: Now measure a *redshifted* timescale; infer *redshifted* mass:

$$\int f(t) dt \rightarrow (1 + z)\tau_{\text{orbit}} \propto (1 + z)\mathcal{M}$$

Redshift is degenerate with masses.

True when taken to higher order as well ...
cannot infer redshift from GW measurables.

Pieces of inspiral waveform

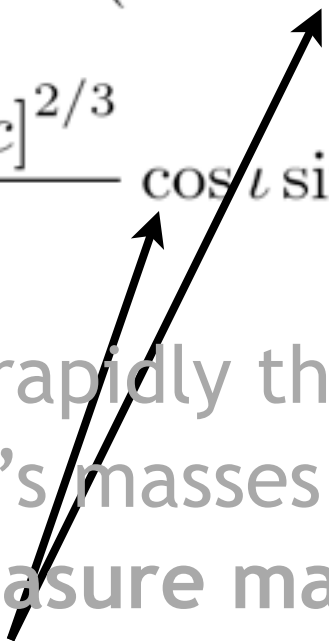
$$h_+ = \frac{[GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} (1 + \cos^2 \iota) \cos \left[2\pi \int f(t) dt \right]$$

$$h_\times = \frac{2 [GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \cos \iota \sin \left[2\pi \int f(t) dt \right]$$

1. Phase. Depends on how rapidly the orbit evolves.
Rate is controlled by binary's masses and spins.

Measure the phase, measure masses and spins.

Pieces of inspiral waveform

$$h_+ = \frac{[GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} (1 + \cos^2 \iota) \cos \left[2\pi \int f(t) dt \right]$$
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1. Phase. Depends on how rapidly the orbit evolves.
Rate is controlled by binary's masses and spins.

Measure the phase, measure masses and spins.

2. Inclination of orbital plane to line of sight.

Measure both polarizations, you measure this angle.

Pieces of inspiral waveform

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1. Phase. Depends on how rapidly the orbit evolves. Rate is controlled by binary's masses and spins.

Measure the phase, measure masses and spins.

2. Inclination of orbital plane to line of sight.

Measure both polarizations, you measure this angle.

3. Luminosity distance. Sets amplitude, once masses and inclination are determined.

Pieces of inspiral waveform

$$h_+ = \frac{[GM/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} (1 + \cos^2 \iota) \cos \left[2\pi \int f(t) dt \right]$$

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Pieces of inspiral waveform

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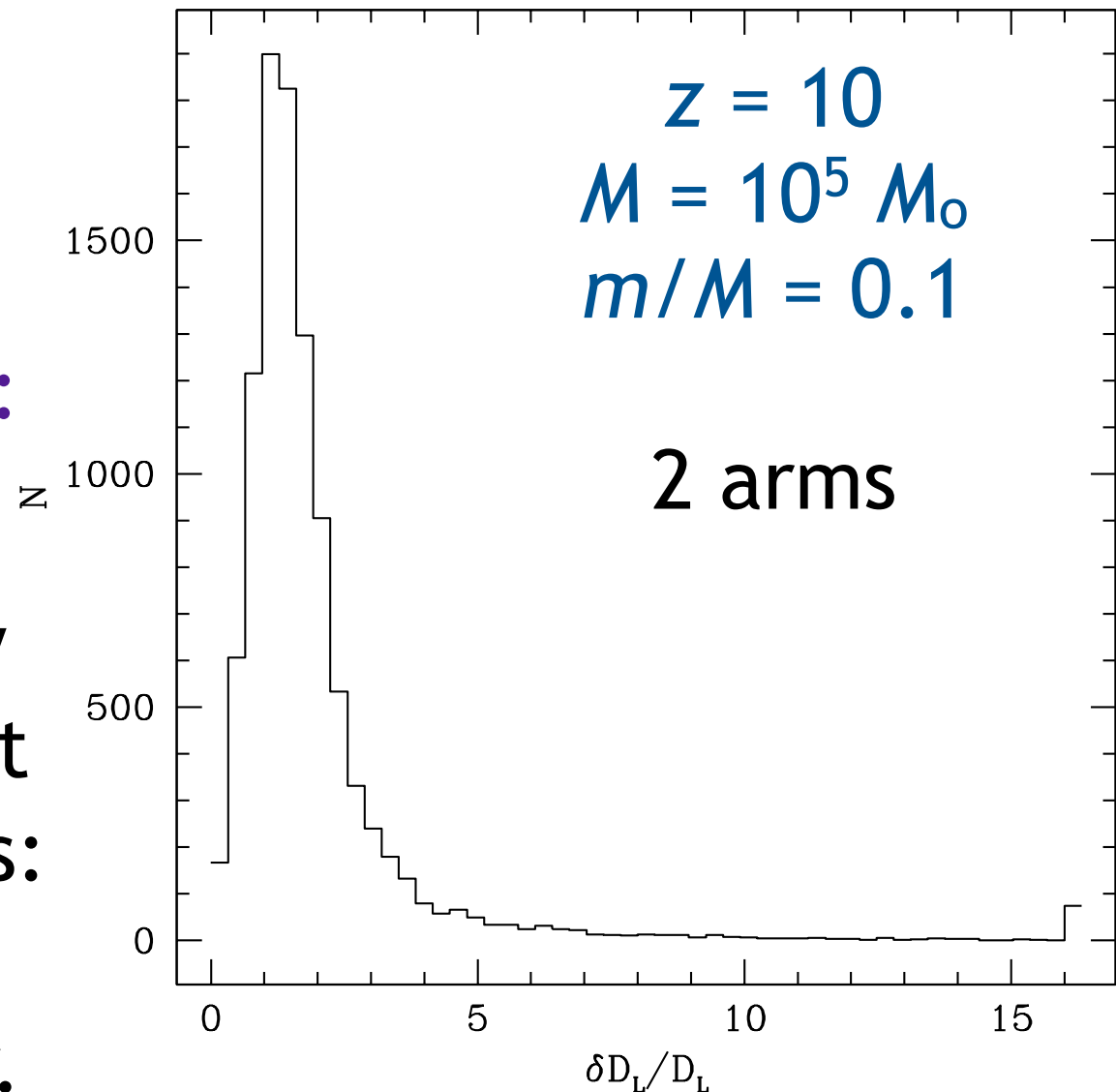
Once we know distance, can get z by inverting distance-redshift relation, assuming the cosmography ... Lets us break degeneracy and determine *rest frame* parameters of binary.

Both polarizations and directly measuring distance requires 3 arms

Example: distribution of errors in distance for 2-armed eLISA

Median distance error:
 $\delta D/D = 1.5$

Distance is essentially unconstrained for most of these measurements:
Not good enough to break m - z degeneracy.

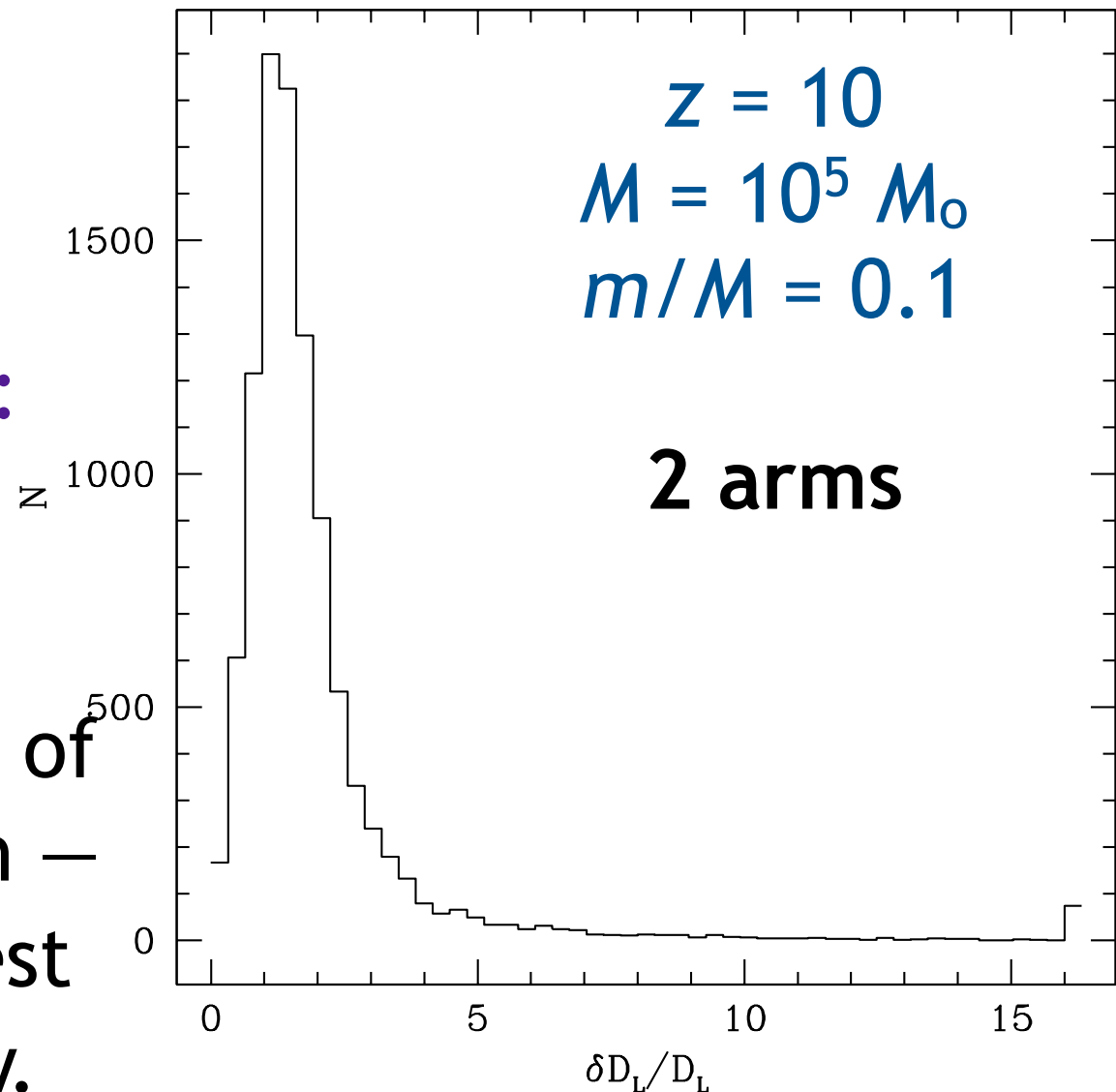


Both polarizations and directly measuring distance requires 3 arms

Example: distribution of errors in distance for 2-armed eLISA

Median distance error:
 $\delta D/D = 1.5$

Many events make it possible to test models of early black hole growth — but not as well as if rest frame masses directly.

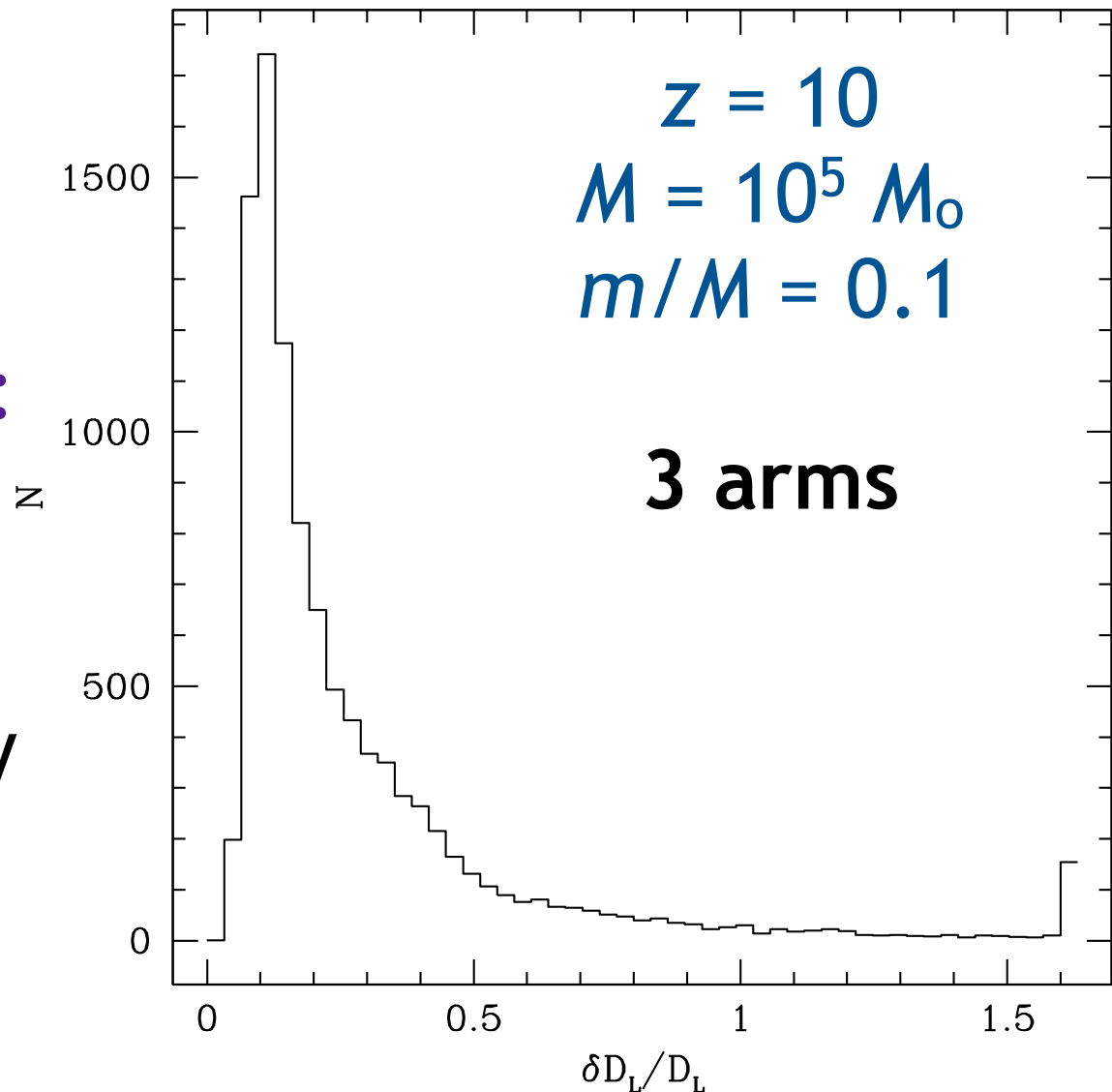


Both polarizations and directly measuring distance requires 3 arms

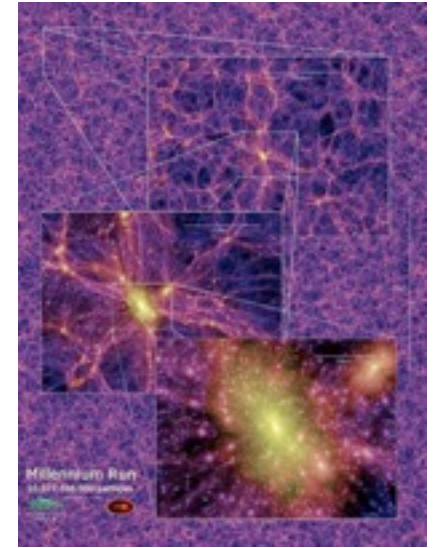
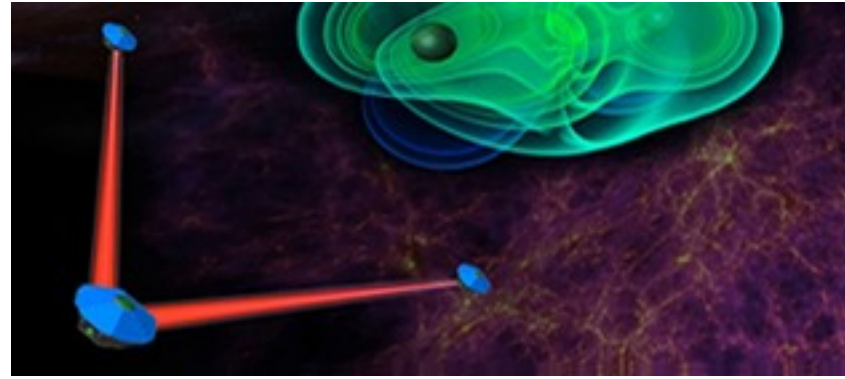
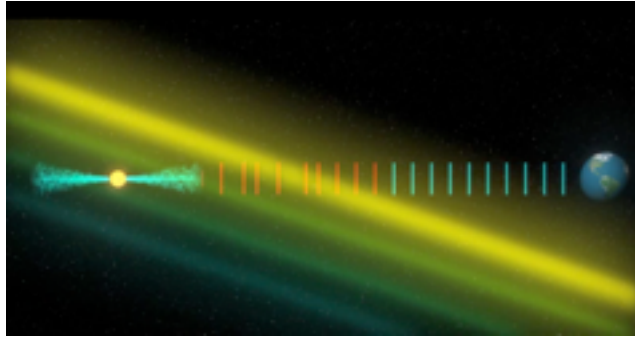
Example: distribution of errors in distance for 3-armed eLISA

Median distance error:
 $\delta D/D = 0.17$

Accurate enough to break m - z and directly track rest frame evolution of masses.



Conclusion

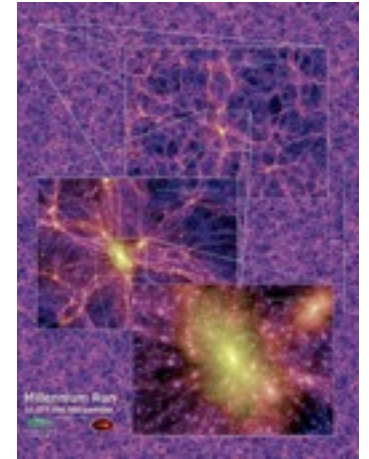
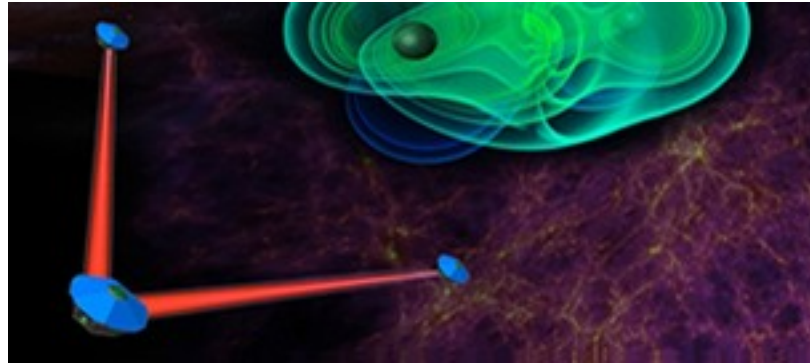
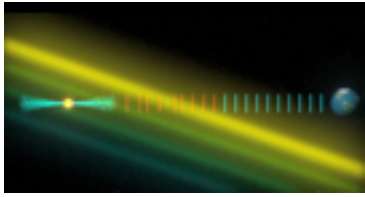


GWs from MBH binaries information rich:

- * Precise (redshifted) masses
- * Good information about spin parameter
- * *Possibly* distances accurately enough to break mass-redshift degeneracy.

Nature appears to be giving us the binaries ...
“just” need to start measuring these waves.

Conclusion



***Let's grab
some babies!!***



<http://www.etsy.com/listing/67649622/baby-black-hole-now-with-adoption>