Bounds on spin-two KK masses

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2109.11560 + 2212.02511 + WIP
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Introduction

KK spectrum: one of the most important piece of data associated to a compactification

- Full spectrum: relevant for holography

[Kim, Romans, Van Nieuwenhuizen '85; Fabbri, Fré, Gualtieri, Termonia '99; Ceresole, Dall'Agata, D'Auria, Ferrara '99...]

- Smallest masses: scale separation, massive graviton models

[Lüst, Palti, Vafa '19; Klaewer, Lüst, Palti '18...]
[Karch, Randall '00...]

Explicit computation relies on symmetries:

- Homogeneous spaces

[Kim, Romans, Van Nieuwenhuizen '85; Fabbri, Fré, Gualtieri, Termonia '99; Ceresole, Dall'Agata, D'Auria, Ferrara '99...]

- Exceptional/generalized geometry:

[Malek, Samtleben, '19; Malek, Nicolai, Samtleben, '20...]

- gauge fixing; disentangling different spins; ...

  ⇨ problem is reduced to eigenvalues of internal diff. operators

Example: Freund–Rubin

review: [Duff, Nilsson, Pope '86]

Table 5
Mass operators from the Freund–Rubin ansatz

Spin	Mass operator
2^+	Δ_0
$(3/2)^{(1),(2)}$	$\not{D}_{1/2} + 7m/2$
$1^{-(1),(2)}$	$\Delta_1 + 12m^2 \pm 6m(\Delta_1 + 4m^2)^{1/2}$
1^+	Δ_2
$(1/2)^{(4),(1)}$	$\not{D}_{1/2} - 9m/2$
$(1/2)^{(3),(2)}$	$3m/2 - \not{D}_{3/2}$
$0^{+(1),(3)}$	$\Delta_0 + 44m^2 \pm 12m(\Delta_0 + 9m^2)^{1/2}$
$0^{+(2)}$	$\Delta_L - 4m^2$
$0^{-(1),(2)}$	$Q^2 + 6mQ + 8m^2$

Laplace–Beltrami

Laplace–de Rham

Lichnerowicz

- For spin-two, operator is always weighted Laplacian

$$\Delta_f \psi \equiv -\mathrm{e}^{-f} \nabla^m (\mathrm{e}^f \nabla_m \psi)$$

[Csaki, Erlich, Hollowood, Shirman'00; Bachas, Estes '11]

total dimension

$$f = (D-2)A$$

$$\mathrm{d}s^2_D = \mathrm{e}^{2A}(\mathrm{d}s^2_d + \mathrm{d}s^2_n)$$

warping

internal 'de-warped' metric

This talk: mathematical bounds on KK masses

- unwarped case: Laplace eigenvalues, long history

  many bounds require assumptions about curvature.

  for example: Ricci positive definite $\Rightarrow \frac{\pi^2}{4\mathrm{diam}^2} \leqslant m_1^2 \leqslant \frac{2n(n+4)}{\mathrm{diam}^2}$ [Li, Yau '80] [Cheng '75]

  (dimension: n; smallest mass: m_1^2)

- warping seemingly spoils the old results

  but more recent ideas: Bakry–Émery geometry, optimal transport [Sturm '06; Lott, Villani '07; Ambrosio, Gigli, Savaré 14...]

  of current mathematical interest!

luckily, Ricci+warping combine in EoM in the 'right' mathematical way.

Plan

- Einstein equations

  Curvature, warping, and the weighted Raychaudhuri equation

- Overview of bounds

  in terms of Planck mass; Cheeger constant; diameter

- Examples and applications

  gravity localization; scale separation

Einstein equations

[De Luca, AT '20]

previous attempts in [Gautason, Schillo, Van Riet, Williams '15]

Consider a higher-dimensional gravity $m_D^{D-2}\int \mathrm{d}^D x\sqrt{-g_D}R_D$ + matter

and a compactification $\mathrm{d}s^2_D = \mathrm{e}^{2A}(\mathrm{d}s^2_d + \mathrm{d}s^2_n)$

max. symmetric

'de-warped' internal

EoM: $R_{MN} = \frac{1}{2}m_D^{2-D}\left(T_{MN} - \frac{1}{D-2}g_{MN}T\right) \equiv \hat{T}_{MN}$

internal:

$$\Lambda - \frac{1}{d}\hat{T}_{(d)} \quad \text{external}$$

$$R_{mn} + (D-2)(-\nabla_m\nabla_n A + \partial_m A\partial_n A) = ((D-2)|\mathrm{d}A|^2 + \nabla^2 A)g_{mn} + \hat{T}_{mn}$$

$$= \Lambda g_{mn} + \left(\hat{T}_{mn} - \frac{1}{d}g_{mn}\hat{T}_{(d)}\right) \geqslant \Lambda g_{mn}$$

non-negative

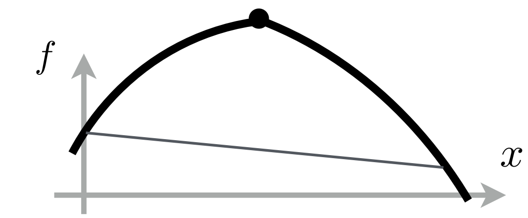
["Reduced Energy Condition"]

- for all bulk fields in type II and $d = 11$ sugra
- potentials
- for brane sources

But sources create singularities. It would be best to avoid derivatives...

- Inspiration: functions of one variable

$f'' \leqslant 0$ — generalize to non-smooth functions: → concavity



- Consider a distribution of particles moving geodesically:

$\rho(x)$ such that $\int_M \sqrt{g}\rho = 1$

Entropy: $S = -\int_M \sqrt{g}\rho \log \rho$

velocity field

$$\partial_t^2 S = -\int_M \sqrt{g}\rho(\nabla_m U_n \nabla^m U^n + R_{mn} U^m U^n)$$

$R_{mn} \geqslant 0$ — generalize to non-smooth spaces: → $\partial_t^2 S \leqslant 0$

- Weighted 'Tsallis entropy': homogeneous (rather than extensive) [$\sim$ log Rényi entropy]

[Havrda, Charvat '67; Patil, Taillie '82; Tsallis '88]

$$S_{N,f} \equiv N\left(1 - \int_M \sqrt{g}\,\mathrm{e}^f \rho^{\frac{N-1}{N}}\right)$$

$$\partial_t^2 S_{N,f} \leqslant -\int_M \sqrt{g}\,\mathrm{e}^f \rho^{\frac{N-1}{N}} \underbrace{\left(R_{mn} - \nabla_m\nabla_n f + \frac{1}{n-N}\nabla_m f \nabla_n f\right)}_{\equiv\; R_{mn}^{N,f}} U^m U^n$$

"Bakry–Émery curvature"

[Bakry, Émery '85]

N 'effective dimension':
played $\sim$ role of rank of $\nabla_m U_n$

$$R_{mn}^{N,f} \geqslant 0 \quad \xrightarrow{\text{generalize to non-smooth spaces:}} \quad \partial_t^2 S_{N,f} \leqslant 0$$

this leads to the 'Riemann-Curvature-Dimension' [RCD] condition

[Sturm '06; Lott, Villani '07; Ambrosio, Gigli, Savaré 14]

[One can also reformulate the Einstein equations in this language]

[McCann '19; Mondino, Suhr '19; De Luca, De Ponti, Mondino, AT '22]

- our earlier EoM bound in terms of BE curvature:

$$R^{N,f}_{mn} = R_{mn} + (D-2)(-\nabla_m \nabla_n A + \partial_m A \partial_n A) \geq \Lambda g_{mn}$$

$f = (D-2)A$

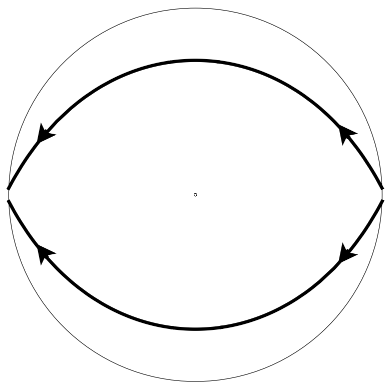
$N = 2 - d < 0$

but still OK

- for brane singularities, generalized to ‘RCD(N, Λ)’: $\sim$ concavity of entropy

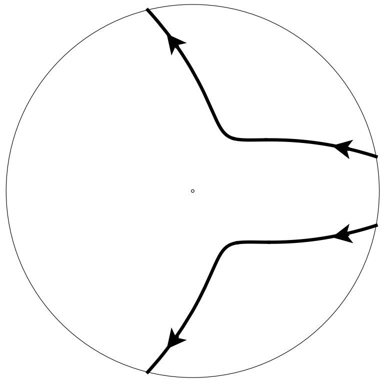
[De Luca, De Ponti, Mondino, AT ‘22]

geodesics attracted by a D-brane...



- for O-planes this doesn't work.

... but repelled by an O-plane



Here we use ‘infinitesimally Hilbertian spaces’: broader class useful for bounds that don't require information about curvature.

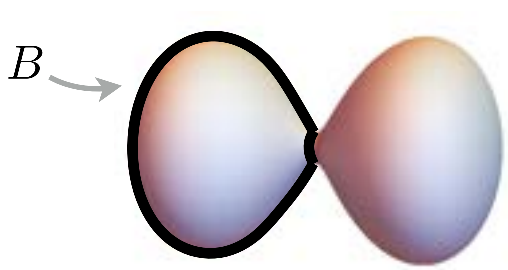
[De Luca, De Ponti, Mondino, AT ‘22]

Overview of bounds

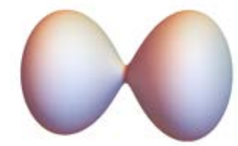
[De Luca, AT '20; De Luca, De Ponti, Mondino, AT '21,'22]

- Upper, lower bounds
- On lightest or on higher masses
- Different degrees of generality: smooth spaces, branes, O-planes
- In terms of
  - 4d Planck mass $M_4^2 \sim M_D^{D-2} \int_M \sqrt{g} e^{(D-2)A}$ [if unwarped: int. volume]
  - diameter: max. distance between any two points in M
  - Cheeger constant h_1: small when space has small 'neck'

$h_1 = \min_B \frac{\mathrm{vol}_A(\partial B)}{\mathrm{vol}_A(B)}$ 'min. of $\frac{\text{perimeter}}{\text{area}}$'



$\mathrm{vol}_A(B) \equiv \int_B \sqrt{g} e^{(D-2)A}$

	4d Planck mass	Cheeger [presence of necks] 	diameter
upper	m_k [smooth; warp.]	m_1 [D-branes; warp.] $\frac{m_k}{m_1^2}$ [O-planes]	m_k [smooth; warp.]
lower		m_k [O-planes]	m_1 [D-branes]

- [smooth] $\subset$ [D-branes] $\subset$ [O-planes]

  spaces with D-brane sing. — spaces with O-plane sing.

- [warp.]: bound contains $\sigma \equiv \sup_M (D-2)|\mathrm{d}A|$ $\longrightarrow$ sometimes bound becomes useless for M noncompact

- common technique: minimizing $\frac{\int e^f \sqrt{g}|\nabla\psi|^2}{\int e^f \sqrt{g}|\psi|^2}$ [e.g. Cheeger: test ψ related to B]

Gravity localization

- If weighted volume $\mathrm{vol}_A(M) = \int_M \sqrt{g} e^{(D-2)A} = \infty$, no massless graviton.

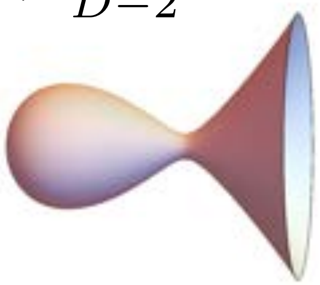
smallest mass: $m_1^2 \leqslant \max\left\{\frac{21}{20} h_1 \sqrt{K}, \frac{22}{20} h_1^2\right\}$

$h_1 = \min_B \frac{\mathrm{vol}_A(\partial B)}{\mathrm{vol}_A(B)}$

$K \equiv |\Lambda| + \frac{\sigma^2}{D-2}$

adapting
[De Ponti, Mondino '19]

[includes D-branes]

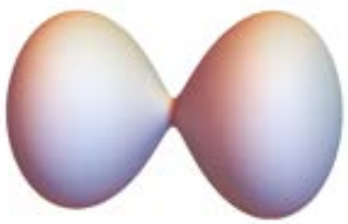


small 'neck' ⇨ Light massive spin-two

- If M is compact, lightest spin-two above graviton:

[relevant for 'bigravity' models]

smallest mass: $m_1^2 \leqslant \max\left\{\frac{21}{10} h_1 \sqrt{K}, \frac{22}{5} h_1^2\right\}$



[De Luca, De Ponti, Mondino, AT '21,'22]

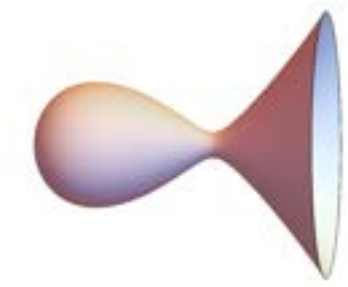
- For localization we would also like $m_2 \gg m_1$.

$$\frac{h_k^2}{Ck^6} < m_k^2$$ [includes O-planes]

'higher' Cheeger detect presence of 'more necks'

$$h_k \equiv \inf B_0, \ldots, B_k \max_{0 \leqslant i \leqslant k} \frac{\mathrm{vol}_A(\partial B_i)}{\mathrm{vol}_A(B_i)}$$

for example m_2 large on

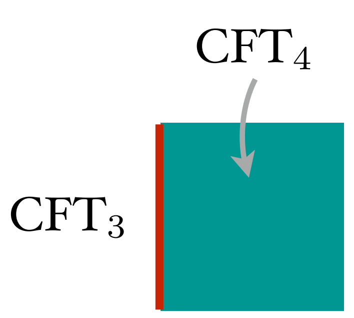


[D'Hoker, Estes, Gutperle '07]
[Assel, Bachas, Estes, Gomis '11]
[Bachas, Lavdas '17, '18]

- Concrete class of examples: $\mathcal{N} = 4$ AdS$_4$ vacua
  - we can show quickly that gravity localizes
  - string realization of Karch–Randall model [Karch, Randall '00]

Cheeger constant has a holographic interpretation:

$$h_1 \propto \frac{\mathcal{F}_0(\mathrm{CFT}_4)}{\mathcal{F}_0(\mathrm{CFT}_3)}$$



- bonus: no localization with non-constant zero modes

[De Luca, De Ponti, Mondino, AT *WIP*]

- On the other hand:

+ previous result on m_1

$$m_k^2 \leqslant \frac{128k^2 m_1^4}{h_1^2} \quad \Rightarrow \quad m_k^2 < 600k^2 \ \mathbf{max} \left\{ m_1^2, |\Lambda| + \tfrac{\sigma^2}{D-2} \right\}$$

[includes O-planes]

so: $m_1^2 > |\Lambda| + \frac{\sigma^2}{D-2} \quad \Rightarrow \quad m_k^2 < 600k^2 m_1^2$

above this threshold, no mass hierarchy;
in agreement with 'Spin-2 conjecture'

$\Rightarrow$ KK scale $\sim m_1$

[Klaewer, Lüst, Palti '18]
[de Rham, Heisenberg, Tolley '18]
[Bachas '19]

Scale separation

- Can we make $\sqrt{\Lambda} \ll m_1$ for AdS vacua?

our earlier upper bounds don't exclude this, not even without sources

[unless a nontrivial bound on diameter is found]

empirical bound on *d* among SE's:
[Collins, Jafferis, Vafa, Xu, Yau '22]
and among sphere quotients:
[Gorodski, Lange, Lytchak, Mendes '19]

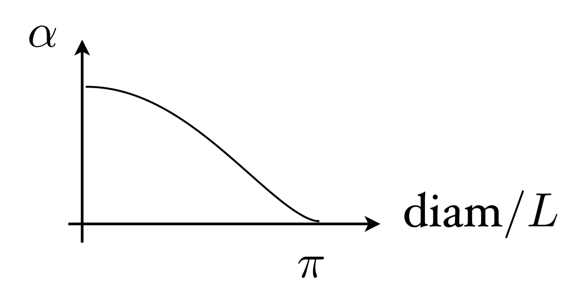
- lower bounds can be useful for establishing scale separation in a given solution.

$$\frac{h_1^2}{4} \leqslant m_1^2$$

[includes O-planes]

$$\alpha\left(\frac{\text{diam}}{L_{\text{AdS}}}\right)\frac{1}{\text{diam}^2} \leqslant m_1^2$$

[includes D-branes]



[De Luca, De Ponti, Mondino, AT '21,'22];
diam. bound inspired by
[Calderon '19]

- famous example: f_0, f_4, h (NSNS 3-form), O6 [DeWolfe, Giryavets, Kachru, Taylor '05]

violates the REC ⇨ no curvature bound

⇨ can use internal CY

large $N \propto \int F_4$

$r \sim N^{1/4} \ll r_{\rm AdS} \sim N^{3/4}$ ✓

∃ approximate 10d 'uplift' [Acharya, Benini, Valandro '06; Junghans '20; Marchesano, Palti, Quirant, AT '20]

estimate of Cheeger constant:

- B tube around O6: $\frac{\mathrm{per}(B)}{\mathrm{vol}(B)} \to \infty$
- B tube elsewhere: $\frac{\mathrm{per}(B)}{\mathrm{vol}(B)} \sim 1/r$ ✓

$m_1^2 \geqslant \frac{h_1^2}{4} \sim N^{-1/2}$

$\gg$

$|\Lambda|$ ✓

- A potentially simpler example: T-dualize, lifting to M-theory

⇨ smooth $\mathrm{AdS}_4 \times (\mathrm{weak}G_2)_7$? [Cribiori, Junghans, Van Hemelryck, Van Riet, Wrase '19]

$F_4 \sim N \mathrm{vol}_{\mathrm{AdS}_4}$ $L \sim N^{7/6}$ diam $\sim N^{11/12}$

$$m_1^2 \geqslant \frac{c}{\mathrm{diam}^2} \sim N^{-11/24} \gg |\Lambda| \sim N^{-7/12} \quad ✓$$

- Another way to obtain scale separation: Casimir energy

[De Luca, De Ponti, Mondino, AT '22]

similar to [Arkani-Hamed, Dubovsky, Nicolis, Villadoro '07]

violates the REC ⇨ no curvature bound

⇨ can use internal torus

'Freund–Rubin' $\mathrm{AdS}_4 \times T^7$:

$$\langle T^{\mathrm{Cas}}_{\mu\nu} \rangle = \frac{\ell_P}{R_7^{11}} g_{\mu\nu} \qquad \langle T^{\mathrm{Cas}}_{mn} \rangle = -\frac{4}{7}\frac{\ell_P}{R_7^{11}} g_{mn}$$

⇨ $L \sim N^{11/3}$ diam $\sim N^{2/3}$

Conclusions

- Einstein equations imply bound on 'Bakry–Émery curvature' (Ricci+warping)

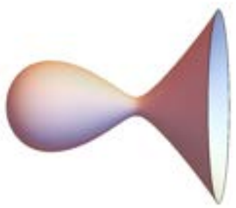
- 'Synthetic' point of view: local curvature as entropy concavity

  ⇨ bound still makes sense on 'attractive' singularities

- Upper and lower bounds on spin-2 KK masses

  in terms of Planck mass; Cheeger constant; diameter

  ⇨ gravity localization in solutions with 'necks'



  ⇨ confirms scale separation for some approximate solutions

Backup Slides

- mass bounds in terms of the diameter

[largest distance between any two points]

empirical bound on *d* among SE's: [Collins, Jafferis, Vafa, Xu, Yau '22] and among sphere quotients: [Gorodski, Lange, Lytchak, Mendes '19]

- $m_k^2 \leqslant n\left(|\Lambda| + \frac{D-1}{D-2}\sigma^2\right) + \gamma(n)\frac{k^2}{\mathrm{diam}^2}$

[De Luca, AT '21] using [Setti '98]

for now, M_n smooth.

[Using RCD$(N<0)$ it *might* be possible to get rid of σ.]

[De Luca, De Ponti, Mondino, AT: *WIP*]

- $m_1^2 \geqslant \frac{c(d)}{\mathrm{diam}^2}$

so small diameter does imply scale separation for spin-2. For now, no O-planes

[Calderon '19; De Luca, De Ponti, Mondino, AT: '22 *to appear*]

- bounds in terms of **Cheeger constant** $h_1(M_n) \equiv \inf_B \frac{\int_{\partial B} \sqrt{\bar g_{\partial B}}\, e^{(D-2)A}\, d^{n-1}x}{\int_B \sqrt{\bar g}\, e^{(D-2)A}\, d^n x}$

[De Luca, De Ponti, Mondino, AT '21]

'min. of $\frac{\text{perimeter}}{\text{area}}$'

a space where h_1 is small has a small 'neck':

- smallest mass: $\frac{1}{4}h_1^2 \leqslant m_1^2 \leqslant \max\left\{\frac{21}{10}h_1\sqrt{K},\ \frac{22}{5}h_1^2\right\}$

broad class, including O-planes

[De Luca, De Ponti, Mondino, AT '22 *to appear*]

RCD(K,∞) sing. [recall: includes D-branes]

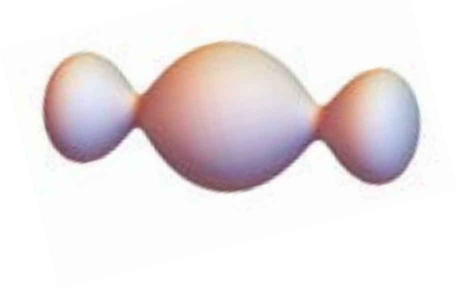
adapting [De Ponti, Mondino '19]

$K \equiv |\Lambda| + \frac{\sigma^2}{D-2}$

- higher masses: $\frac{h_k^2}{Ck^6} < m_k^2 < 600k^2 \max\left\{K, 2\sqrt{K}h_k, 5h_k^2\right\}$

$h_k \equiv \inf_{B_0,\ldots,B_k} \max_{0\leqslant i\leqslant k} \frac{\int_{\partial B_i} e^f\, d\overline{\mathrm{vol}}_{n-1}}{\int_{B_i} e^f\, d\overline{\mathrm{vol}}_n}$

a space where h_2 is small (but not h_3):



here h_1 small, h_2 large:

Raychaudhuri/Bochner eq.

- Consider now a distribution of particles moving along geodesics

expansion $\theta \equiv \nabla_\mu U^\mu$: behavior of density around a point

tangent vector: U^μ

Raychaudhuri eq.: $-\nabla_U \theta = \nabla_\mu U_\nu \nabla^\mu U^\nu + R_{\mu\nu} U^\mu U^\nu$

$[\mathrm{Tr} M^2 \geqslant \frac{1}{r} (\mathrm{Tr} M)^2]$

rank

in many applications: $\nabla_\mu U_\nu \nabla^\mu U^\nu \geqslant \frac{1}{r}\theta^2$

in 4d, $r=1$ (massive) or $r=2$ (massless)

- with measure $\int \sqrt{g} \mathrm{e}^f$, more natural $\theta_f \equiv \mathrm{e}^{-f} \nabla_\mu (\mathrm{e}^f U^\mu)$

$\Rightarrow$ weighted Raychaudhuri: $-\nabla_U \theta_f = \nabla_\mu U_\nu \nabla^\mu U^\nu + (R_{\mu\nu} - \nabla_\mu \nabla_\nu f) U^\mu U^\nu$

$N < 0$

$\geqslant \underbrace{(R_{\mu\nu} - \nabla_\mu \nabla_\nu f + \tfrac{1}{n-N} \nabla_\mu f \nabla_\nu f)}_{\equiv R^{N,f}_{\mu\nu}} U^\mu U^\nu + \frac{1}{N} \theta_f^2$

'negative effective dimension'

$R^{N,f}_{\mu\nu}$ "Bakry–Émery curvature"

[Bakry, Émery '85]