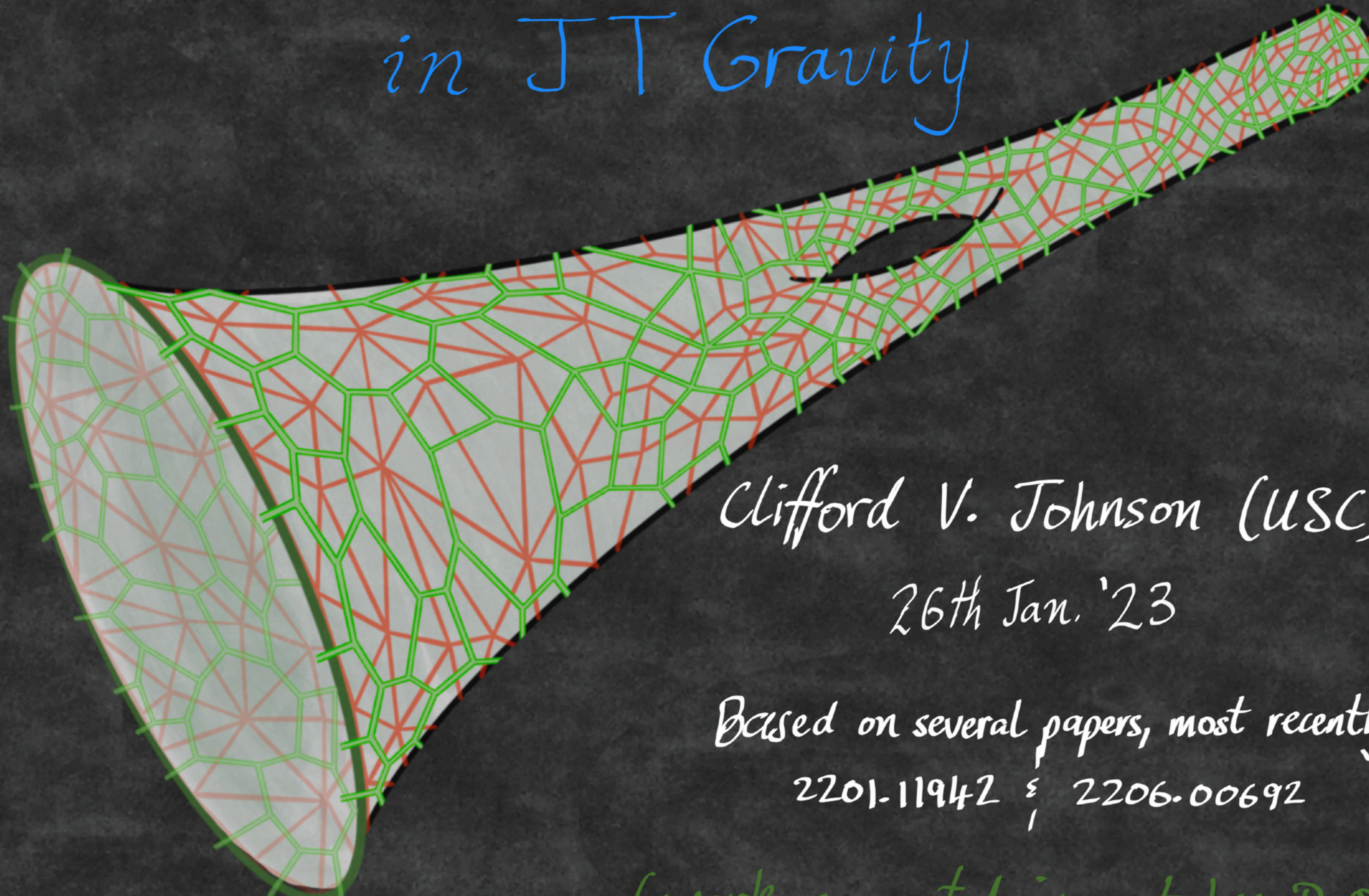
The Distribution of Ground States in JT Gravity



Clifford V. Johnson (USC)

26th Jan. '23

Based on several papers, most recently:

2201.11942 & 2206.00692

(work supported in part by D.O.E)

Plan:

Key: Elevate Wigner to the same level as 't Hooft in how we interpret matrix model results.

Slogan: Foregrounding the matrices in random matrix models

See also the talk by J. Sonner in this conference.
→ ETH perspective

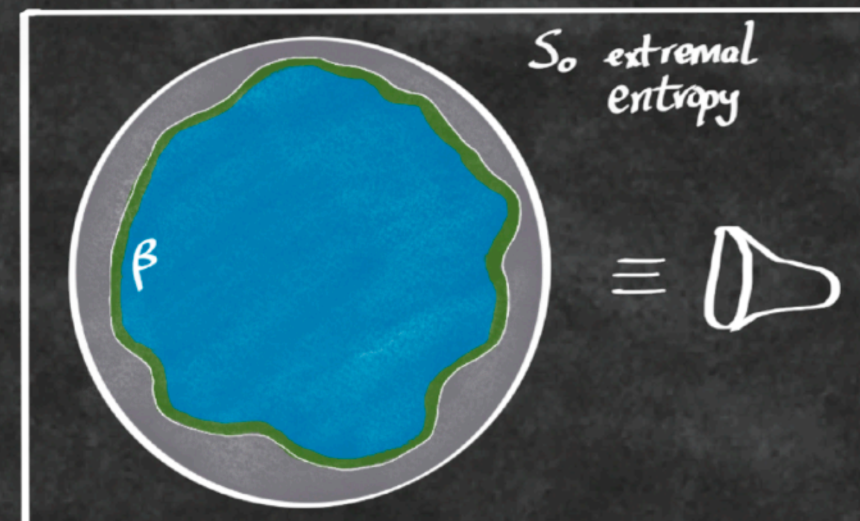
A brief reminder of SSS, etc

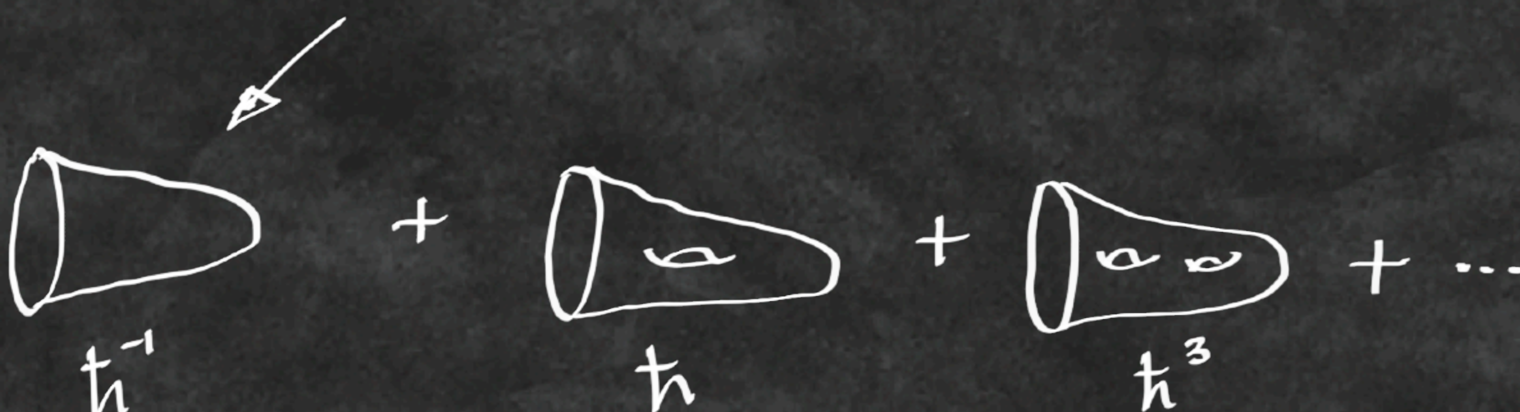
JT gravity (Euclidean)

$$I = -\frac{1}{2}\int_{\mathcal{M}} \sqrt{g}\,\phi(R+2) - \int_{\partial\mathcal{M}} \sqrt{h}\,\phi_b (K-1) - S_0 \underbrace{\left[\frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{g}\,R + \frac{1}{2\pi}\int_{\partial\mathcal{M}} \sqrt{h}\,K\right]}_{\chi(\mathcal{M}) = 2-2g-b}$$

$$Z_0(\beta) = \frac{e^{S_0} e^{\pi^2/\beta}}{4\sqrt{\pi}\,\beta^{3/2}} = \int dE\, \rho_0(E) e^{-\beta E}$$

$$\rho_0(E) = e^{S_0} \frac{\sinh(2\pi\sqrt{E})}{4\pi^2} \qquad \hbar = e^{-S_0}$$





$$Z(\beta) = \underbrace{\sum_{g=0}^{\infty} Z_g(\beta)}\ + \underbrace{\cdots}_{NP} \qquad \rho(E) = \sum_{g=0}^{\infty} \rho_g(E) + \underbrace{\cdots}_{NP}$$

Saad Shenker Stanford: perturbatively given by RMM!

Note:

- All geometries + topologies!
- Continuous spectrum.

- $\dim \mathcal{H} \sim e^{S_0}$, Where is the discreteness?

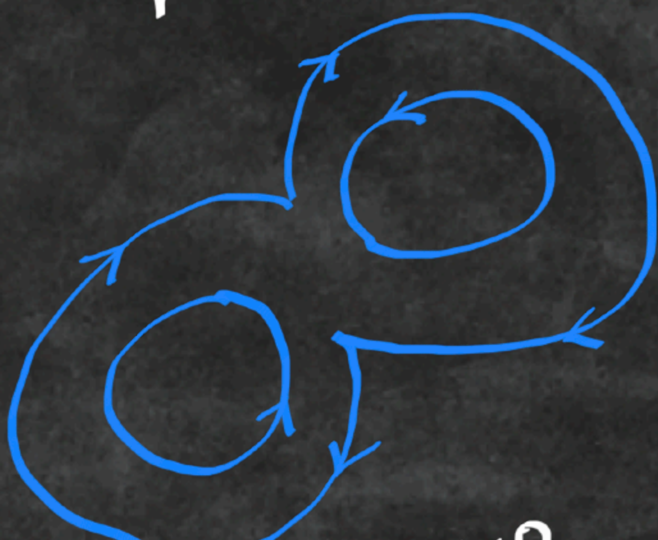
Matrix model approach:

$\rho_0(E)$ is leading piece of a spectral density of an Hermitian matrix model in the "double scaling limit". (DSL)

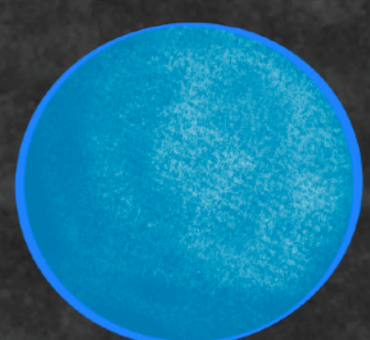
$$\tilde{Z}(g_4) = \int DM \exp\{-N \,\mathrm{Tr}\, V(M)\} \qquad V(M) = \frac{M^2}{2} + g_4 M^4$$

M is $N \times N$

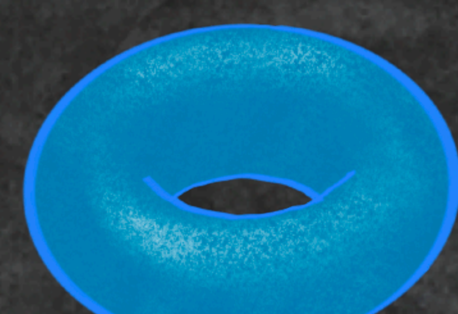
'**t Hooft:**



N^0



N^{-2}

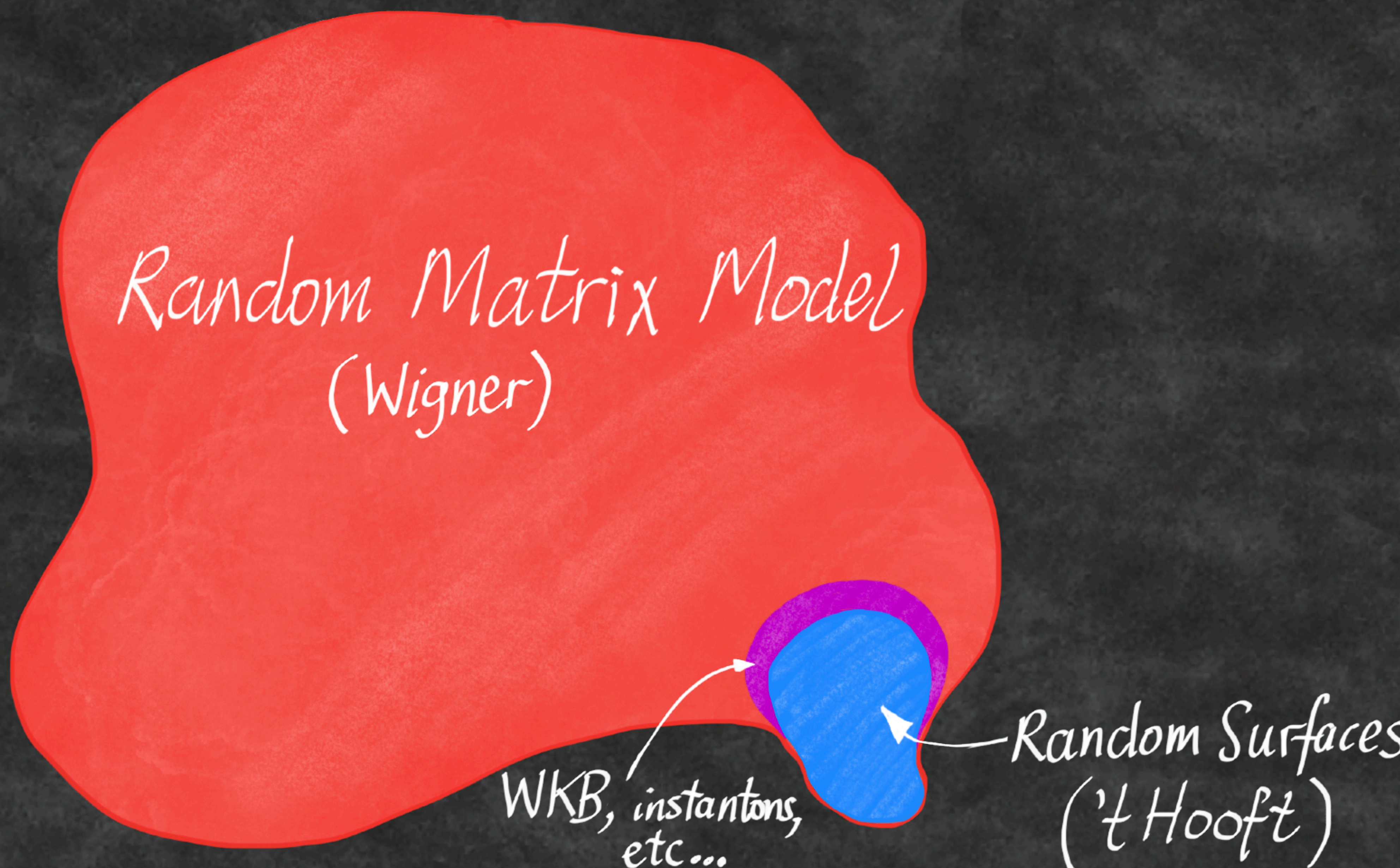


- large N captures topology
- $g_4 \to g_4^c$ captures large smooth surfaces

Brezin Kazakov
Douglas Shenker ~'90
Gross Migdal

Over the last few years, I've developed the tools needed to formulate and explore the full RMM of JT...

$$Z(\beta) = \sum_{g=0}^{\infty} Z_g(\beta) + \underbrace{\cdots}_{NP} \qquad \rho(E) = \sum_{g=0}^{\infty} \rho_g(E) + \underbrace{\cdots}_{NP}$$



...The picture that emerges is highly instructive.

It suggests a new perspective!

The output of non-perturbative formulation:

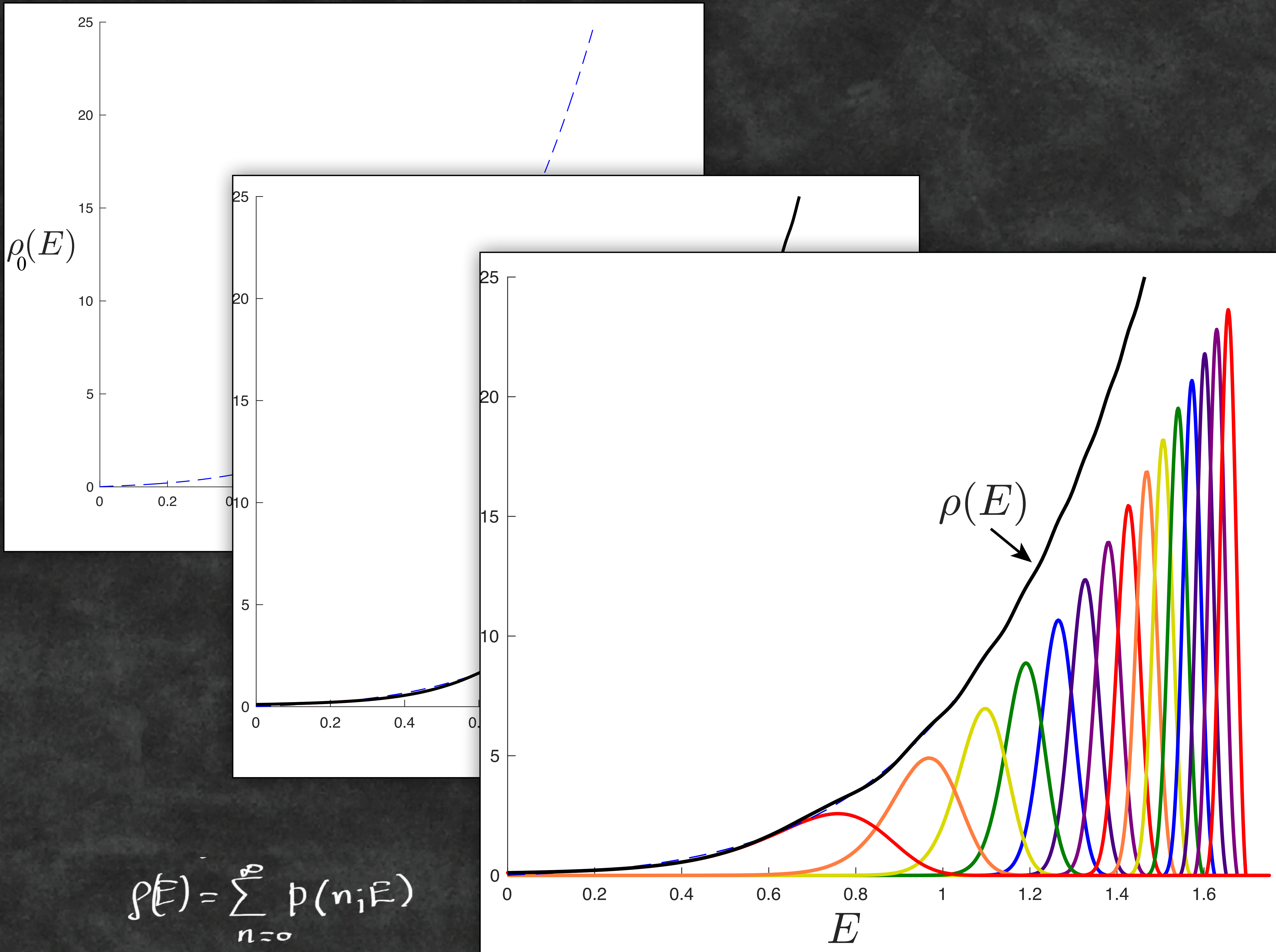
See C. V. Johnson:

1912.033637, 2005.01893, 2006.10959,
2104.02733, 2106.09048, 2112.00766,
2201.11942, 2206.00692

also:
CVJ and F. Rosso 2011.06026
CVJ, F. Rosso, A. Svesko 2102.02227

Further examples:

See also work by F. Rosso + collaborators at UBC on flat space examples



$$\rho(E) = \sum_{n=0}^{\infty} p(n; E)$$

Observations

- Gaps between levels set by $\hbar = e^{-S_0}$ → $\boxed{\text{Exactly what's expected from quantum gravity!}}$

$\dim \mathcal{H} \sim e^{S_0}$

- Peaks narrow swiftly toward higher energy
- So do the gaps. $\boxed{\rightarrow \text{Schwarzian spectrum at high } E}$

Schwarzian spectrum is secretly discrete.
There's an infinite family of spectra that resemble it asymptotically.
The matrix model is made from them.

$$\boxed{\{E_n\} : \int_0^{E_n} \rho_0(E)\,dE = n}$$

RMM $\rho(E)$ is smooth because using all the ensemble of spectra (Wigner) "fills in the gaps" at all E.

This is what enables smooth surfaces ('t Hooft) to emerge, as needed for Euclidean quantum gravity!

What would Wigner Do?

(WWWD)

My suggested interpretation.

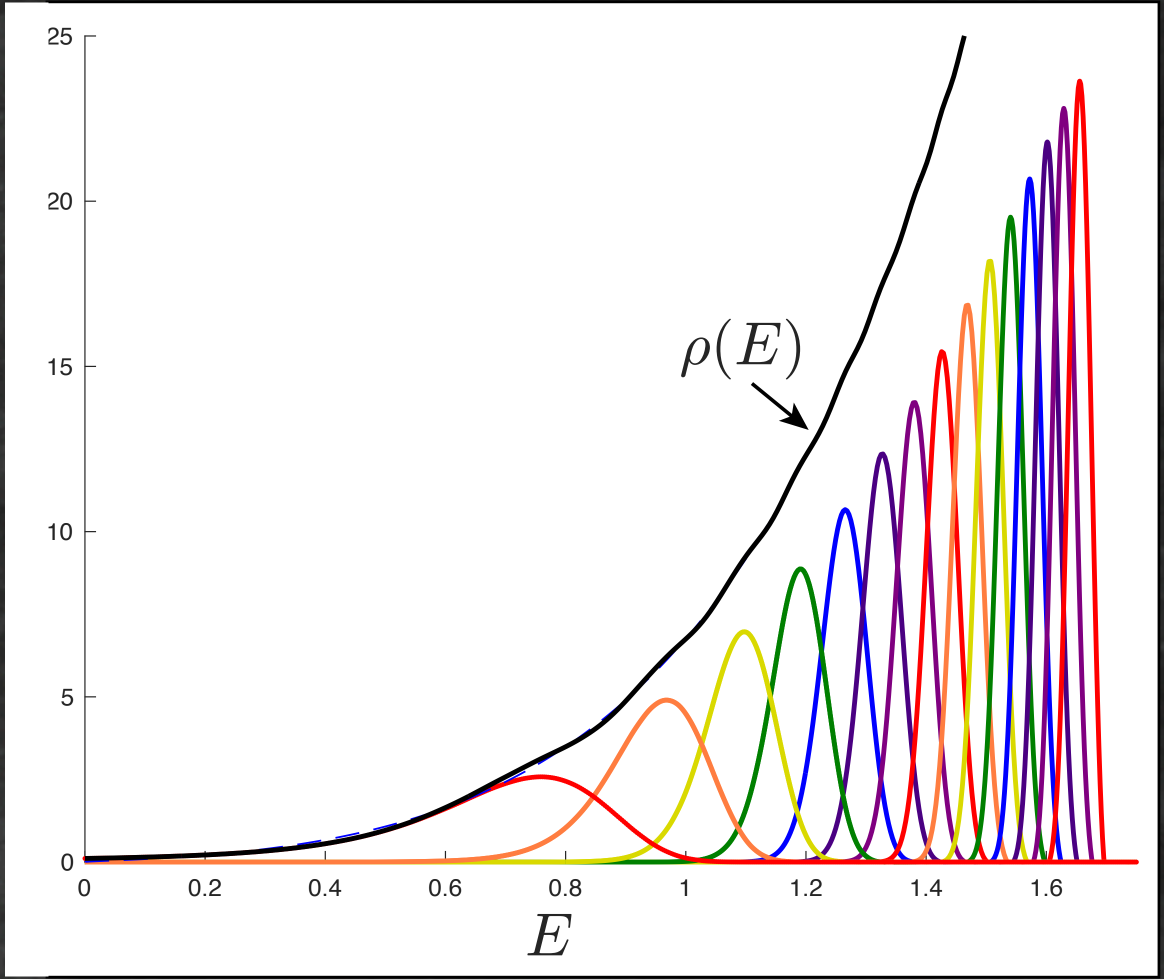
(WWWD – "what would Wigner do?")

WWWD: An Alternative History

Wigner:

- There's a Hilbert space: $\dim \mathcal{H} \sim e^{S_0}$
- There's some quantum gravity computation that yields discrete spectrum.
- Finding H, or its spectrum, is hard. But it exists.
- Try to characterize H statistically. Just as for atomic nuclei.
- Study all H with leading spectrum $\rho_0(E) \sim \sinh(2\pi\sqrt{E})$

This is what he finds: →



NB: Just like he did not conclude that atomic nuclei don't exist, and only are an ensemble, (!) ...

... he wouldn't do that here.

Summary Remarks

- The Random matrix model is an ensemble of JT gravity theories.
- Each member is an holographic dual compatible with the leading Schwarzian data.
- There's no factorization problem – holography is safe

So what is RMM for?

- Beautiful method for perturbation theory (Euclidean path integral).
- Shows that quantum gravity is much more than smooth surfaces.
- Shows how sensible thermodynamics emerges.
- Reveals and characterized discrete spectrum!

Strongly hints that Euclidean path integral should not be used as a definition of QG.

It is extremely useful for excavating universal features of the family of possible spectra.

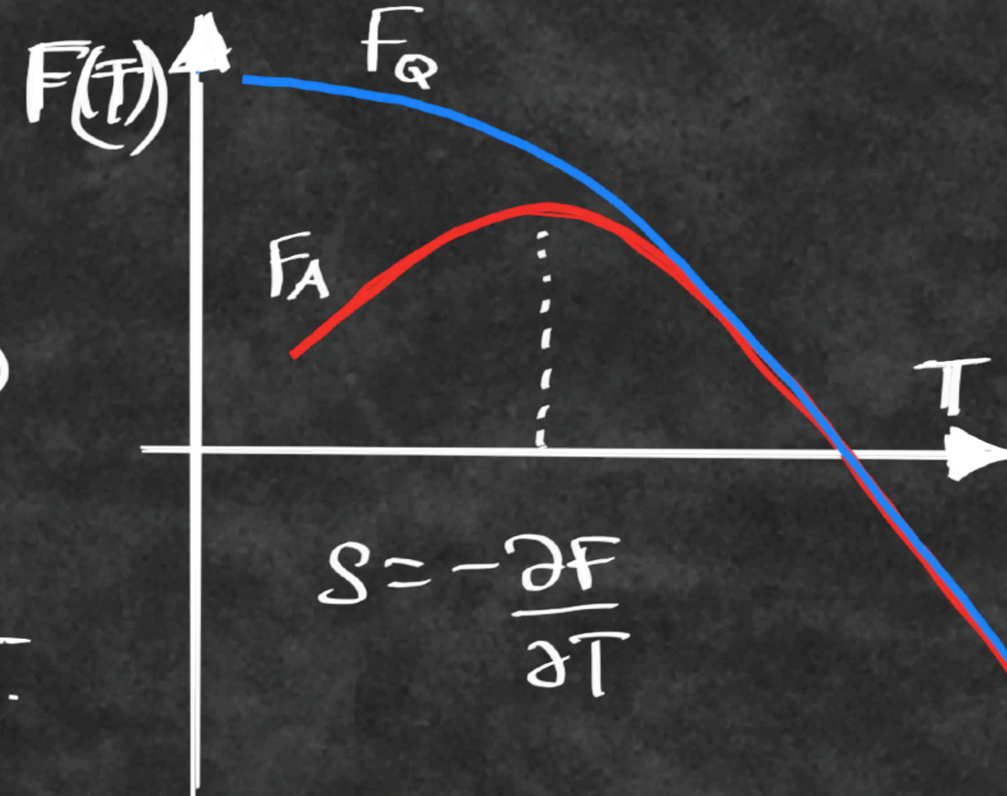
Thermodynamics

Thermodynamics Explained!

Take $Z(\beta) = \int \rho(E)\, e^{-\beta E}\, dE$

and compute $F = -T \ln Z(T)$

Problem: eventually $S(T)$ goes negative at some finite T.



Can't seem to get to $T=0$ where $S \to S_0$ Why?

$F_Q(T)$ better captures the thermodynamics of the typical $\{E_n\}$

Ensemble explanation: $Z(T) = \langle Z(T) \rangle$ $\hookrightarrow Z(T) = \sum_n e^{-\beta E_n}$

and above is $F_A(T) = -T \ln \langle Z(T) \rangle$

Really should compute $F_Q(T) = -T \langle \ln Z(T) \rangle$

Engelhardt et al. 2007.07444 attempted ...

Partial results: Okuyama, '21
Janssen + Mirbabayi, CVJ

RMM prototype: Gaussian Unitary Ensemble

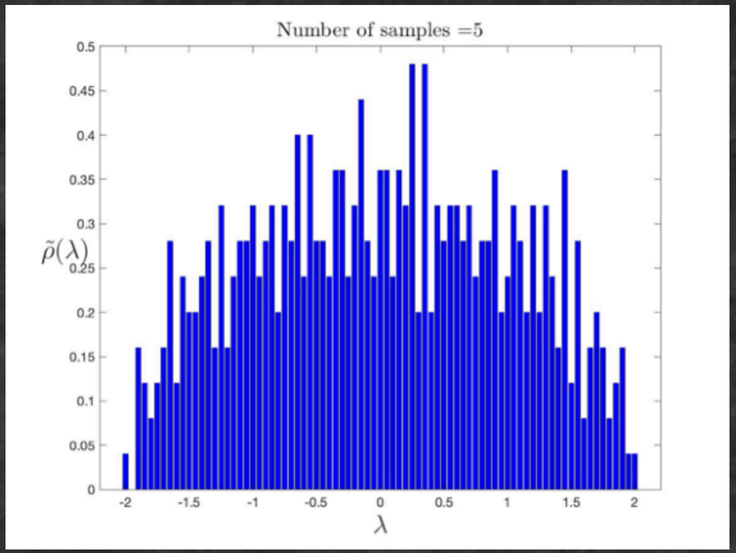
Try this at home!

- sample hermitian matrices M with probability

$$p(M) = e^{-\frac{\mathrm{Tr}\, M^2}{2}}$$

- compute eigenvalues, λ_i $\; i=1\ldots N$; histogram them:

$$\lambda \to \frac{\lambda}{\sqrt{N}} \; ; \; \tilde{\rho}_0(\lambda) = \frac{\sqrt{4-\lambda^2}}{2\pi}$$ Wigner semi-circle



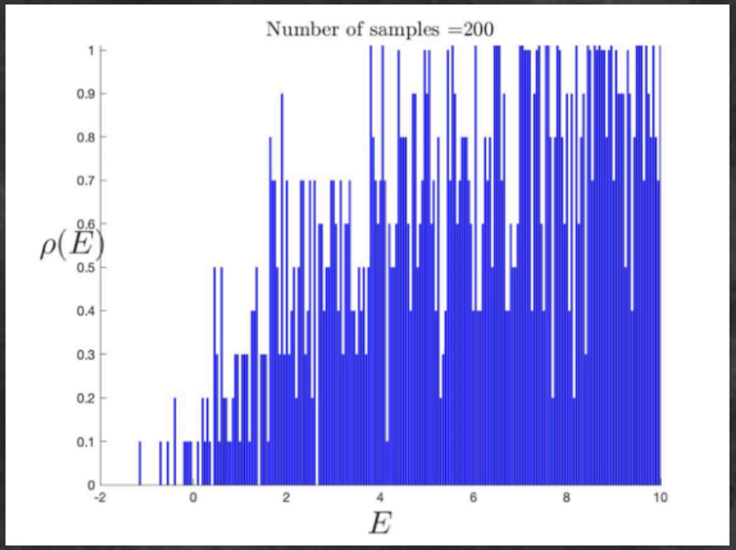
- focus on an endpoint and zoom in:

$$\lambda = -2 + \frac{\delta^2 E}{N^{2/3}}$$

$$\boxed{\tilde{\rho}_0(\lambda) \to \frac{1}{N^2}\frac{E^{1/2}}{\pi \hbar} = \frac{1}{N^2}\rho_0(E)}$$

$$\hbar = \frac{1}{N\delta^3}$$

$\delta \to \infty$ as $N \to \infty$



- the magnification reveals undulations/wiggles:

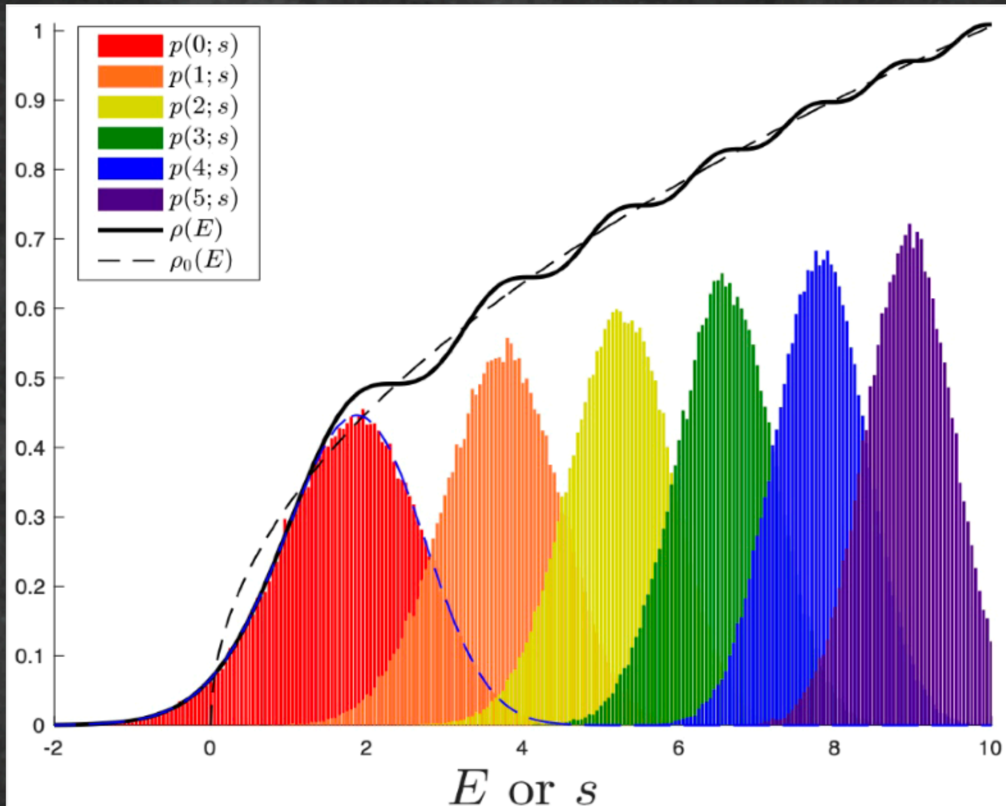
$$\rho(E) = \hbar^{-2/3}\left(Ai'(\zeta)^2 - \zeta\, Ai(\zeta)^2\right) \text{ for } \zeta = -\hbar^{-2/3}E$$

$$= \rho_0(E) + \rho_1(E) + \rho_2(E) + \ldots \text{ (non-pert.)}$$

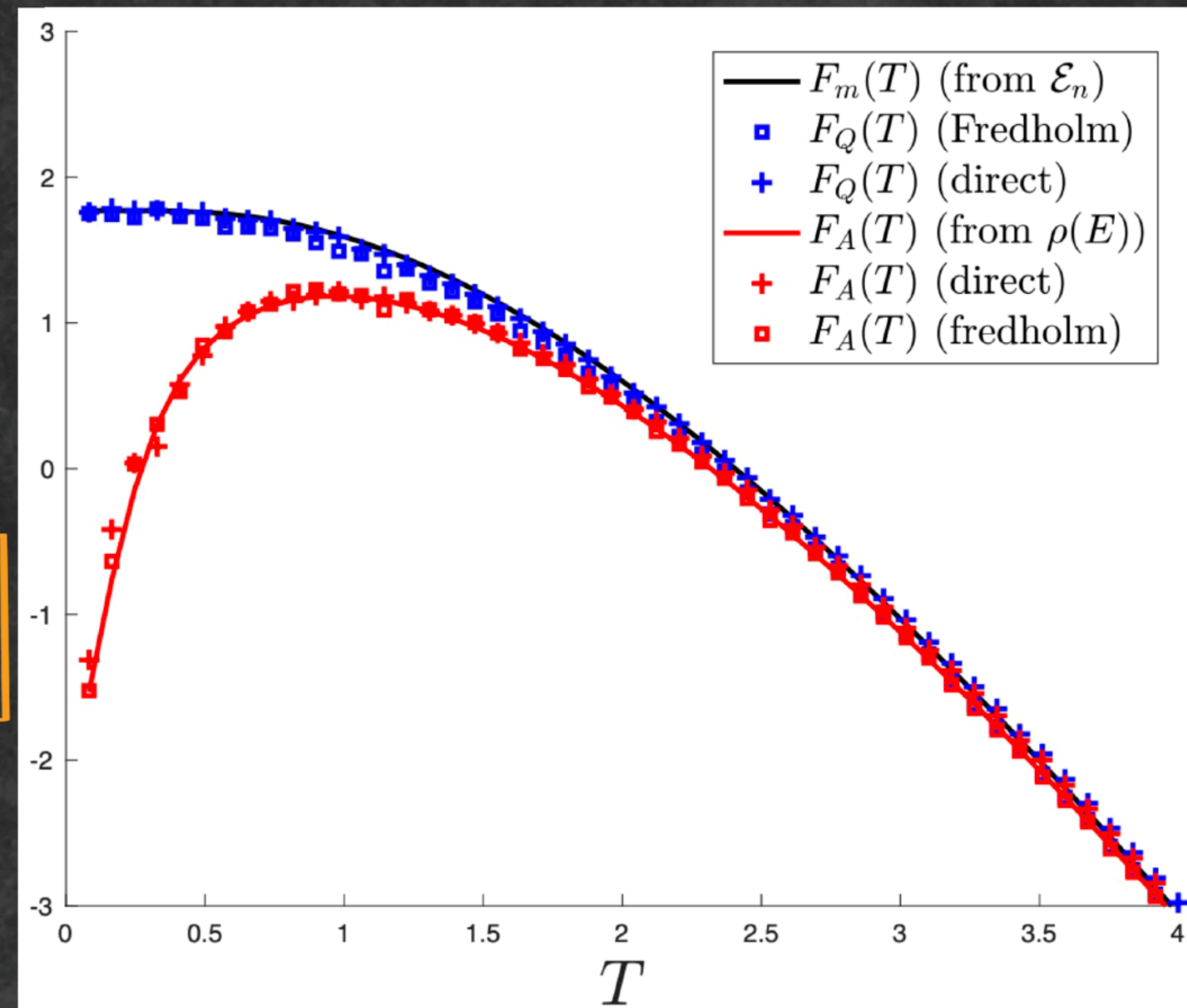
This is now the "Airy model".

Check everything in GUE / Airy (with actual matrices!)

Try this at home!



Result: cvj 2104.02733



- For each sample simply compute the quantity $\boxed{Z(\beta) = \sum_i e^{-\beta E_i}}$
- Then explicitly compute:

$$F_A(\beta) = -T \ln \langle Z(\beta) \rangle$$

"annealed"

and

$$F_Q(\beta) = -T \langle \ln Z(\beta) \rangle$$

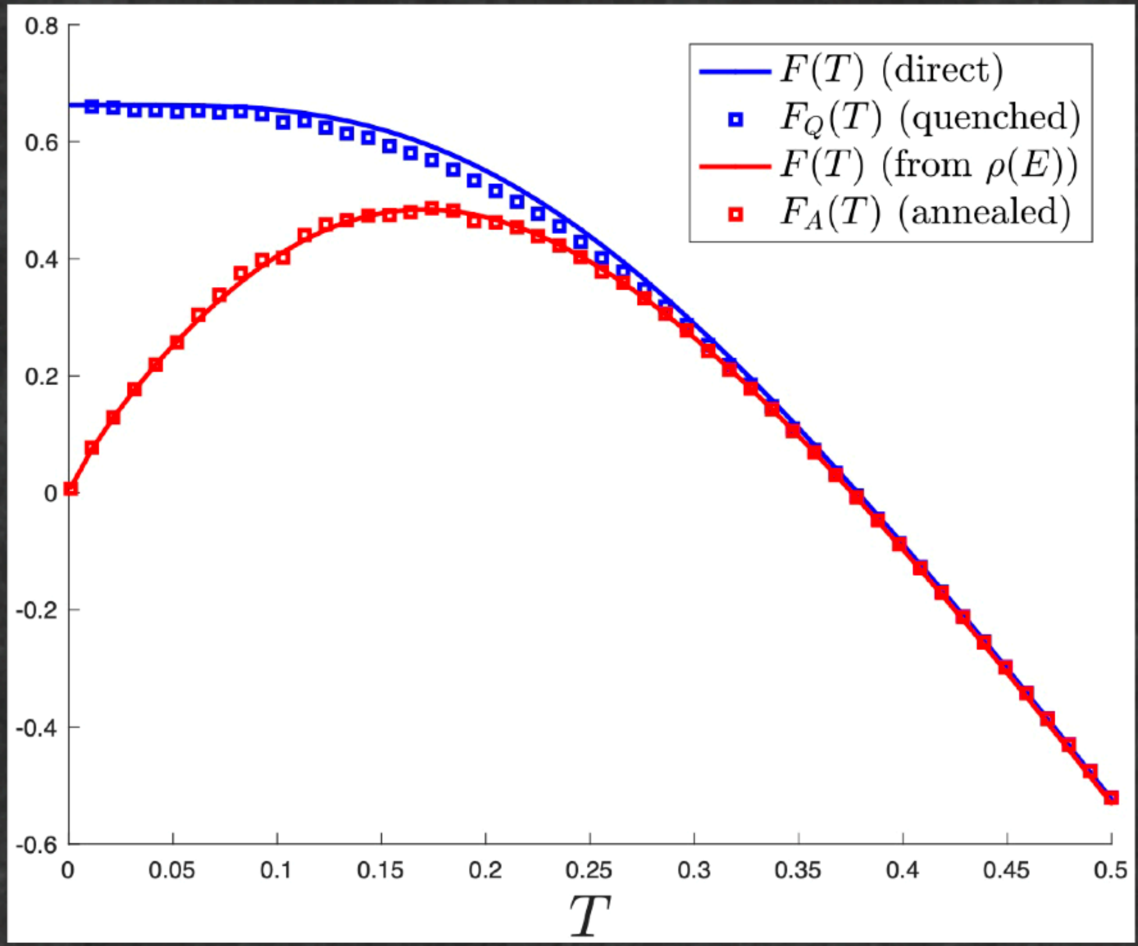
"quenched"

Highly instructive prototype... Now for JT!

cvj (2106.09048 & 2201.11942)

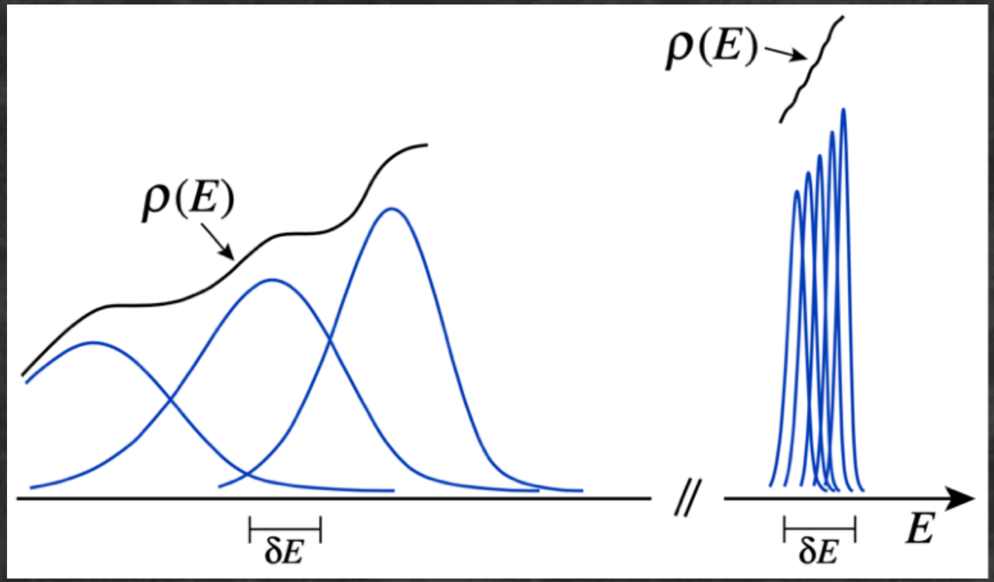
So F_Q computable using "spectroscopic" approach.

It gives a good measure of the typical behaviour of a single spectrum.



Generally (see literature on disorder)

F_A gets distorted by increasingly atypical configurations.



Exactly what's going on here!

Variance of peaks increases at low E

"Wignerian" mixing of spectra makes its presence known in low E parts of $\rho(E)$.

Defining the full RMM (How to)

- Rewrite matrix computations in terms of $P_n(\lambda)$, orthogonal wrt $d\lambda\, e^{-V(\lambda)}$

$$\int P_n(\lambda) P_m(\lambda) e^{-V(\lambda)} d\lambda = h_n \delta_{nm} \; ; \quad |n\rangle = \frac{|P_n\rangle}{h_n} \qquad \int \to \langle n|m\rangle = \delta_{nm}$$

- $\lambda P_n(\lambda) = P_{n+1}(\lambda) + R_n P_{n-1}(\lambda)$ recursion relation (even V)

(Gaussian case: P_n are Hermite polynomials
$\lambda H_n = H_{n+1} + n H_{n-1} \qquad R_n = n$)

(beware convention)

- Everything can be computed from the R_n.
- At large N: $\frac{n}{N} \to X$; $P_n(\lambda) \to P(X,\lambda)$; $R_n \to R(X)$
- In DSL (zoom an endpoint) scaling pieces survive:

$\lambda \to E$
$X \to x \in \mathbb{R}$
$\frac{1}{N} \to \hbar$
$P \to \psi(E,x)$
$R \to u(x)$

$$\hat{\lambda} \to -\hbar^2 \frac{\partial^2}{\partial x^2} + u(x) \equiv \mathcal{H}$$

$$\boxed{\mathcal{H}\psi(x,E) = E\psi(x,E)}$$

$$\boxed{\rho(E) = \int_{-\infty}^{0} |\psi(E,x)|^2 dx}$$

'89, '90
Brezin-Kazakov
Douglas-Shenker
Gross-Migdal

Banks et al.
Moore

Gaussian: $u(x) = -x$: $\psi(E,x) = \hbar^{-2/3} Ai(-(E+x)\hbar^{-2/3})$

The partition function of interest is the "loop operator"

$$Z(\beta) = \langle \mathrm{Tr}(e^{-\beta M}) \rangle = \int_{-\infty}^{0} \langle x | \, e^{-\beta \mathcal{H}(u)} | x \rangle \, dx$$

insert $\int |\psi\rangle\langle\psi| = 1$ w/ $\mathcal{H} |\psi_E\rangle = E|\psi_E\rangle$

$$Z(\beta) = \int dE \int_{-\infty}^{0} dx \, \underbrace{\langle x|\psi_E\rangle \langle \psi_E | x\rangle}_{|\psi(E,x)|^2} e^{-\beta E}$$

$$\boxed{Z(\beta) = \int dE \, \rho(E) \, e^{-\beta E}}$$

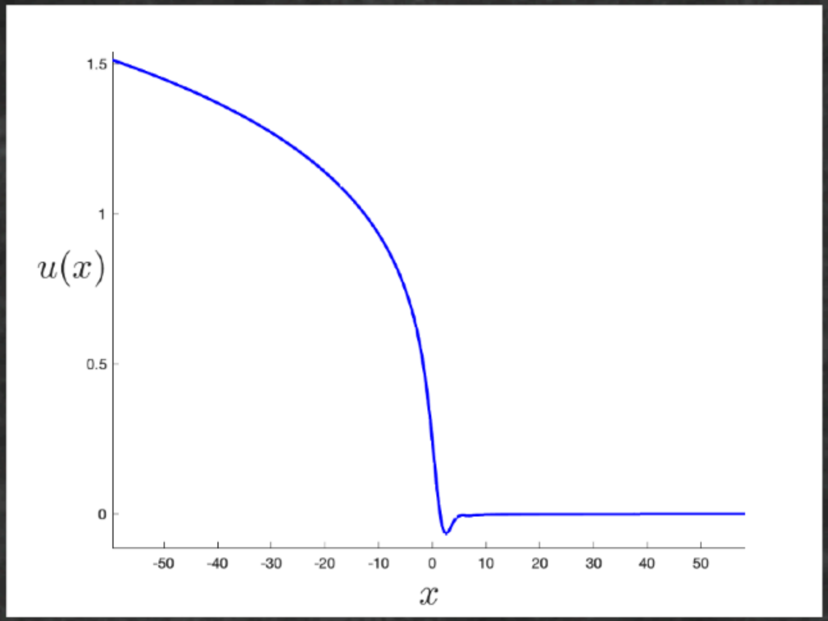
Can do this for full JT gravity!

$$\mathcal{H} = -\hbar^2 \frac{\partial^2}{\partial x^2} + u(x)$$

↑ find eqn for this

⋮

solve it



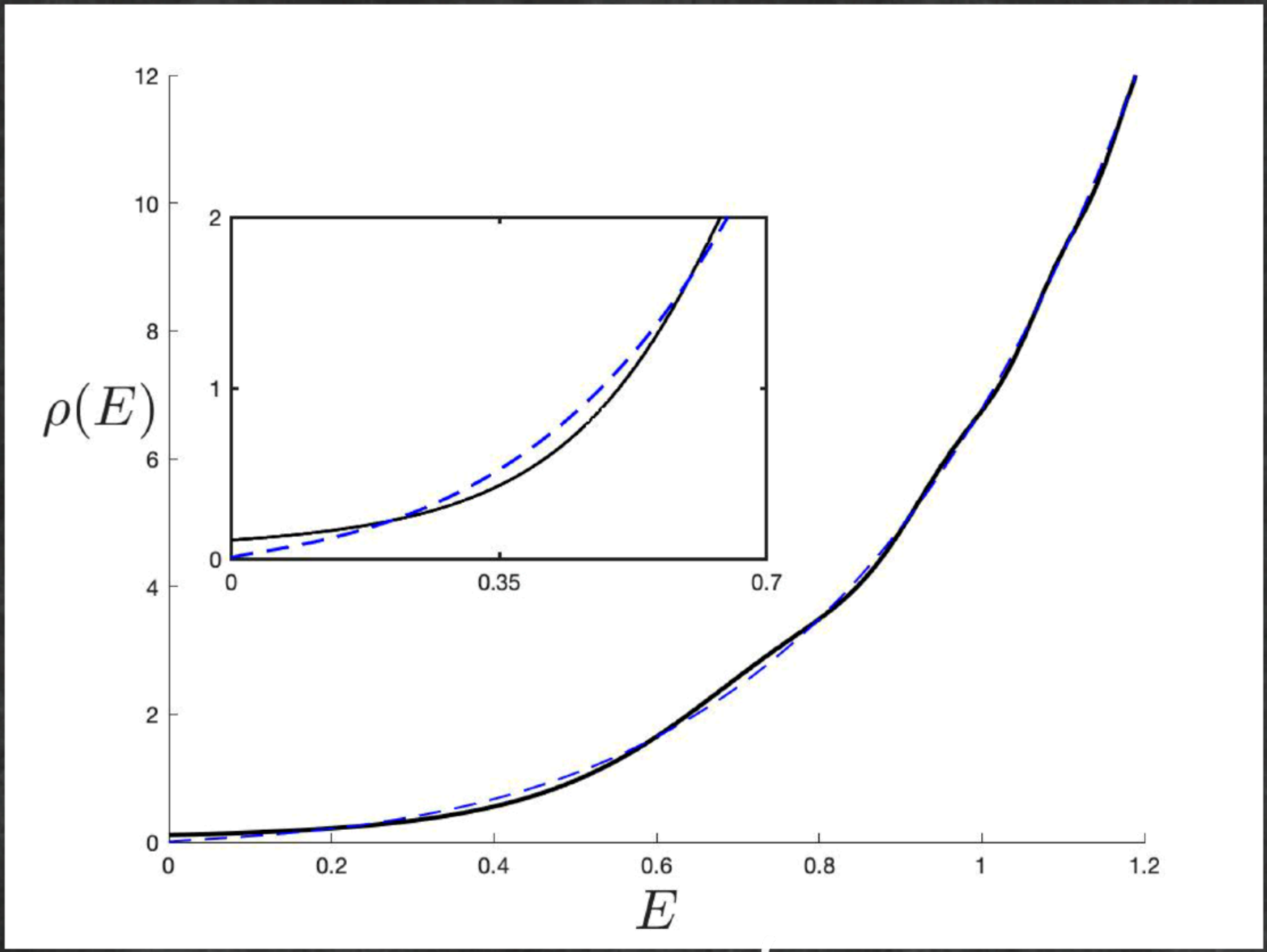
$$\underbrace{\mathcal{H}\psi(E,x) = E\,\psi(E,x)}$$

↑ solve this

⋮

then compute this

$$\rho(E) = \int_{-\infty}^{0} |\psi(E,x)|^2 dx$$



CVJ 1912.03637 & 2006.10959

Revealing the discreteness (How to)

There's much more to matrix models!

Beyond what we usually do for gravity RMM.

- Already have the components: $\psi(E,x)$
- Build the "Kernel" $K(E',E) = \int_{-\infty}^{0} \psi(x,E)\psi(x,E')\,dx$

[famous as Airy Kernel for GUE]

- All questions about RMM nicely answered in terms of it.

$$\rho(E) = K(E,E)$$

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Beyond what we usually do for gravity RMM.

- Already have the components: $\psi(E,x)$
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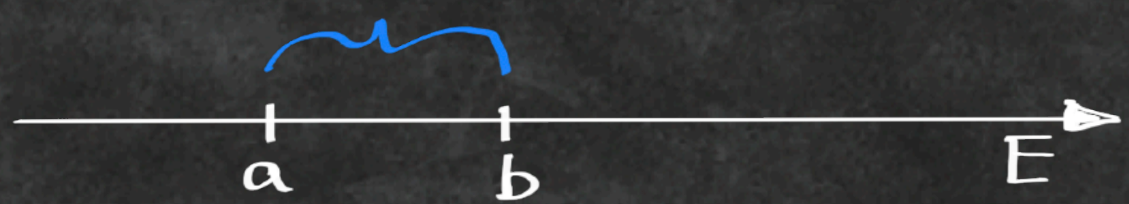
[famous as Airy Kernel for GUE]

- All questions about RMM nicely answered in terms of it.

$$\rho(E) = K(E,E)$$

- $\det(\mathbb{1} - K_{(a,b)})$ arises naturally for "gap probabilities"

(Gaudin 1961)



where: $K_{(a,b)}$: $\int_a^b dE'\, K(E,E')\, f(E') = g(E)$ (Fredholm 1903)

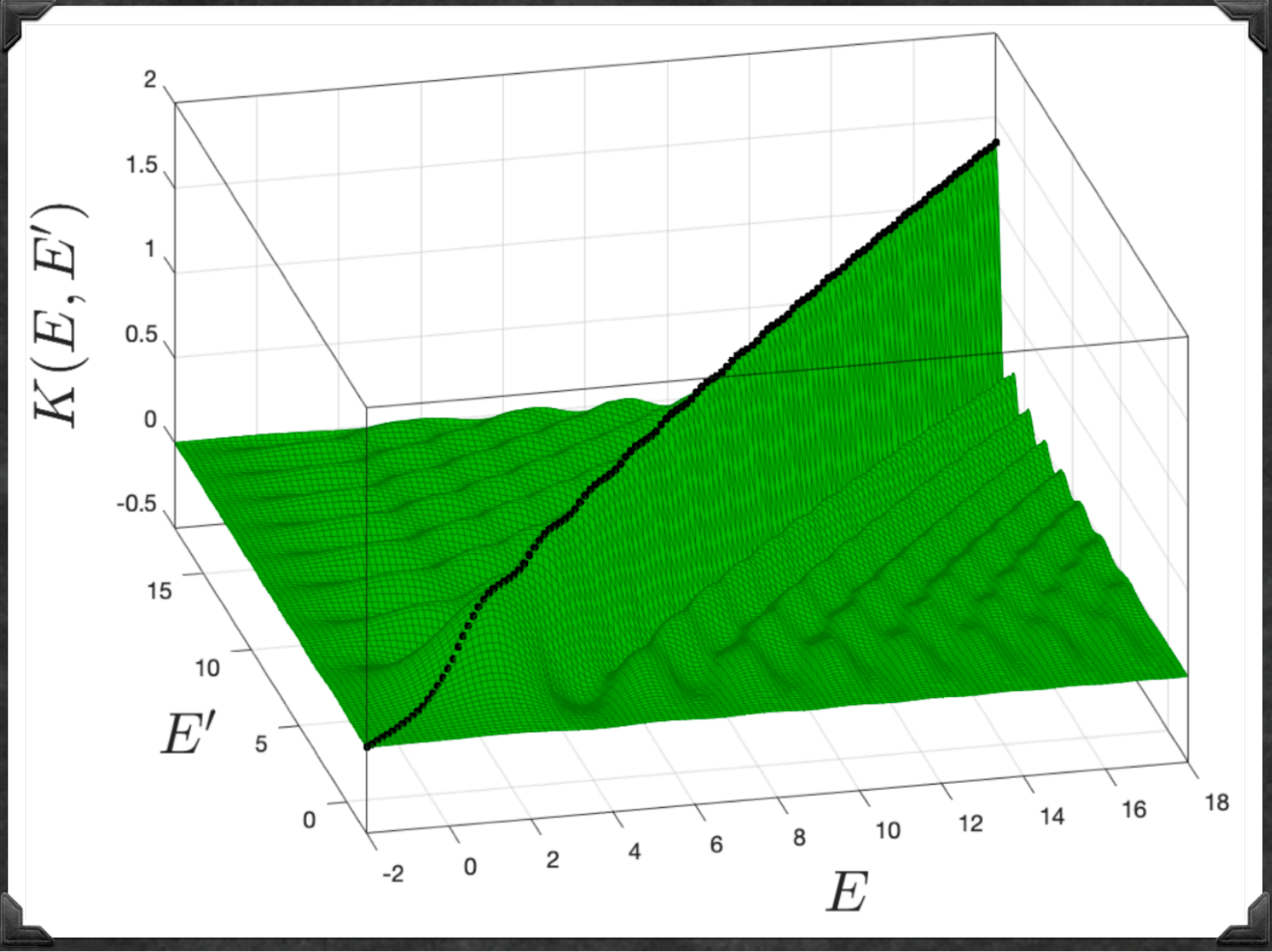
- Can use this to compute statistics of energy levels

Fredholm Determinant

The Airy Model

$$\psi(E,x) = \hbar^{-2/3} Ai\left(-(E+x)\hbar^{-2/3}\right)$$

$$K(E',E) = \int_{-\infty}^{0} \psi(x,E)\,\psi(x,E')\,dx$$



The Airy Model

(See books of Mehta, and Forrester)

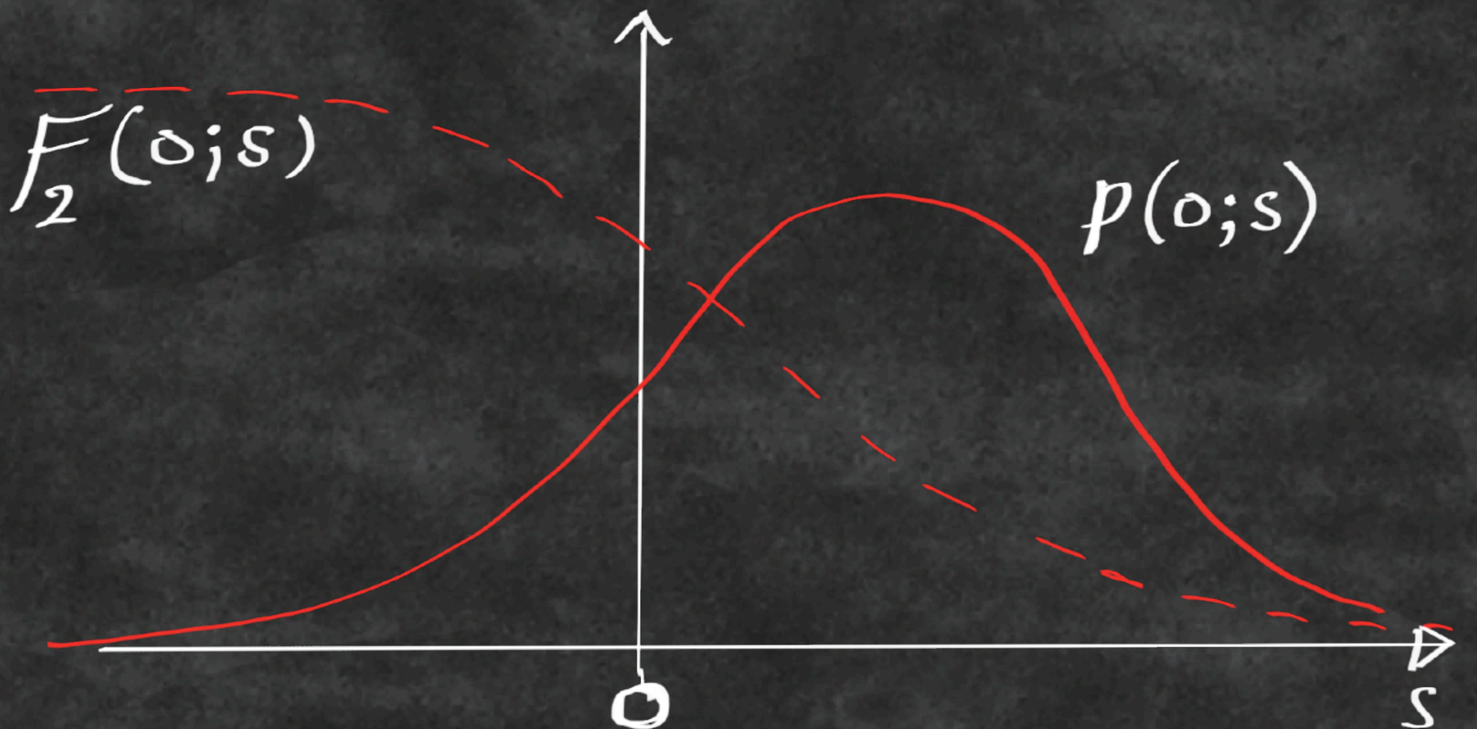
Probability of not finding an eigenvalue in $(-\infty, s)$:

$$\boxed{F_2(0;s) = \det\left[\mathbb{1} - K\big|_{(-\infty,s)}\right]}$$

Probability distribution:

$$p(0;s) = -\frac{dF_2(0;s)}{ds}$$

(This is the ground state. Can do this recursively for other levels.)



- Computing F_2 is difficult though.

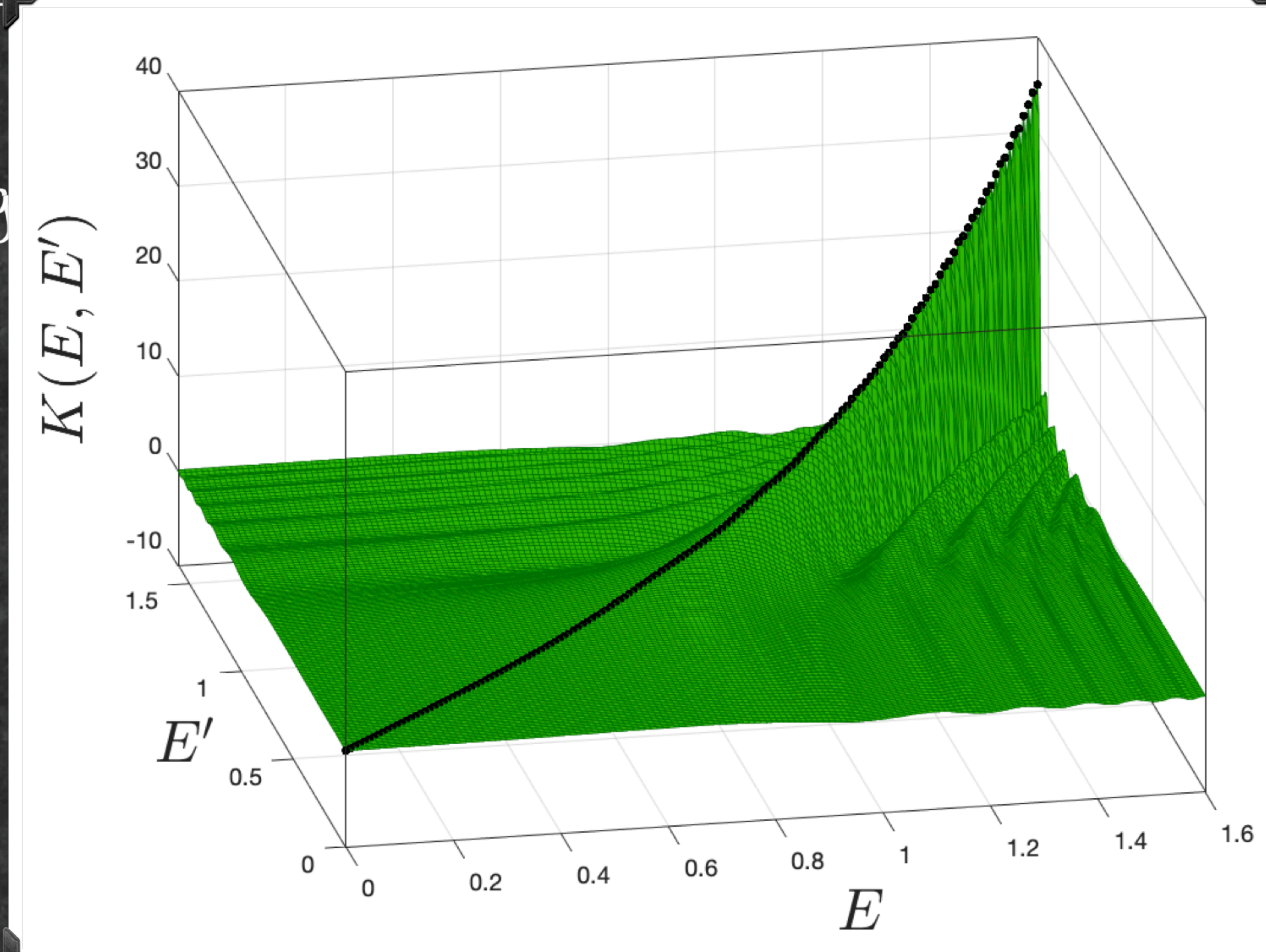
- It can be done analytically in a few situations.

Gaudin'61 Dyson'62, Mehta'67

However, Bornemann (0804.2543) developed quadrature methods for Fredholm determinants

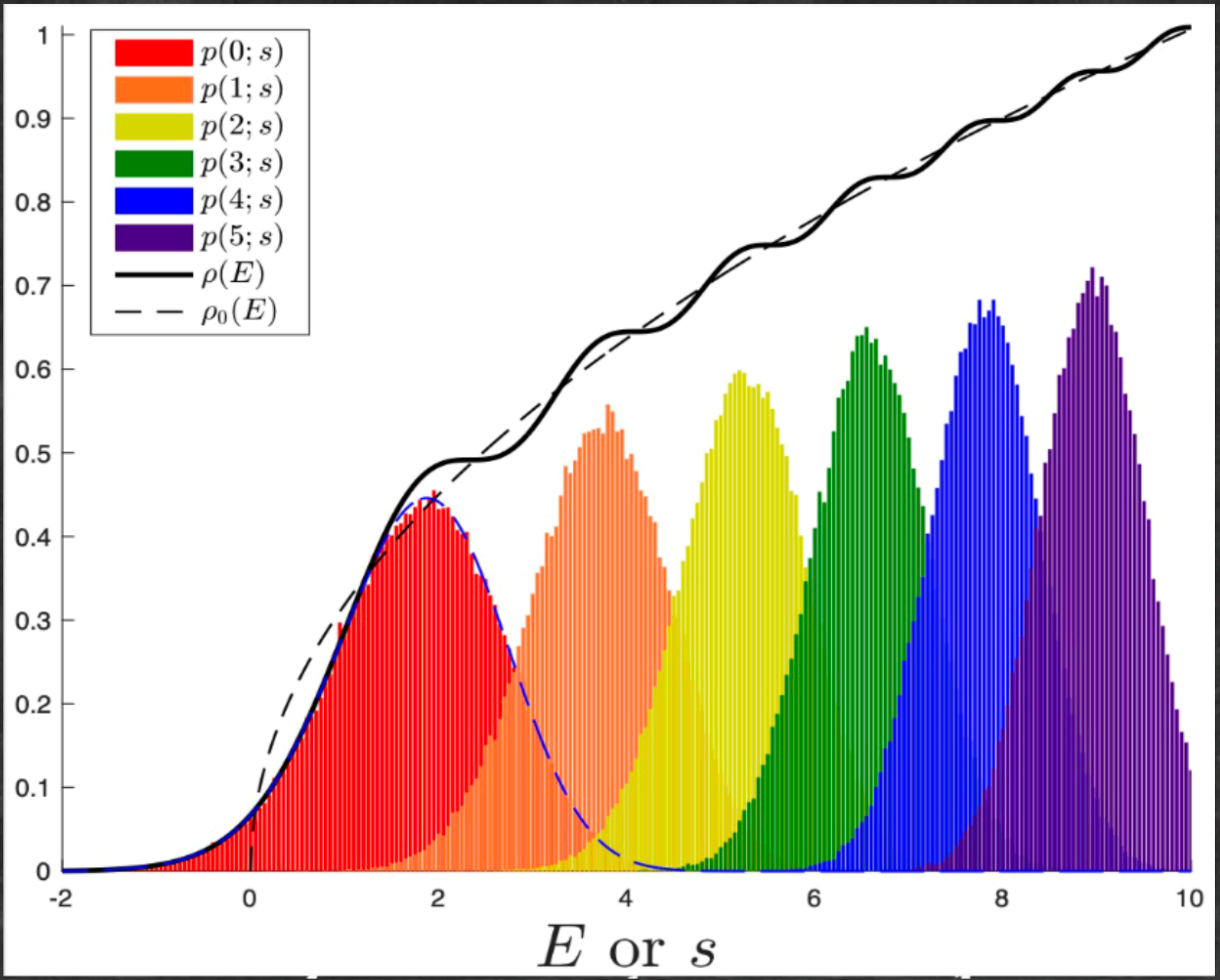
$$\int_0^s f(E)\,dE \approx \sum^{m} w_i f(e_i) \Longrightarrow \det\left(\mathbb{1} - K\big|_0^s\right)$$

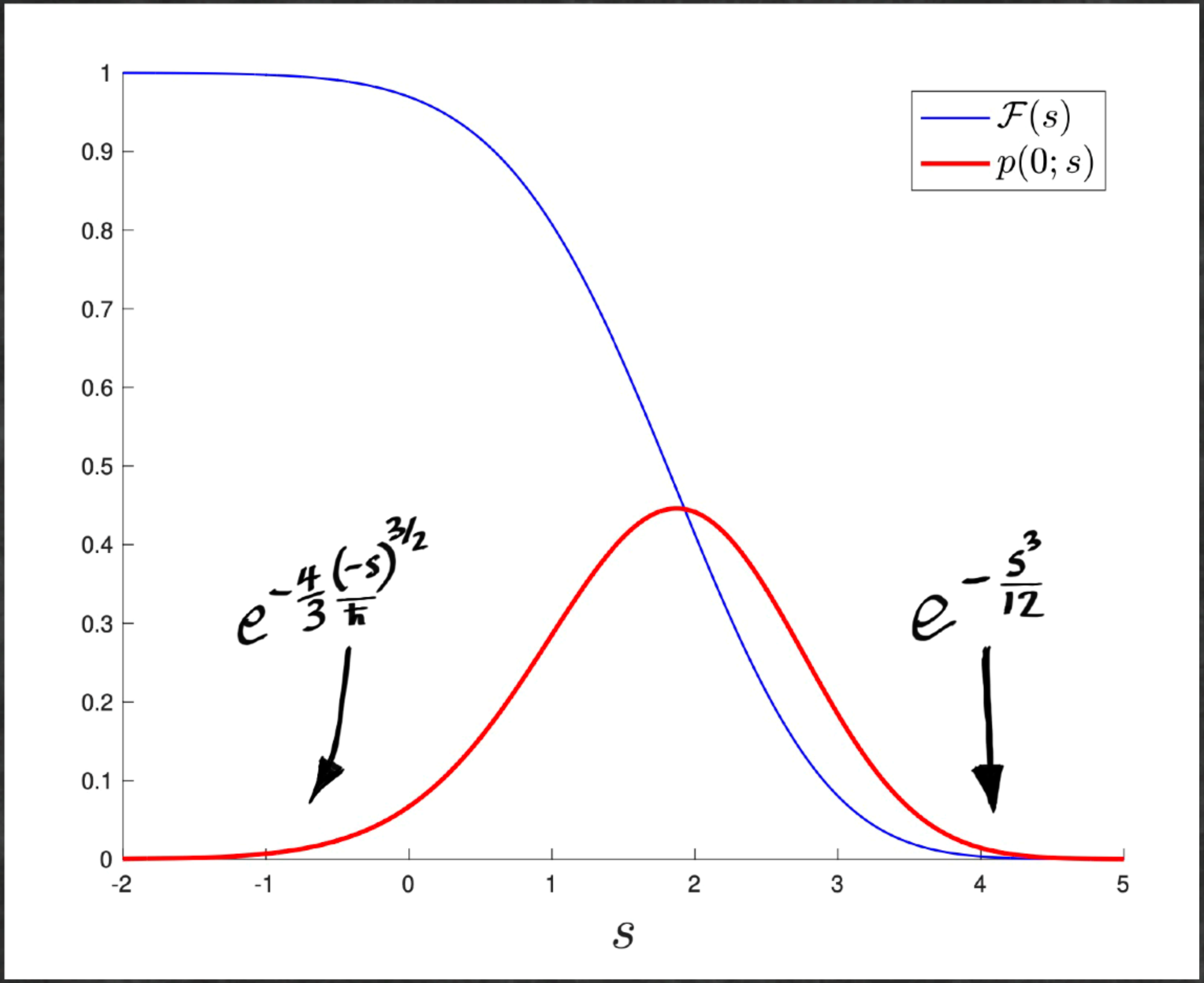
Do this for g



Newest Results: Distribution of (S)JT Ground States

The Airy Model





- The Tracy-Widom Distribution. '94
  - Shows up in many fields of physics and mathematics (theoretical and experimental)
  - Universality!
  - Connected to 3^{rd} order phase transition: Gross-Witten-Wadia.
  - RMM: Describes smallest energy of Gaussian distribution.

- Key: TW show that Fredholm determinant is equivalent to solution of an ODE → Painlevé II
- Many analytic expressions can then be derived.

Writing the cdf $F(s) = e^{-f(s)}$

TW '94 See also Jimbo, Miwa, Mori, Sato '80 Forester '93

$f''(z) = q^2(z)$ where $\boxed{\hbar^2 q''(z) - z q(z) - 2q^3 = 0}$

Painlevé II

with $q(z) \to \sqrt{\frac{-z}{2}} + \dots$ $z \to -\infty$

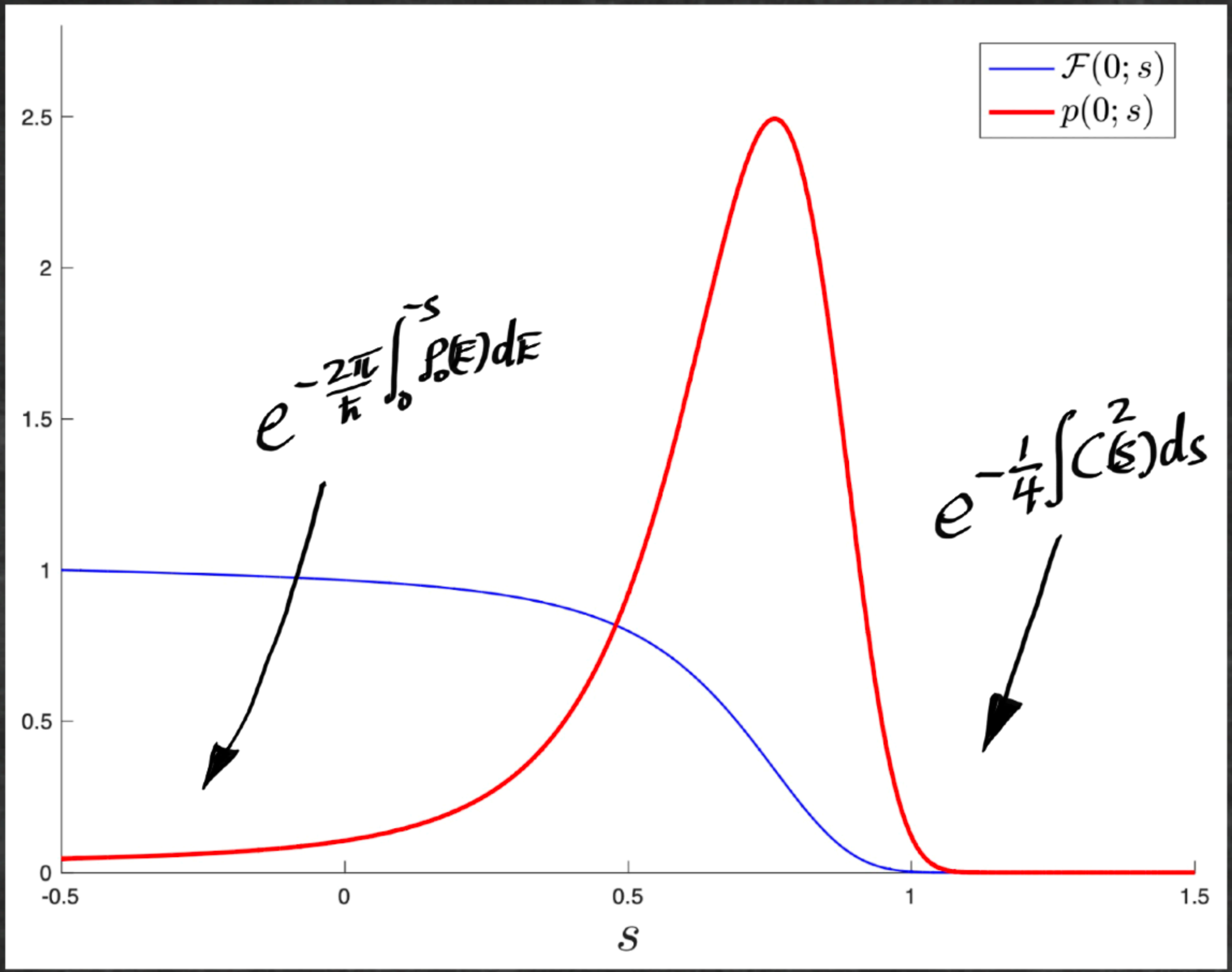
$\to A_i(\hbar^{-2/3} z)$ $z \to +\infty$

(Hastings - McLeod solution '80)

$e^{-\frac{1}{12}s^3}$ tail

$e^{-\frac{4}{3}\frac{(-s)^{\frac{2}{3}}}{\hbar}}$ tail.

Can this be done for RMM of JT gravity?



- A new distribution, for gravity!
  (cvj, 2206.00692)

- Also expect (a new kind of) universality, closely connected to Schwarzian universality...

- (... perhaps even beyond gravity?)

- Have recently derived it from an ODE family! (Painlevé XXXIV heirarchy)

- Many analytic results can be extracted!

$$\left(C(s) = \frac{\sqrt{s}}{2\pi} I_1(2\pi\sqrt{s}) \quad \text{(also controlled by Schwarzian)} \right)$$

(Analogous work can be done for super JT, etc...)

Derivation of ODE

A key reformulation first:

- Imagine the lowest energy of ensemble is s.

$$\left[-\hbar^2\frac{\partial^2}{\partial x^2}+u(x)\right]\psi(s,x)=s\,\psi(s,x)$$

($u(x,s)$ solves an ODE given below)

$$\left[\hbar\partial_x+v(x)\right]\left[\hbar\partial_x-v(x)\right]\psi(s,x)=0$$

(Generalized Miura)

where $\quad v^2+\hbar v'+s=u$

write $\psi(s,x)\propto\exp\left\{\frac{1}{\hbar}\int^{x}v(x')\,dx'\right\}$

- The equation for $u(x,s)$ was discovered long ago:

$$\boxed{(u-s)R^2-\frac{\hbar^2}{2}RR''+\frac{\hbar^2}{4}(R')^2=0}$$

Dalley, CVJ, Morris, '90

where $\quad R\equiv u+x$

- It implies an equation for $v(x,s)$

DJM + Watterstam '92

- $v^2 + \hbar v' + s = u$ gives

$$\frac{\hbar^2}{2} v'' - (x+s)v - v^3 + \frac{\hbar}{2} = 0$$

A different, deformed PII, with a constant.

- can also write directly the inverse:

$$v(x,s) = \frac{\hbar}{2} \frac{R'[u]}{R[u]}$$ which gives $$\boxed{\psi(x,s) = \frac{1}{\sqrt{2}} R[u]^{1/2}}$$

CVJ 2206.00692

- So now we can construct the density for this level s as

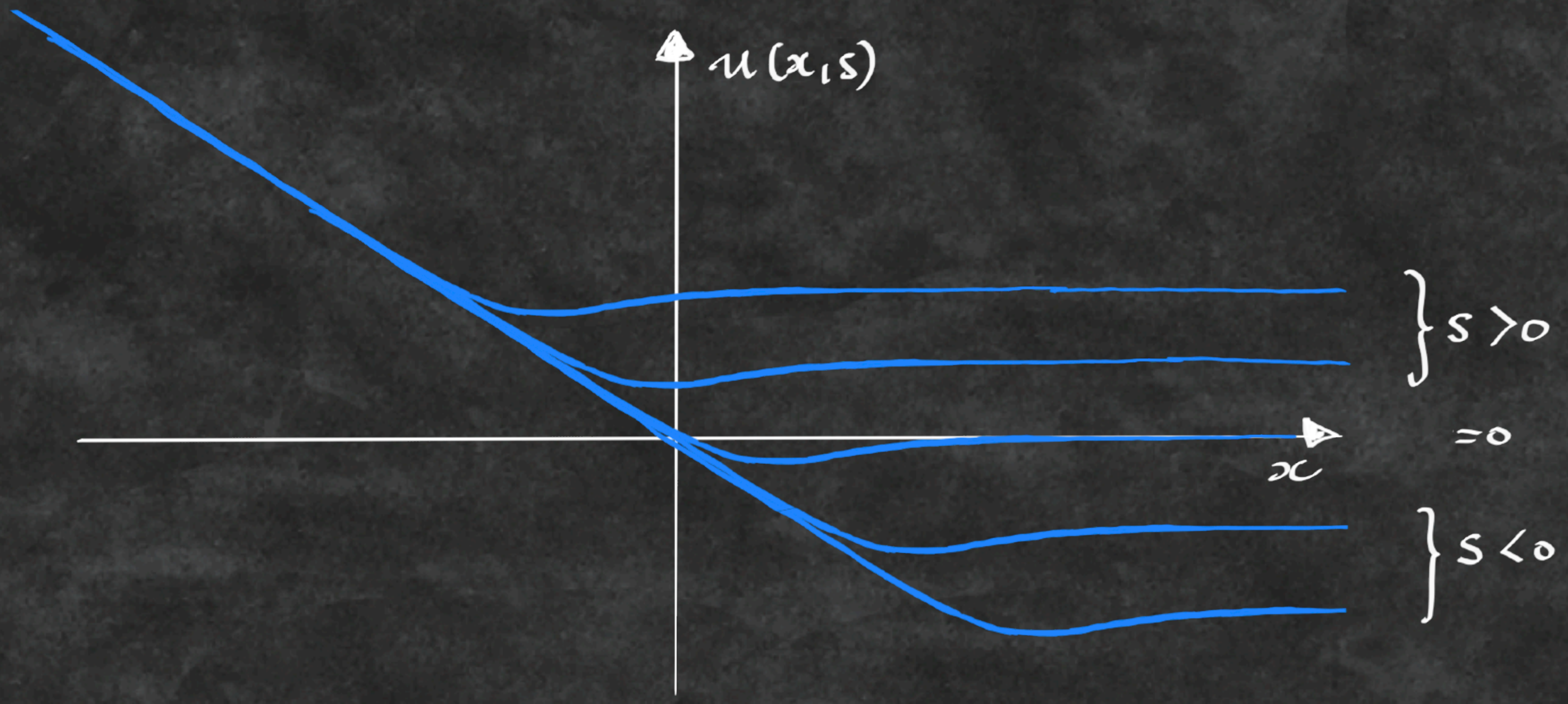
$$\frac{df}{ds} = \rho(s) = \int_{-\infty}^{0} \psi(s,x)^2 \, dx = \frac{1}{2}\int_{-\infty}^{0} R[u(x,s)] \, dx$$

and hence $f(s)$ can be constructed.

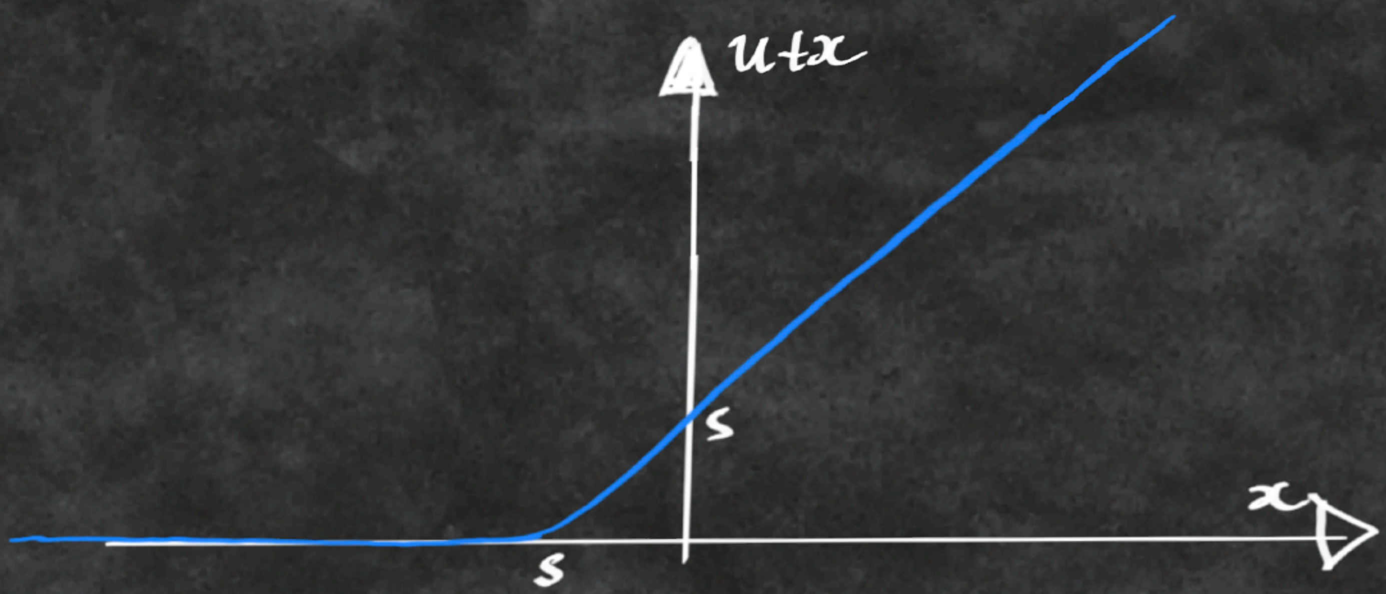
The solution for $u(x,s)$ obeys the asymptotics:

$$u(x,s) \to s \quad \text{as} \quad x \to +\infty$$

$$u(x,s) \to -x \quad \text{as} \quad x \to -\infty$$



Meanwhile $R = u + x$ vanishes as $x \to -\infty$ and becomes $x + s$ as $x \to +\infty$



So asymptotically in large +ve s

$$\frac{df}{ds} \to \int_{-\infty}^{0} R[u]\, dx = \frac{s^2}{4}$$

$e^{-\frac{s^3}{12}}$ tail!

other tail controlled by instanton correction to R vanishing everywhere

equivalence to TW route:

$$u = z^{1/2}(q^2 + z) \qquad z = -2^{1/3}(x - s)$$

get PII in q, with zero constant.

(See also Nadal + Majumdar, 2011)

- Finally, can generalize to any "kth" minimal model:

$$R = R_k[u] + x$$

where $R_k[u]$ is a Gel'fand-Dikii polynomial in $u, u', u'' \ldots u^{(2k-2)}$

$R_1[u] = u$

- $k=1$ turns out to be Painlevé XXXIV!
- Generally a Painlevé XXXIV heirarchy

$$(u-s)R^2 - \frac{\hbar^2}{2}RR'' + \frac{\hbar^2}{4}(R')^2 = 0$$

(See also math-phys refs in my paper)

DJM '90

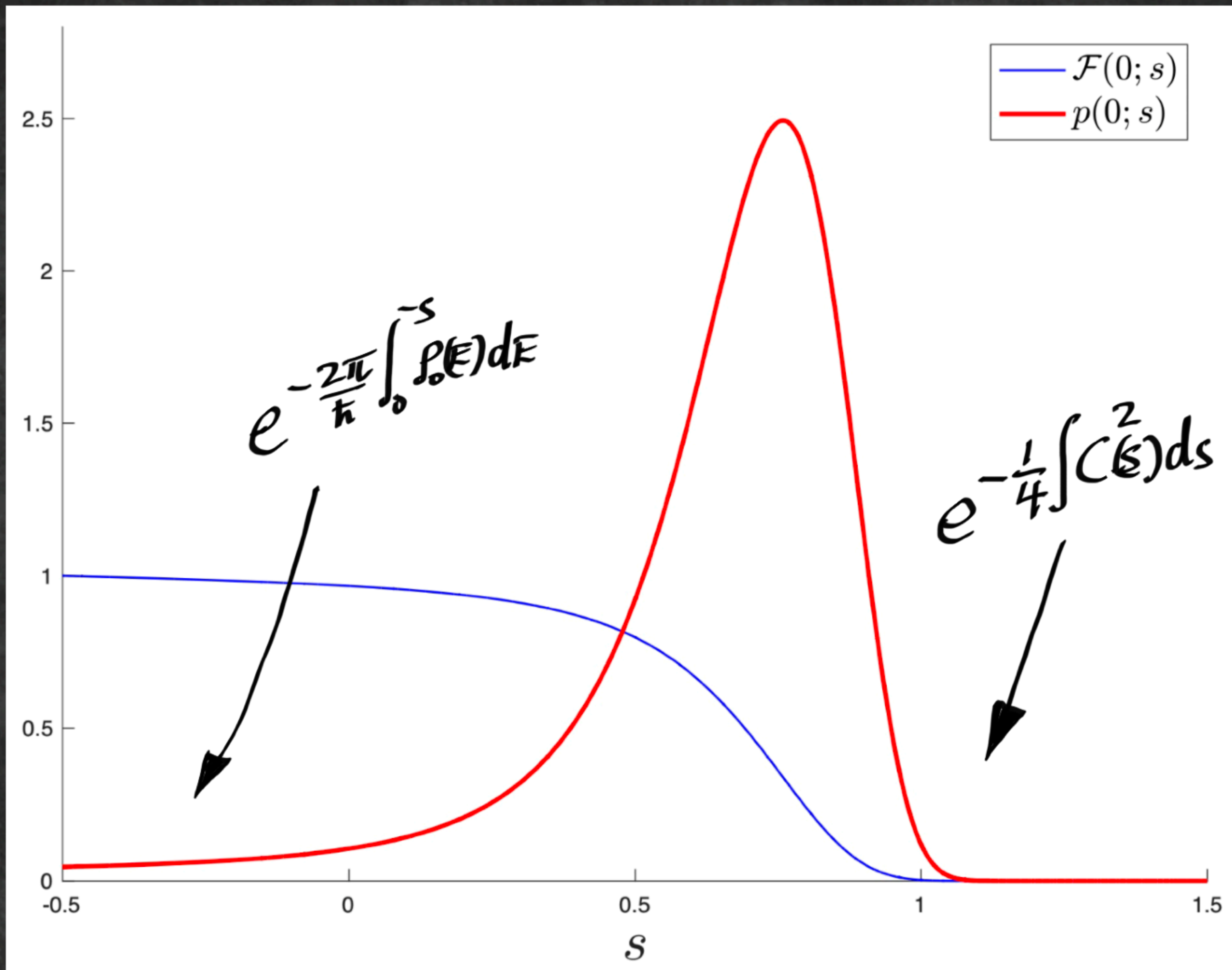
- For JT, there is a specific "mix" of these:

$$R = \sum_{k=1}^{\infty} t_k R_k[u] + x$$

$$t_k \equiv \frac{\pi^{2k-2}}{2\,k!\,(k-1)!}$$

- ∞ order ODE, but successively higher orders contribute successively less...

(truncate at order 15 to display)



$$C(s) = \frac{\sqrt{s}}{2\pi} I_1(2\pi\sqrt{s})$$

(Analogous work can be done for super JT, etc...)

Thank You!

—cvj