3d Gravity and Teichmüller TQFT

Lorenz Eberhardt

IAS Princeton

Based on work in progress with Scott Collier & Mengyang Zhang

Motivation

- The relation of 3d gravity with $\Lambda < 0$ to $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ Chern-Simons theory has been much discussed in the literature. [Witten '88;...]
- The relation between the two theories is subtle and we understand neither side particularly well.
- In particular, the relation is only of limited use for the purpose of understanding and solving 3d gravity at the quantum level.
- Currently there is a patchwork of different (bulk) approaches to 3d gravity, none of them are entirely satisfactory or applicable in general situations. [Maloney, Witten '07; Giombi, Maloney, Yin '08; Cotler, Jensen '18-'20; LE '22;...]
- There is also a large number of partially conflicting proposals for the nature of the boundary CFT. [Witten '07; Maloney, Witten '07; Benjamin, Collier, Maloney '20; Cotler, Jensen '20; Maxfield, Turiaci '20; Chandra, Collier, Hartman, Maloney '22; ...]

This talk

- In this talk, we propose a more precise correspondence of 3d gravity with a much more well-defined TQFT: Teichmüller TQFT. [Verlinde '89; Teschner '02 - '05; Kashaev, Andersen '11-'18; Mikhaylov '17; . . .]
- Teichmüller TQFT still has a number of subtleties related to its continuous spectrum of Wilson lines, but they can be overcome.
- We will explain that this relation is computationally very useful and leads in particular to a complete solution of 3d gravity on all *hyperbolic* 3-manifolds, i.e. those admitting a classical saddle.

Outline

I. The phase space of 3d gravity

II. Teichmüller TQFT

III. Concrete applications

I. The phase space of 3d gravity

The classical story

- The relation of 3d gravity and SL$(2, \mathbb{R})$ Chern-Simons theory follows classically by setting [Witten '88]

$$A_\mu^{\pm,a} = \omega_\mu^a \pm \ell^{-1} e_\mu^a \ .$$

spin connection

dreibein

SL$(2, \mathbb{R})$ gauge fields

AdS length

- $A_\mu^\pm$ transform like SL$(2, \mathbb{R})$ gauge fields under infinitesimal coordinate transformations and local Lorentz transformations.
- This change of variables maps the Einstein-Hilbert action to the SL$(2, \mathbb{R}) \times$ SL$(2, \mathbb{R})$ Chern-Simons action with level

$$k = \frac{\ell}{4G} = \frac{c}{6} \ .$$

[Brown, Henneaux '86]

Invertibility

- We forgot about an additional constraint in the gravity theory: we need a positive definite metric everywhere.
- In particular, e_μ^a needs to be invertible. No such condition exists in $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory.
- We will argue that this problem is completely resolved by considering Teichmüller TQFT instead.
- Luckily, Teichmüller TQFT is a much better understood TQFT than $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons theory and this makes the theory actually simpler.

Global problems

- There are further global problems with the gauge group.
- In gravity, there can be large diffeomorphisms (3-dimensional analogues of modular transformations).
- They are formalized in the mapping class group

$$\mathsf{MCG}(M) = \mathsf{Diff}(M)/\mathsf{Diff}_0(M)\ .$$

- While these are gauged in gravity, they remain ungauged on the TQFT side.
- For hyperbolic manifolds, we can resolve this problem by hand.

The phase space

- It is useful to understand the phase space of 3d gravity on a spatial manifold Σ.
- In Chern-Simons theory, we have

  CS Phase space on $\Sigma =$ Flat SL$(2,\mathbb{R}) \times$ SL$(2,\mathbb{R})$ bundles on Σ .

- However, not all flat bundles correspond to good initial conditions for gravity, since they do not all lead to smooth regular solutions.
- The moduli space of flat (P)SL$(2,\mathbb{R})$ bundles on Σ is actually a disconnected space, whose components are labelled by a certain topological number.

The phase space

- One of these components is Teichmüller space which describes hyperbolic structures on Riemann surfaces.
- Gauge fields in this subspace, describe precisely regular metrics. [Krasnov, Schlenker '05; Scarinci, Krasnov '11; Kim, Porrati '15]
- Thus

$$\text{Gravity phase space on } \Sigma = (\text{Teichmüller space on } \Sigma)^2 \ .$$

The Hilbert space

- To pass to the quantum theory, we need to quantize phase space.
- In particular, the Hilbert space of 3d gravity on Σ is obtained by quantizing Teichmüller space.
- This was achieved in a series of papers [Verlinde '89; Kashaev '98; Teschner '02 -'05]. The main result is

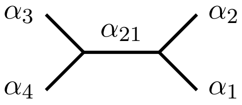
  $$\text{Quantization of Teichmüller space} = \text{Liouville conformal blocks}\,.$$

- Thus the Hilbert space of 3d gravity is

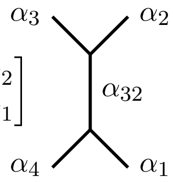
  $$\mathcal{H}_{\text{gravity}} = \mathcal{H}_\Sigma \otimes \overline{\mathcal{H}}_\Sigma\ ,$$
  $$\mathcal{H}_\Sigma = \text{Liouville conformal blocks on } \Sigma\ .$$

Crossing transformations

- $\mathcal{H}_\Sigma$ enjoys additional structure. Since conformal blocks transform under crossing transformations, we get a (unitary, centrally extended) representation of the 2d mapping class group on $\mathcal{H}_\Sigma$.
- The most important move is the four-point crossing transformation



$$= \int \mathrm{d}\alpha_{32}\, \mathbb{F}_{\alpha_{21},\alpha_{32}} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}$$



.

↓ Virasoro fusion kernel

[Teschner, Ponsot '99]

- They satisfy the Moore-Seiberg consistency conditions.

[Moore, Seiberg '89]

II. Gravity and Teichmüller TQFT

A 3d TQFT

- From the point of view of gauge theory, we restricted the phase space in an arbitrary way and only quantized Teichmüller space, since it corresponds to smooth 3d gravity solutions.
- In principle, this does not have to lead to something consistent.
- But the fact that Liouville conformal blocks behave nicely under crossing transformations means that we have all the data we need to define a 3d TQFT.
- This is called Teichmüller TQFT. Its relation to Liouville theory is analogous to the relation of CS theory to WZW models.

Subtleties due to non-normalizable vacuum

- This TQFT satisfies all the usual axioms, except for one: it does not possess a trivial line.
- Said differently, the vacuum block is not normalizable. Thus, various partition functions of the theory will diverge. For example,

$$Z_{\mathsf{Teich}}(\mathrm{S}^3) = \mathbb{S}_{\mathsf{vac},\mathsf{vac}}$$

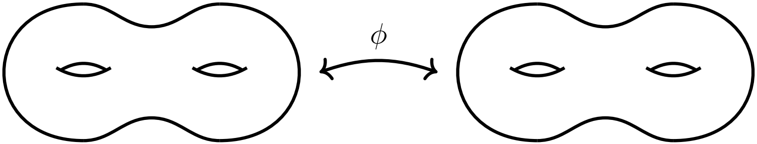
  is not well-defined.
- Naively performing surgery as in CS-theory usually leads to ill-defined expressions.

Heegaard splitting

- One way to compute partition functions in Teichmüller TQFT is by using Heegaard splitting.
- For closed M (without Wilson lines), we can always write

$$M = H_1 \cup_\phi H_2$$

for two handlebodies H_1 and H_2, which are identified along their common boundary Σ. We can apply a non-trivial modular transformation ϕ before gluing the two handlebodies.



A prescription for Teichmüller TQFT partition functions

- The path integral on a handlebody prepares a state in the boundary surface Σ. As in CS theory, this state is simply the conformal block in the respective channel as specified by the handlebody.
- Thus for $M = H_1 \cup_\phi H_2$ a Heegaard presentation of M, its partition function is given by

$$Z_{\mathsf{Teich}}(M) = \langle \mathcal{F}_{\mathsf{vac}} \,|\, U(\phi) \,|\, \mathcal{F}_{\mathsf{vac}} \rangle \ ,$$

  where $U(\phi)$ is the unitary representation of the mapping class group element ϕ on $\mathcal{H}_\Sigma$ and $\mathcal{F}_{\mathsf{vac}}$ is the vacuum block.
- We skipped the definition of the inner product in Teichmüller TQFT, but see [Verlinde '89].
- This can be generalized to deal with Wilson lines and/or boundaries.

Finiteness

- Since $\mathcal{F}_{\text{vac}}$ is not normalizable, this inner product may or may not be finite.
- We conjecture that it is always finite for hyperbolic 3-manifolds, since those are the manifolds admitting a classical solution.
- This holds up in the explicit examples we considered.

A prescription for 3d gravity

- Knowing how to compute Teichmüller TQFT partition functions immediately leads to a prescription for 3d gravity.
- In the path integral, we divide by the diffeomorphism group $\mathsf{Diff}(M)$ when we compute the partition function Z_{gravity}.
- Thus we propose for the gravity partition on a fixed topology

$$Z_{\mathsf{gravity}}(M) = \frac{1}{|\mathsf{MCG}(M)|} \left| Z_{\mathsf{Teich}}(M) \right|^2$$

- As a consequence of the rigidity theorems of hyperbolic 3-manifolds, the mapping class group is finite in this case.
- To get the full gravity partition function, we should of course also sum over all different topologies with a fixed conformal boundary. We have nothing new to say about this sum.

III. Concrete applications

The Euclidean wormhole

- An important topology for holography purposes is the Euclidean wormhole $\Sigma \times \mathbb{R}$. If we take the moduli of the two boundaries to coincide, it supports a simple hyperbolic metric [Maldacena, Maoz '05]

$$\mathrm{d}s^2_{\Sigma\times\mathbb{R}} = \mathrm{d}\rho^2 + \cosh^2(\rho)\mathrm{d}s^2_\Sigma$$

  with $\mathrm{d}s^2_\Sigma$ a hyperbolic metric on Σ.
- From the TQFT perspective, this topology represents the identity on the Hilbert space $\mathcal{H}_\Sigma$.
- Thus

$$Z_{\mathsf{Teich}}(\Sigma \times \mathbb{R}) = \sum_{\substack{\text{conformal blocks} \\ \mathcal{F}_\Sigma \text{ on } \Sigma}} \frac{\mathcal{F}_\Sigma\,\overline{\mathcal{F}_\Sigma}}{\langle \mathcal{F}_\Sigma | \mathcal{F}_\Sigma \rangle}\ .$$

The Euclidean wormhole (cont'd)

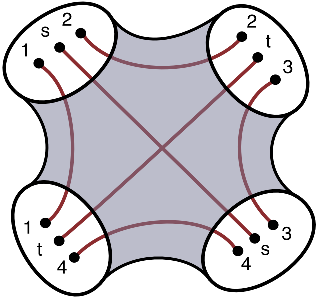
- Using the explicit form of the inner product, this is simply the conformal block expansion of a Liouville correlation function/partition function on Σ.
- The moduli on both sides of the wormhole can in principle be different.
- The mapping class group is trivial.
- To get the full gravity answer for this topology, we have to sum over the boundary modular transformations. In this case, we only need to sum over say the modular transformations of the right boundary.
- So for $\boldsymbol{m}_{\mathsf{L}}$ and $\boldsymbol{m}_{\mathsf{R}}$ the left and right moduli we propose

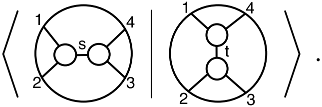
$$Z_{\text{gravity}}(\Sigma \times \mathbb{R}) = \sum_{\phi \in \text{MCG}(\Sigma)} |Z_{\text{Liouville}}(\boldsymbol{m}_{\mathsf{L}}, \phi \cdot \boldsymbol{m}_{\mathsf{R}})|^2 \ .$$

Compare with [Cotler, Jensen '20]

The four-boundary wormhole

- A less trivial example is the following four-boundary wormhole. It controls the leading non-Gaussianities in the proposal of a ensemble dual of 3d gravity [Chandra, Collier, Hartman, Maloney '22].
- We can cut the picture in half, so that we are computing the inner product of an s-channel and a t-channel conformal block:





The four-boundary wormhole (cont'd)

- The inner product is essentially the Virasoro fusion kernel by definition. Thus we have

$$Z_{\mathsf{Teich}}(\text{4 bdry wormhole}) = \underbrace{\rho(P_t)^{-1}}_{\text{Plancherel measure}} \underbrace{C_0(P_1,P_2,P_s)C_0(P_3,P_4,P_s)}_{\text{DOZZ structure constants}}\,\mathbb{F}_{st}\begin{bmatrix}3 & 2\\ 4 & 1\end{bmatrix}\,.$$

- The right hand side has full tetrahedral symmetry. This is manifest in the geometry, but is non-trivial from the point of view of crossing symmetry.

Other examples and properties

- 3d gravity partition functions are in principle topological invariants of the respective 3-manifolds.
- They lead in particular to knot invariants of hyperbolic links such as the figure 8 knot, the Borromean rings, ...
- Their semiclassical limit $c \to \infty$ computes the (renormalized) hyperbolic volume of these manifolds.

Conclusions

- We have given a definition of 3d gravity partition functions in terms of Teichmüller TQFT.
- This definition is computationally extremely useful and trivializes many computations that were often laboriously done in the literature.
- Our prescription works for all hyperbolic 3-manifolds.
- Very optimistically, this technology hopefully helps to settle the question whether 3d gravity makes sense at the quantum level.

Thank you!