Topological spin models

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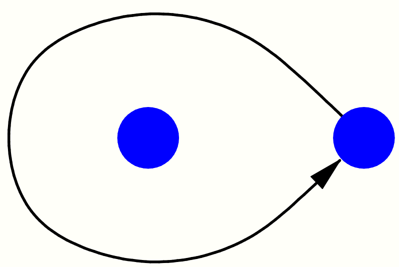
1. Introduction: topological phases
2. Concrete Hamiltonians:
   a) (Perturbed) toric code;
   b) The honeycomb lattice model and
      Yao-Kivelson model.
   c) Planar model.
3. Derivation of the planar model (on the blackboard).

Introduction: Quantum topological phases

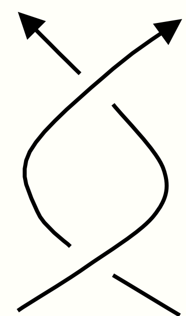
Motivation: Topological quantum computation

- Computation is done by moving anyons around each other (braids in space-time)

Top view (looking from the future)



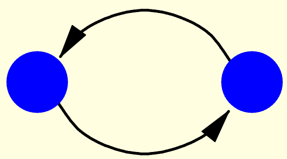
Side view (times goes up)



- Anyons are quasiparticles in some medium, called *quantum topological phase*.

What are anyons?

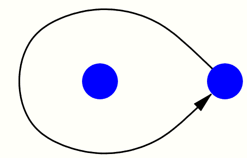
Anyons are particles with nontrivial statistics



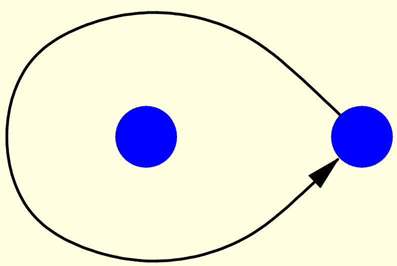
Bosons: $|\Psi\rangle \mapsto |\Psi\rangle$

Fermions: $|\Psi\rangle \mapsto -|\Psi\rangle$

In both cases,



is trivial



Abelian anyons: $|\Psi\rangle \mapsto e^{i\varphi}|\Psi\rangle$

Non-Abelian anyons: $|\Psi\rangle \mapsto U|\Psi\rangle$

(This is only possible in 2 dimensions.)

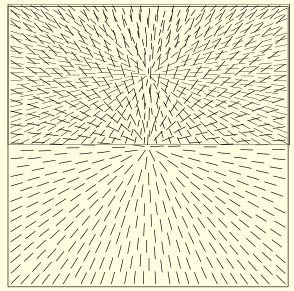
Anyons are stable particle with some *fusion rules*

e.g., $\varepsilon \times \varepsilon = 1, \quad \varepsilon \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \varepsilon$

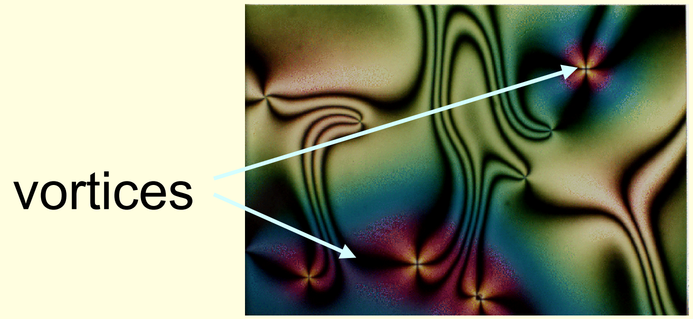
different particle types

Anyons are topological defects (like vortices)

Classical vortex (theoretical):

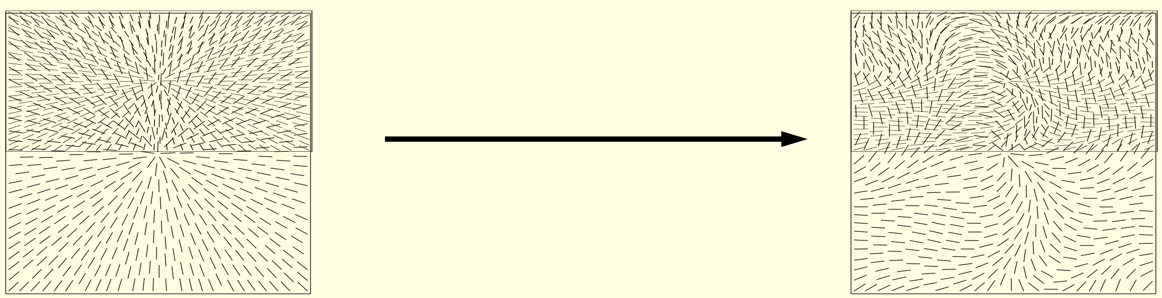


An order parameter is defined in each point and varies in space



Texture in a nematic film (seen through crossed polarizers)

- Quantum vortices: the local order parameter disappears while the topological defect remains.



- There is still some intangible, nonlocal order in the surrounding space.

Fundamental perspective: conservation laws

• Particles may be conserved due to *topological* reasons, like vortices. (Some 150 years ago Thomson and Maxwell speculated that atoms might be a kind of knots…)

• Standard paradigm: Conservation laws are due to *symmetry*. (Noether's theorem and its quantum analogue: Particle types are irreducible representations of the symmetry group.)

• For *bosons and fermions*, conservation laws (or fusion rules) are equivalent to a symmetry *group*. (Doplicher-Roberts theorem.)

• *Anyons* are not described by a group but rather, *unitary modular category*.

• But anyons are just excitations, or defects in a *topological quantum phase*, whose properties are even richer.

What properties of anyons should we look for to detect them experimentally?

- Spin or charge fractionalization, or completely new quantum numbers, which are not dictated by any symmetry of the underlying Hamiltonian. (“Emergent symmetry”.)

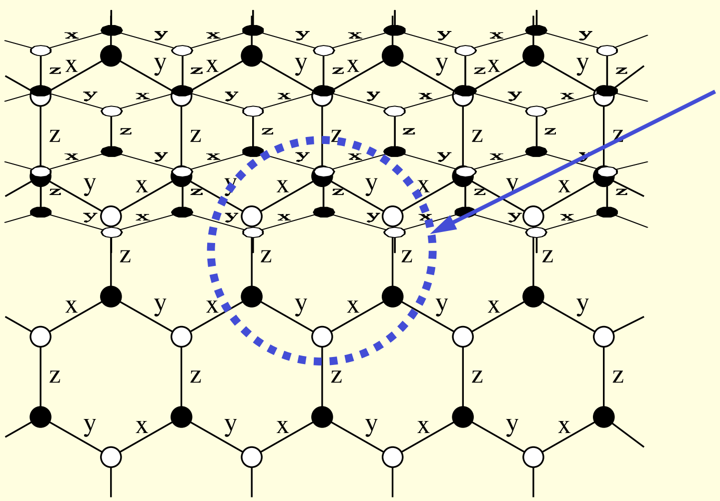
- Local (quasi)conservation of the new quanum numbers – fusion rules.

- Nontrivial braiding.

Detecting anyons in a spin system

- Anyons can only be created in pairs, hence energy- and momentum-resolved absorption spectra have no sharp features. In comparison, the generation of single particles is only possible for $\varepsilon = \varepsilon(p)$.

- An anyon trapped in a potential well does not decay. Its presence can be measured locally.



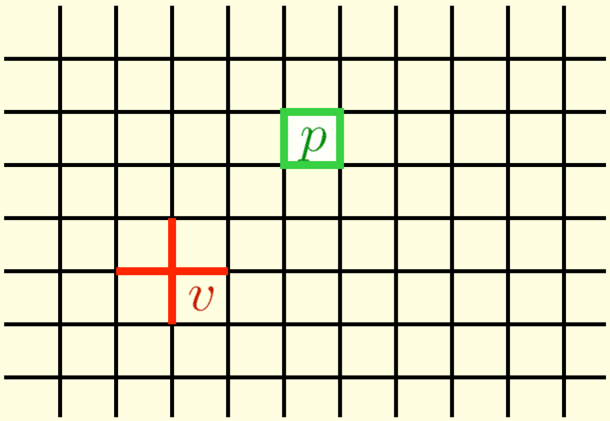
A weak spot (where the spins coupling is decreased by a constant factor) traps all kinds of excitations

Vacancies in the honeycomb lattice trap vortices (Willans, Chalker, Moessner, 2010)

The toric code

Hamiltonian: $$H_{TC} = -J_x \sum_{\text{vertices}} A_v - J_z \sum_{\text{plaquettes}} B_p,$$

where $$A_v = \prod_{\text{star}(v)} \sigma_j^x, \quad B_p = \prod_{\text{boundary}(p)} \sigma_j^z$$



Ground state:

$$A_v|\Psi_{\text{gr.}}\rangle = |\Psi_{\text{gr.}}\rangle, \qquad B_p|\Psi_{\text{gr.}}\rangle = |\Psi_{\text{gr.}}\rangle$$

for all v and p

Excitations:

"Electric charge": $A_v|\Psi_v\rangle = -|\Psi_v\rangle$ for some v

"Magnetic vortex": $B_p|\Psi_p\rangle = -|\Psi_p\rangle$ for some p

(These are $\mathbb{Z}_2$ charges and vortices.)

Toric code (continued)

- Robust properties:

  a) All excitations are gapped;

  b) Four superselection sectors: $1,\ e,\ m,\ \varepsilon = e \times m$;

  c) Fusion rules, eg. $e \times e = m \times m = 1$;

  d) Nontrivial mutual statistics;

  e) Four-fold degenerate ground state on the torus.

- Consequences of exact solvability:

  a) Particles do not move (flat dispersion);

  b) Two-point (and n-point) correlation functions vanish at distances > 1.

  c) The degeneracy on the torus is exact.

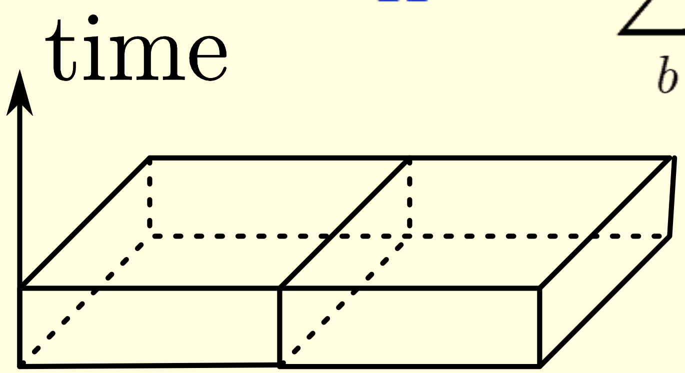
Perturbed toric code

$$H = H_{TC} - h_x \sum_b \sigma_b^x - h_z \sum_b \sigma_b^z$$

Trebst, Werner, Troyer, Shtengel, Nayak (2006) -- with one field,
Tupitsyn, Kitaev, Prokof'ev, Stamp (2008) -- with both fields

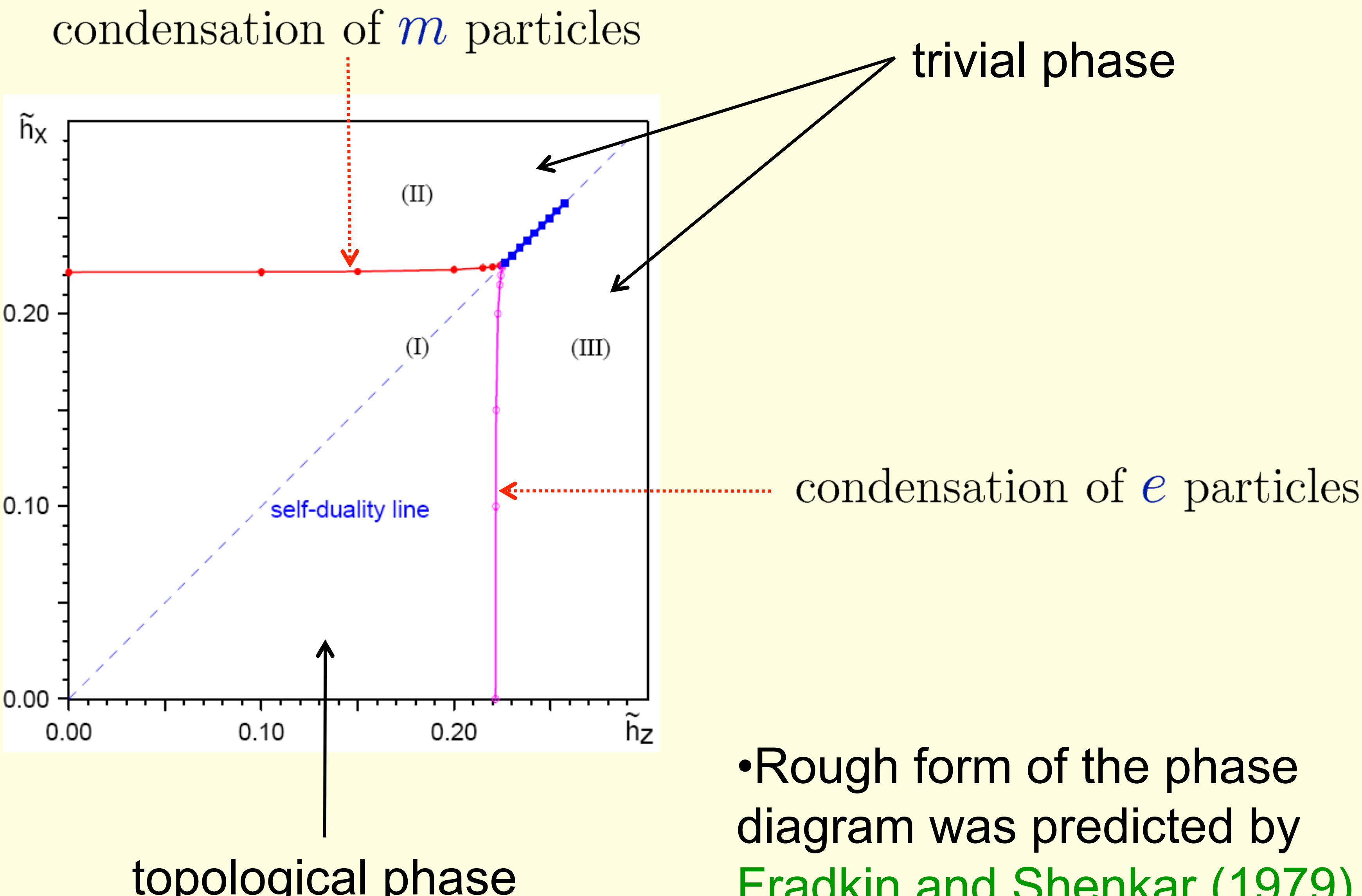
- For small h_x, h_z, the exact solvability is broken
- Transition to the trivial phase at large fields
- Equivalent to the classical *gauge Higgs* (*Wegner*) model in 3D:

$$\widetilde{H} = -\sum_b \lambda_{\text{bond}}^{||,\perp} S_b - \sum_{\text{p}} \lambda_{\text{pl}}^{||,\perp} \prod_{j\in \text{p}} S_j$$



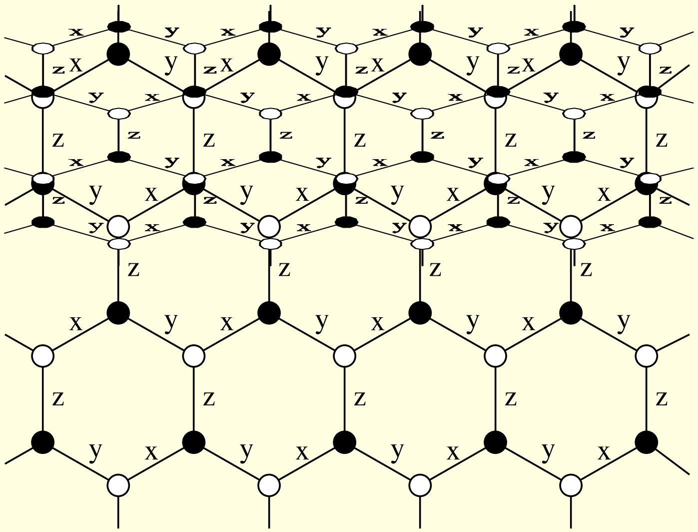
(spins on horizontal and vertical bonds)

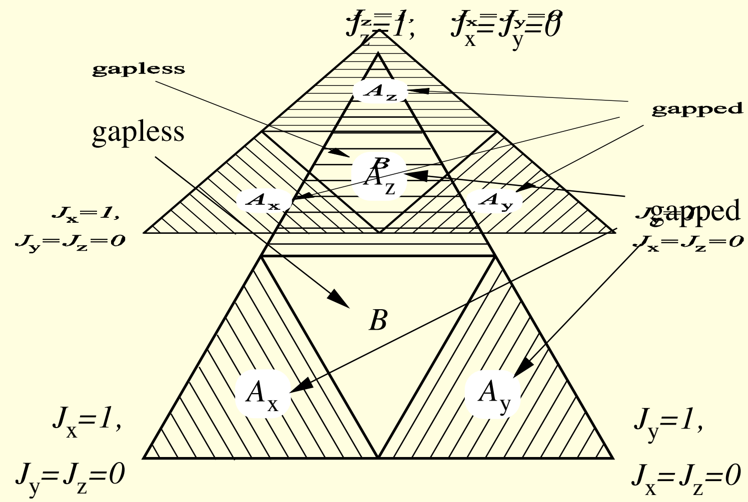
Phase diagram for the perturbed toric code



- Rough form of the phase diagram was predicted by Fradkin and Shenkar (1979)

The honeycomb lattice model





$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z,$$

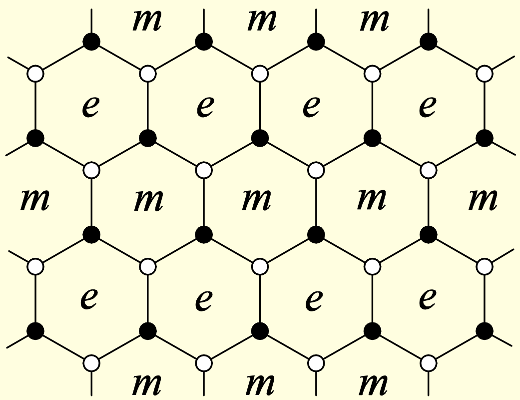
• The model is solved exactly (Kitaev, 2005). An optical lattice realization was proposed by Demler, Duan, Lukin (2003).

• The gapped phases are in the universality class of the toric code.

• In a magnetic field, a gap opens in the B phase. The new phase carries *non-Abelian anyons.*

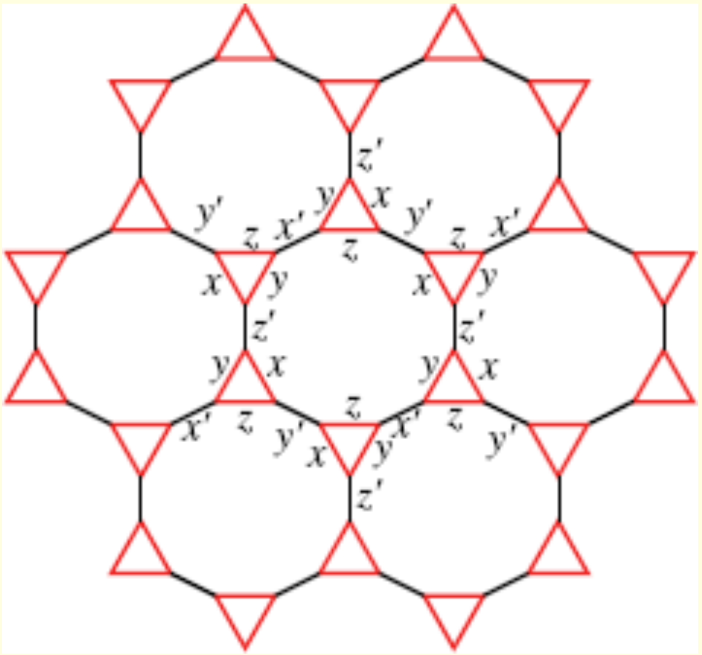
Excitations (without magnetic field)

- Fermions (ε):
  - Move on lattice sites (nontrivial dispersion)
  - Gapless in the B phase (for $J_x \sim J_y \sim J_z$);
  - Gapped in the A phases (i.e. when some of J_α is large).
- Vortices:
  - Located on plaquettes but do no move
    (no dispersion due to the exact solvability);
  - Always gapped;
  - Statistics is undefined in the B phase;
  - In the A phases, equivalent to e and m.



Yao-Kivelson model (2007)

- Defined on the *3-12 lattice*

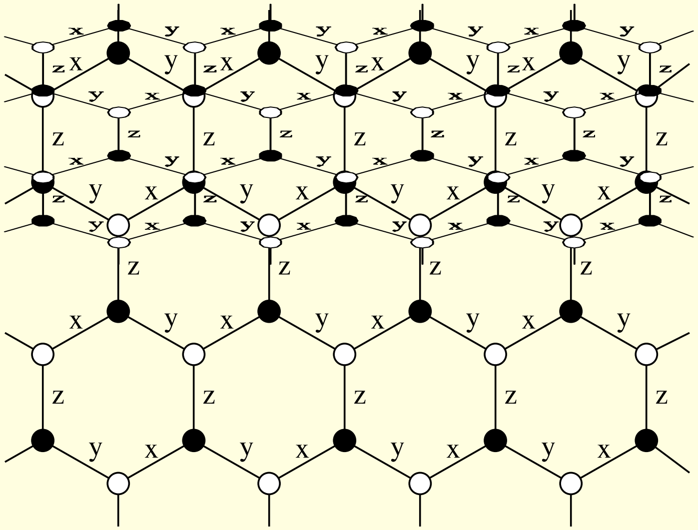


$$H = -\left(J_x \sum_{x\text{-links}} + J'_x \sum_{x'\text{-links}}\right) \sigma_j^x \sigma_k^x - \left(J_y \sum_{y\text{-links}} + J'_y \sum_{y'\text{-links}}\right) \sigma_j^y \sigma_k^y - \left(J_z \sum_{z\text{-links}} + J'_z \sum_{z'\text{-links}}\right) \sigma_j^z \sigma_k^z$$

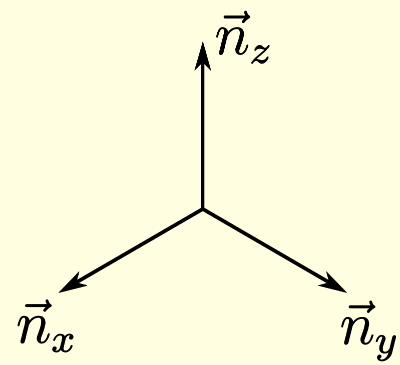
- Exactly solvable

- Properties similar to the B phase of the honecomb model in the magnetic field (*non-Abelian vortices*) are achieved due to spontaneous symmetry breaking.

Planar model on the honeycomb lattice



$$H = -J \sum_{j,k} (\vec{n}_{jk}, \vec{\sigma}_j)\,(\vec{n}_{jk}, \vec{\sigma}_k)$$



- In the original model, $\vec{n}_{jk} = \vec{e}_x$ on x-links, $\vec{n}_{jk} = \vec{e}_y$ on y-links, $\vec{n}_{jk} = \vec{e}_z$ on z-links
- <u>But now, let those vectors lie in the plane:</u>
- The model does not appear to be exactly solvable; its properties are unknown
- Can be realized as a *Heisenberg* model on the 3-12 lattice.