Topological Quantum Computation

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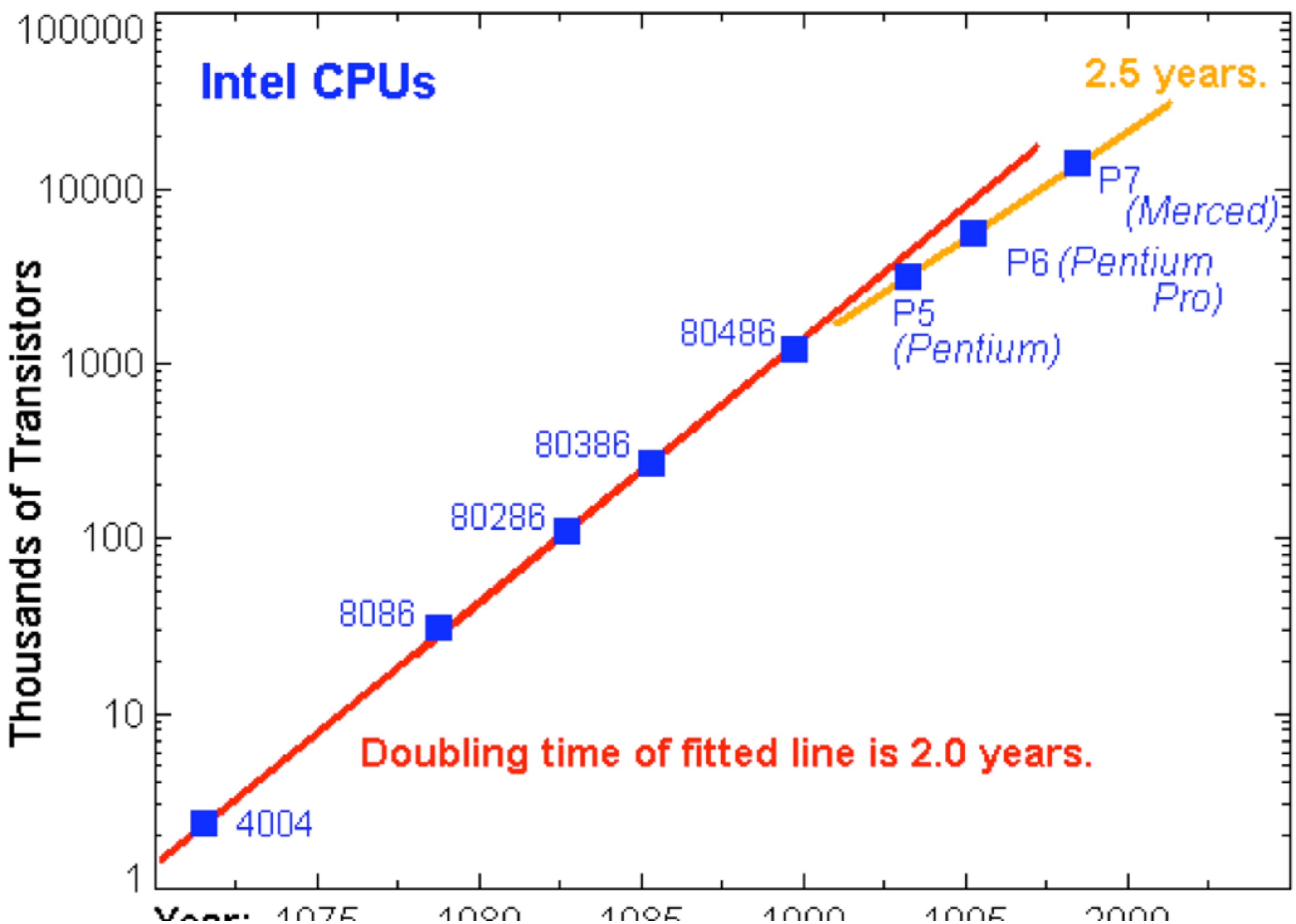
As the components of computers get smaller, we are approaching the limit in which quantum effects become important.



Is this a problem ... or an opportunity?

Feynman ‘81, Deutsch ‘85, Shor ‘94:

A computer which operates coherently on quantum states has much greater power than a classical computer

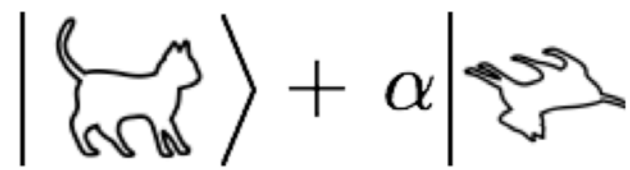
Quantum Computation

Paradigm with four key ingredients:

	Classical bits		Quantum bits
1. Hilbert space $\mathcal{H}$	0110111	$\rightarrow$	$\vert\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\downarrow\rangle$

New feature of qubits: $|\rightarrow\rangle = (|\uparrow\rangle + |\downarrow\rangle)$

customary to use the language of spins,
but could take any quantum system, e.g.:



2. Initial State $\psi_0 \in \mathcal{H}$, e.g. $|\uparrow\uparrow\uparrow\ldots\rangle$

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here,

$$\mathcal{H} = \mathbb{C}^2 \times \mathbb{C}^2 \times \ldots \mathbb{C}^2 \; ; \quad S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; S_z = \Big($$

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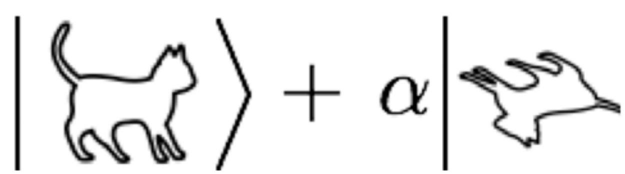
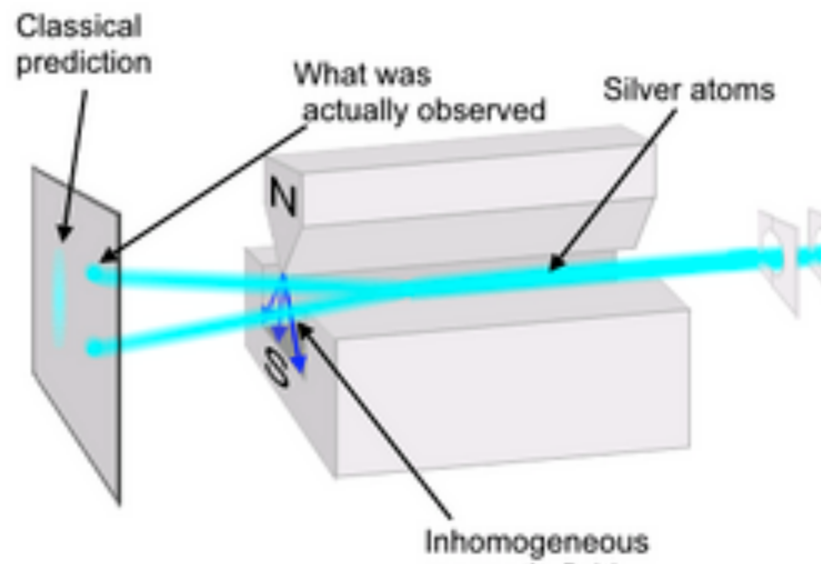


2. Initial State $\psi_0 \in \mathcal{H}$, e.g. $\lvert \uparrow\uparrow\uparrow \ldots \rangle$

3. Unitary Evolution of the State

$\psi_0 \rightarrow U$

4. Measure the State



What Is Q.C. Good For?

Feynman: a classical computer cannot efficiently simulate a quantum system of N spins since it would have to diagonalize a $2^N \times 2^N$ matrix.

A quantum computer can.

(e.g. a quantum system simulates itse

Shor: Classical computers seem unable to efficiently find the prime factors of a large number like:

1807082088687404805951656164405905566278102516769401349170127021450056662540244048387341127590812303371781887966563182013

A quantum computer can:

= 39685999459597454290161126162883786067576449112810064832555157243 ×
4553449864673597218840368689727440886435630126320506960099904

These problems are in the complexity class **BQP**:
Bounded error, **Q**uantum, **P**olynomial time.

$$\mathrm{BPP} \subseteq \mathrm{BQP} \subseteq \mathrm{PP}$$

The only other problem which is known to be in BQP but is suspected to be outside P is the discrete logarithm.

However, searching a database with N entries can be done in $\sqrt{N}$ time (Grover search algorithm).

Graph isomorphism, Hidden subgroup?
Kuperberg: $\exp(b\sqrt{N})$ for dihedral hsp.

Problem: Errors

Quantum phenomena do not occur in a Hilbert space. They occur in a laboratory. - A. Peres

Classical Computers: store multiple copies

Quantum Computers: situation more complex

1. No cloning.

2. If we measure a quantum state at an intermed
   step of a calculation to see if an error has occur
   then we could destroy superposition.

3. Errors can be continuous; not only can a bit
   but an arbitrary phase may be acquired.

Errors: Bete Noir of Q.C.

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Nevertheless, error correction is theoretically possible.

(Shor '95, Gottesman '9

1. Encode information redundantly.
2. Diagnose errors without measuring information.
3. Correct errors.

Represent: $|0\rangle, |1\rangle \rightarrow |000\rangle, |111\rangle$

Diagnose with, e.g. $P = |100\rangle\langle 100| + |011\rangle\langle 011|$

Nevertheless, error correction is theoretically possible.

(Shor '95, Calderbank and Shor '96, Steane

One can: 1. Encode information redundantly.

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3. Correct errors.

Logical Qubits

Physical Qubits

e.g. Represent: $|0\rangle, |1\rangle \rightarrow |000\rangle, |111\rangle$

Diagnose with, e.g. $P = |100\rangle\langle 100| + |011\rangle\langle 01$

However, since errors can occur during the er correction process, the basic error rate must be very l for quant. comp. to be fault-tolerant (various estima are in the range $< 10^{-5}$) or else the process will sh itself in the foot.

Errors are rare in classical computers, but quantu superpositions are delicate and error rates tend to high, often for poorly understood reasons. Ma processes can cause a transition in a quantum syste and it is difficult to make them all small, as discussed Sankar's talk last week for the cases of spins in Si GaAs. Also, it is difficult to enact a precise transf. quantum systems.

Central Problem of Quantum Computatio

And now for something completely different ..

Topology: a branch of mathematics concerned w
those properties of geometric configurations which
unaltered by elastic deformation.

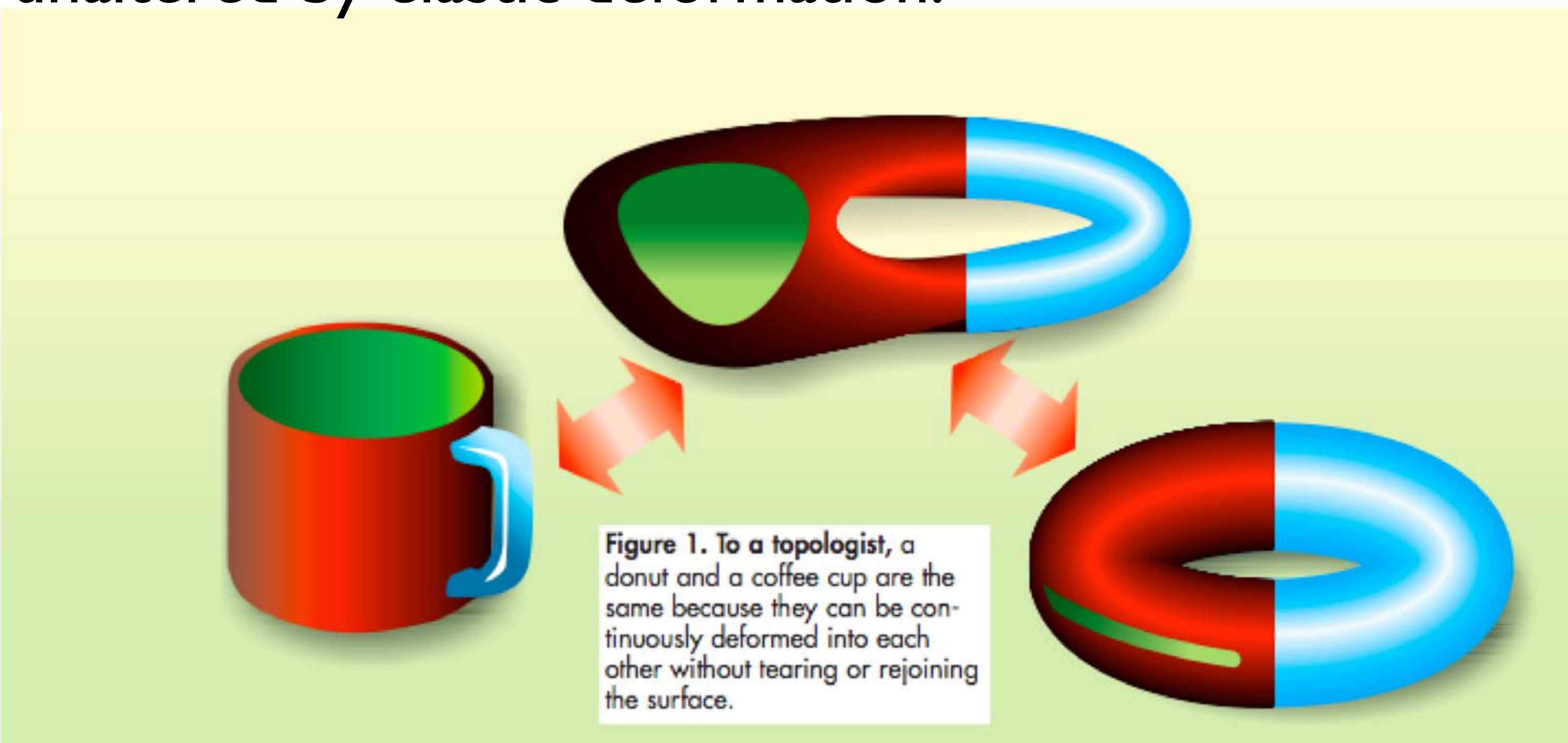


Figure 1. To a topologist, a donut and a coffee cup are the same because they can be continuously deformed into each other without tearing or rejoining the surface.

Topology focuses on certain robust features of
geometry which are stable against *small* perturbation

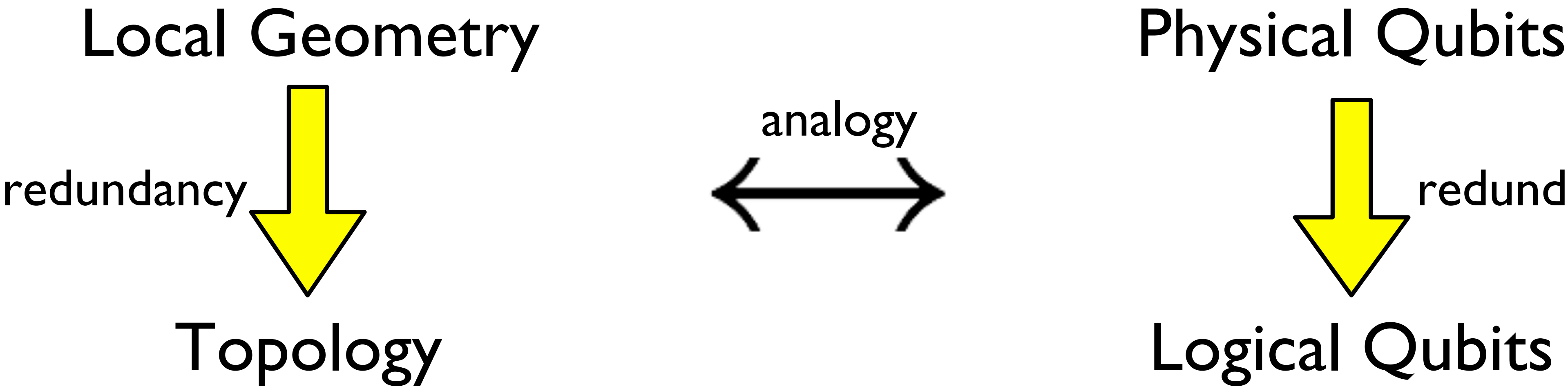
Another example: the first two loops can be deformed into each other without breaking/rejoining, but the third cannot. It is a non-trivial *knot.*

(a) (b) (c)

Lord Kelvin: atoms are vortex loops in the ether.
Different knots = different species of atoms.
Tait: all possible knots = periodic table of the elements.

Moot because of Michaelson-Morley, but knot theory became a fruitful subject in topology, with implications

Idea:



If a physical system were to have *topological degrees freedom* which were *insensitive to local probes*, the information contained in them would be *automatica protected against errors* caused by *local interactions wi the environment.*

A. Kitaev, Ann. Phys. **303**, 2 (200

Problem: topological-invariance is clearly not a symmetry of the underlying Hamiltonian.

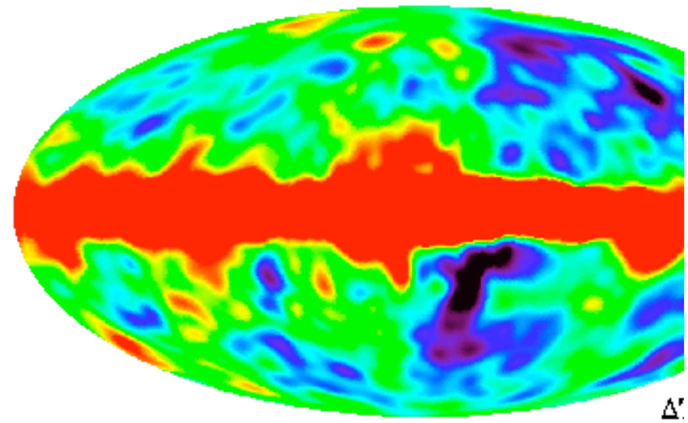
$$H = \sum_i \frac{p_i^2}{2m_e} + \sum_a \frac{P_a^2}{2M} + \sum_{i>j} \frac{e^2}{|r_i - r_j|} + \sum_{a>b} \frac{Z^2 e^2}{|R_a - R_b|} - \sum_{i,a} \frac{Ze}{|r_i -}$$

It must be a symmetry which *emerges* at low energies:

In a topological phase of matter, at *low temperature energies* and *long distances*, the system is insensitive t local perturbations -- in fact, to any notion of length.

- We are accustomed to the idea that the ground state and low-energy, long-wavelength physics might be *less symmetrical* than the microscopic equations of motion (spontaneous symmetry breaking).

  e.g. the world does not appear to be rotationally-invariant.



- The **converse** is also possible:

  Low-energy, long-wavelength physics might be ***more symmetrical*** *than the microscopic equations* (emergent symmetry).

Topological Phases of Matter are an example of this

It is possible for a system to simultaneously exhibit both conventional broken symmetry and topological order.

Chiral p-wave SCs are an example of this: broken Time-Reversal Symmetry *and* Topological Order.

A system is in a topological phase if its low-energy effective field theory is a topological quantum field theory, i.e. if all of its physical correlation functions are topological invariants.

Classic example: Chern-Simons-Witten theory:

$$S = \frac{k}{4\pi} \int \mathrm{tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

Since the metric doesn't appear in the action, we expect topological invariance.

Where/how can such a magical theory arise a the low-energy limit of a complex system of interacting electrons (which is not top. inv.)?

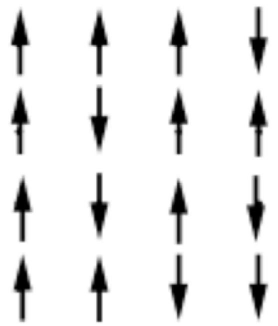
A physical system is in a topological phase if it is described by a TQFT:

The TQFT assigns a vector space $V(\Sigma)$ to a surface Σ. These are the ground states of the system when it is on the surface Σ.

The mapping class group acting on Σ is represented unitarily on $V(\Sigma)$. Think of punctures as quasiparticle excitations; braid group on *n* particles.

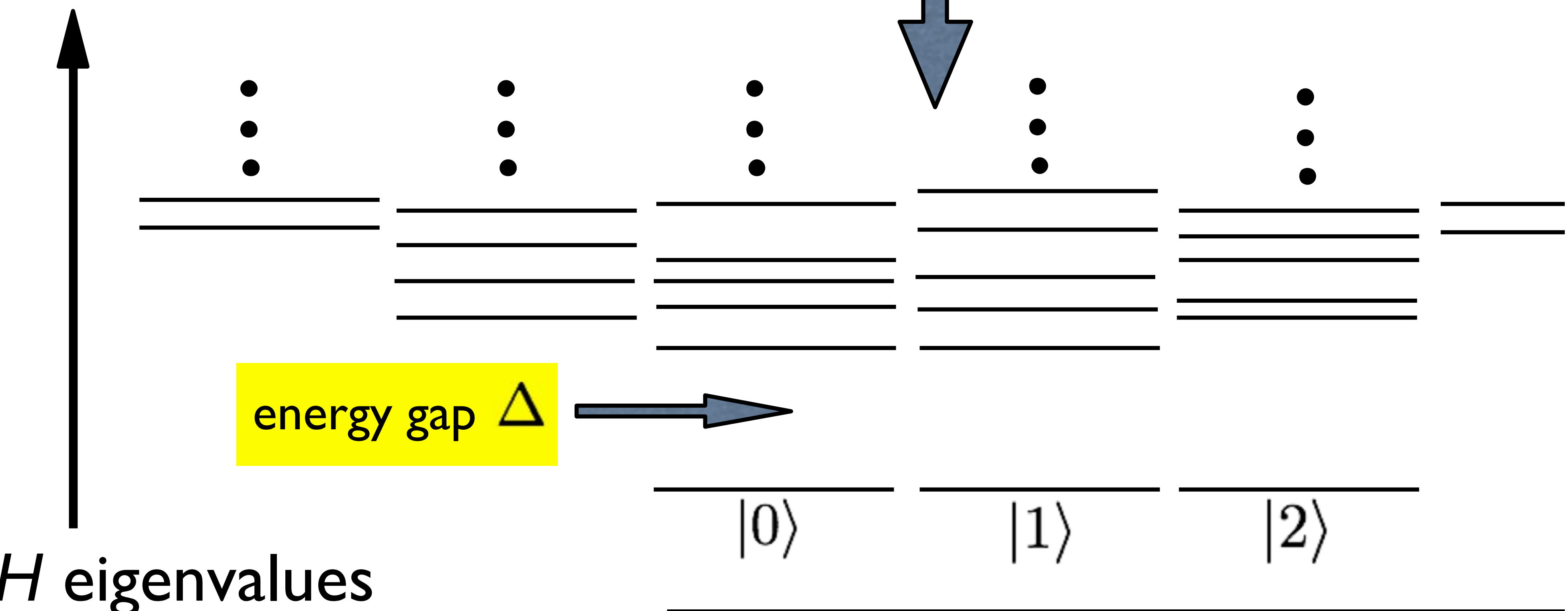
If the system were put on a spacetime manifold $\mathcal{M}^3$, then its partition function would be a Jones-Witten quantum three-manifold invariant (2 spatial dimensions + time) a $\mathbb{C}^2$ for each site of the latti

<u>Spins on a lattice:</u>



$$\mathcal{H} = \mathbb{C}^2 \times \mathbb{C}^2 \times \ldots \mathbb{C}^2$$

Hamiltonian *H* acting on $\mathcal{H}$

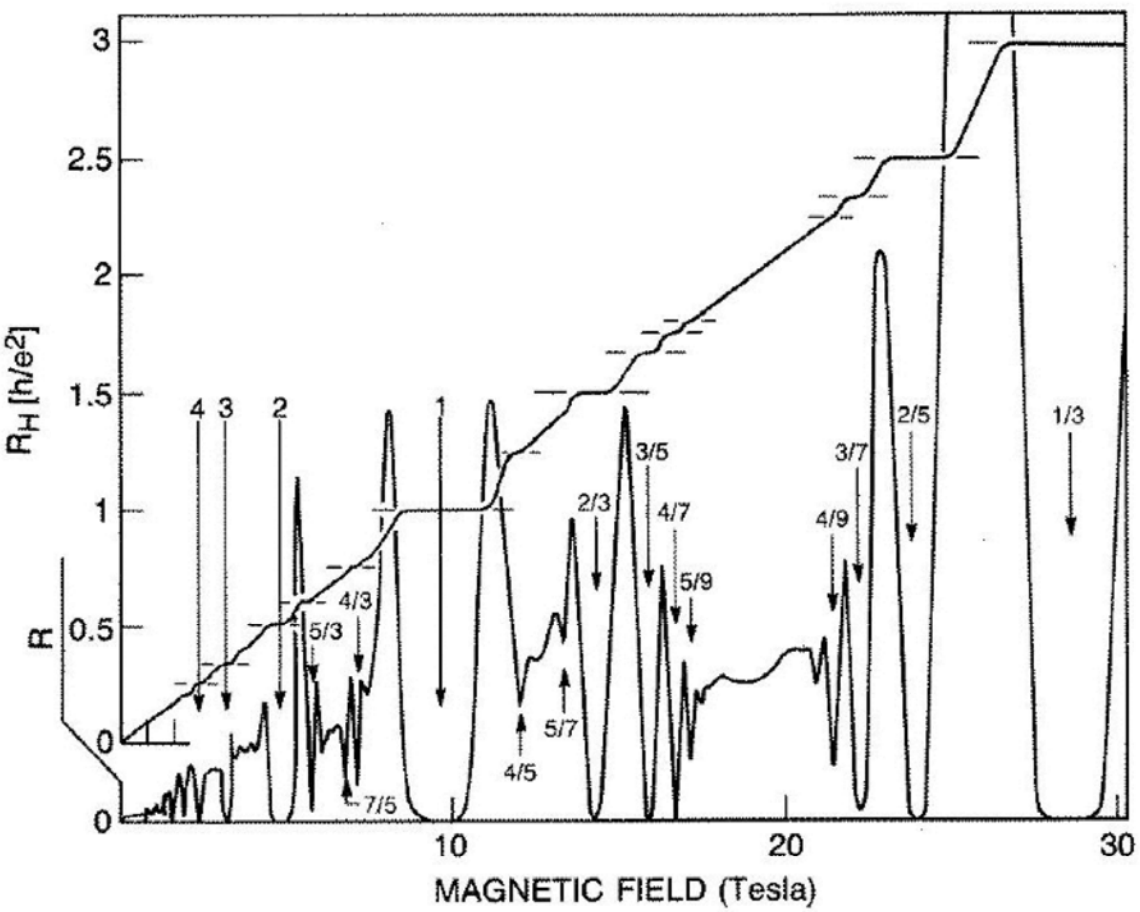


$\mathrm{span}\,\{|0\rangle, |1\rangle, \ldots, |n\rangle\} = V(\Sigma$

Topological Phases in the Quantum Hall Regi

When a 2DEG is placed in a perp. magnetic field a cooled to low temperatures, the electrons organi themselves in a topologically-invariant state.

Tsui, Stormer, Laughlin, 1998 N

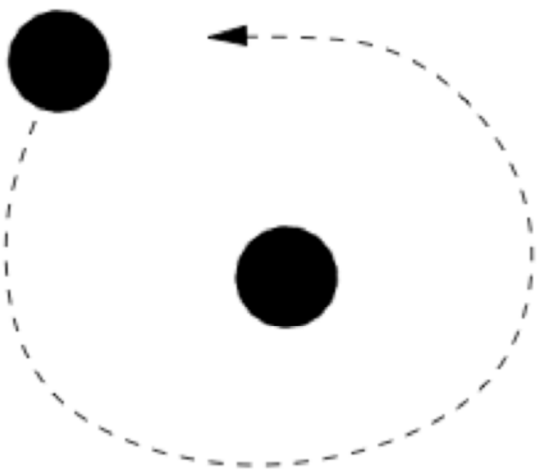


Hall Resistance plateaus are insens to local perturbatio

$$\sigma_{xy} = \frac{p}{q} \frac{e^2}{h}$$

Hallmarks of these States

- Quantized Hall Conductance
- Fractionally Charged Quasiparticles
- Exotic Braiding Statistics



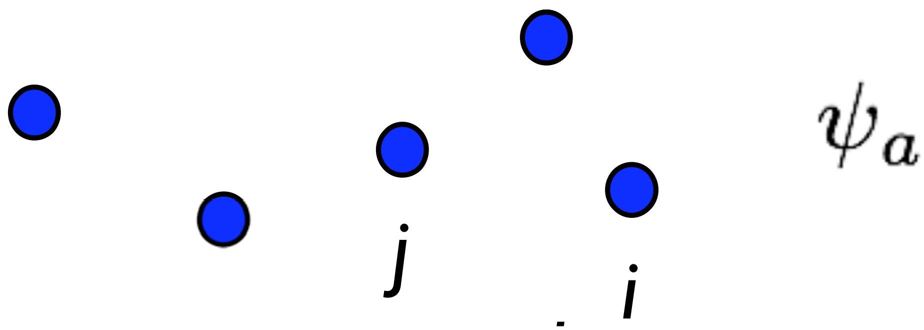
when one quasiparticle go around another, a non-triv phase results, e.g. $e^{2\pi i/3}$ when

$$\sigma_{xy} = \frac{1}{3}\frac{e^2}{h}$$

This is a rather trivial (but errorless) type of unitary operation, not very useful for computatior

Some special topological states are necessary for quantum computation

Non-Abelian Topological States



2. Braiding particles i and j transforms:

$$\psi_a \to M_{ab}\psi_b$$

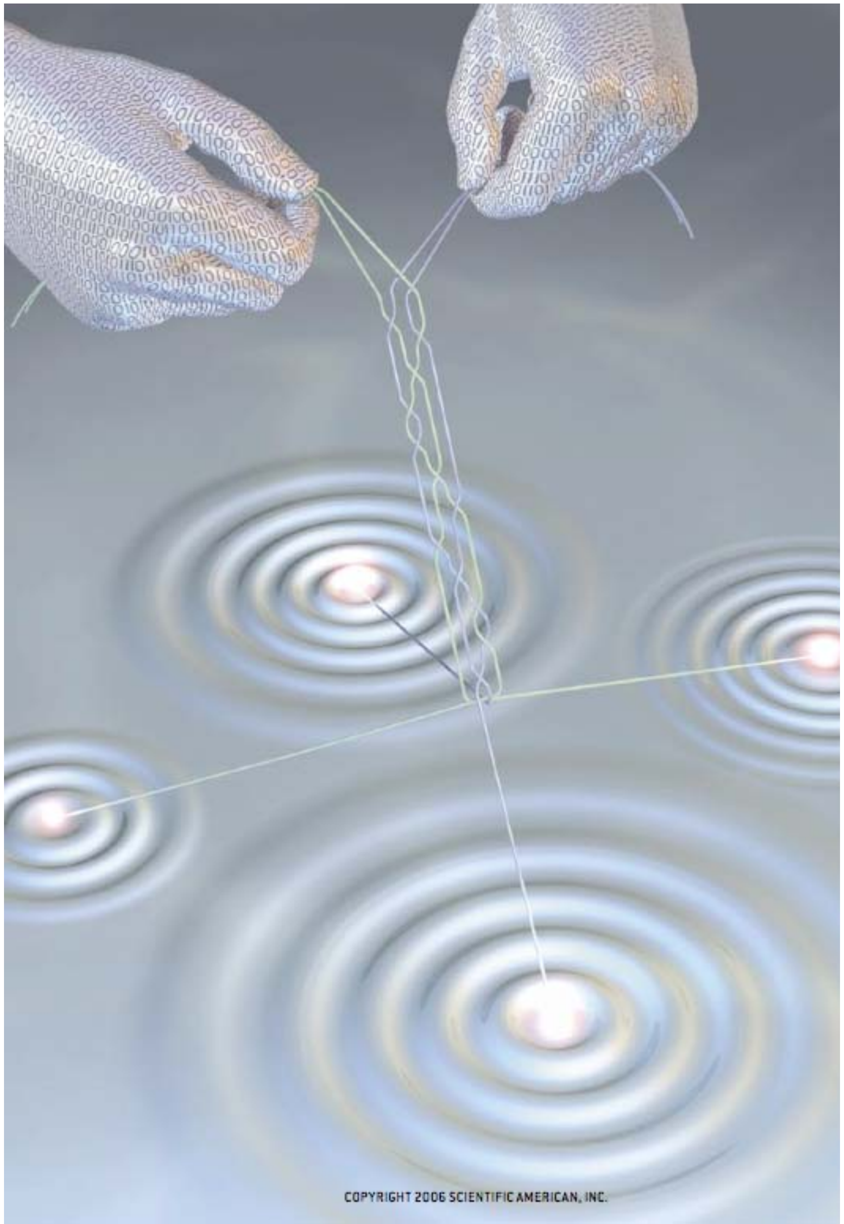
Topological protection: too mu
of a good thing? N

3. Braiding particles j and k: N_{ab} which need not commute with M_{ab}

4. For a large class of states, braiding operations implement all of U(g) to desired accuracy.

M. Freedman *et al.*, Comm. Math. Phys. 227, 605 (2002)

1. Create non-Abelian Quasiparticles (Qubits)
2. Measure their State by Aharonov-Bohm (Initialize)
3. Braid the Quasiparticles (Apply Gates)
4. Measure the Final State by Aharonov-Bohm (Read-out)



An alternative approach, doing away with step 3, will be discussed in M. Freedman's talk

Example: Chiral p-wave SC

- Assume spinless for simplicity

$$\langle \psi(k)\psi(-k)\rangle = \Delta_0(k_x + ik_y)$$

- There are several different types of quasiparticles in a SC:

They are treated differently in BCS theory, but we will put them all on the same footing

bosonic collective modes (e.g. the 'vacu

fermions (e.g. Bogoliubov-de Gennes qp

vortices

- Particles will be distinguished only if they differ topologically, e.g. any bosonic collective mode will be treated as the 'vacuum'.

- Bosons and fermions are simple topologically. However, vortices could have non-trivial topological props.

- In BCS language, the non-trivial properties of vortices derive from zero-energy fermionic vortex core states.

The Hamiltonian of a SC can be written:

$$H = E_0 + \sum_E E\,\Gamma_E^\dagger \Gamma_E$$

(valid at low ene

where $$\Gamma_E^\dagger \equiv \int dr\,[u_E(\mathbf{r})\psi(\mathbf{r}) + v_E(\mathbf{r})\psi^\dagger(\mathbf{r})]$$

and the Bogoliubov-de Gennes eqns for fermionic quasiparticle states in the presence of a vortex are

$$\begin{pmatrix} -\mu(\mathbf{r}) & \frac{i}{2}\{\Delta(\mathbf{r}), \partial_x + i\partial_y\} \\ \frac{i}{2}\{\Delta^*(\mathbf{r}), \partial_x - i\partial_y\} & \mu(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} = E \begin{pmatrix} u(\\ v(\end{pmatrix}$$

with: $\Delta(r,\theta) = |\Delta(r)|e^{i\theta}$

- For every sol'n (*u*,*v*) with energy E, there is a sol'n (*v**,*u**) with -E. Special feature of *p*+*ip* SC: a single E=0 sol'n (*u*,*u**):

$$u_0(r,\theta) = f(r) \begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix}$$

f(*r*) concentrat
in vortex core.

- The corresponding operator commutes with *H*

$$\gamma = \frac{1}{\sqrt{2}} \int dr \; [u_0(r,\theta)\,\psi(\mathbf{r}) + \; u_0^*(r,\theta)\,\psi^\dagger(\mathbf{r})$$

and satisfies $\gamma = \gamma^\dagger$

This is a *Majorana fermion*. Read and Gree

Fusion

- Two vortices: each has its own Majorana zero mo

Two state system: $c = (\gamma_1 + i\gamma_2)/2$

If the vortices are not infinitely far apart, there will be some splitting between the two states:

$$H_t = it\,\gamma_1\gamma_2 = 2t\,(c^\dagger c -$$

- We say that two vortices can 'fuse' in two different ways, $c^\dagger c = 0, 1$. The two different 'fusion channels' form a *qubit*.

- Degeneracy: 2n vortices $\Rightarrow 2^{n-1}$ states

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

$\sqrt{2}$ states per vor

Braiding

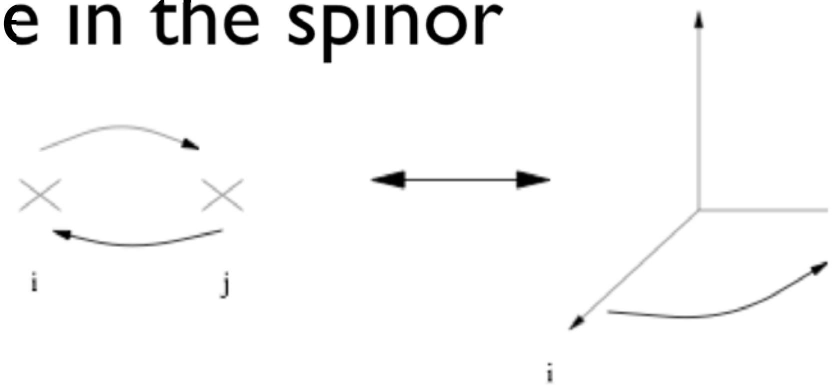
- When one vortex is exchanged with another, the system is transformed within this Hilbert space.

- The basic effect is to exchange the zero modes and to apply minus signs when one zero mode crosses the branch cut from the other vortex.

$$\gamma_1 \rightarrow \gamma_2$$
$$\gamma_2 \rightarrow -\gamma_1$$
$$\gamma_i \rightarrow \gamma_i \ , \qquad i \neq 1, 2$$

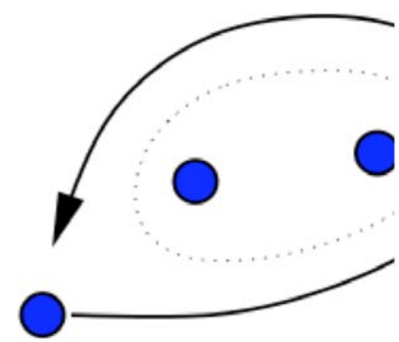
More in M. Stone's talk

- Braiding: Exchanging particles i and j enacts a $\pi/2$ rotation in the i-j plane in the spinor rep. of SO(2n): $T_{ij} = e^{\frac{\pi}{4}\gamma_i\gamma_j}$

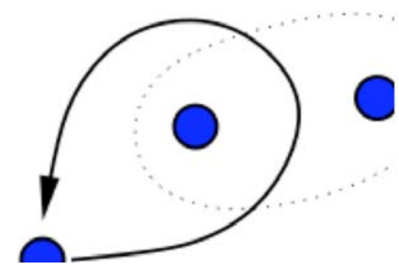


Nayak and Wilczek '96, Ivanov '01, Stern et al. '04, Stone and Chung '05

- A qubit can be measured by taking a third quasiparticle around it.



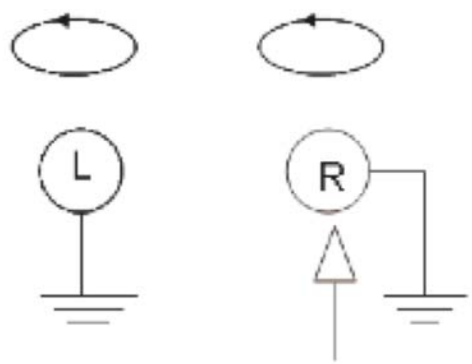
- It can be flipped by taking a third q.p. around one member of the pair



‘Teleportation’

- Since the two Majorana fermions assoc. with two vortices combine to form a single two-level system even when the vortices are well-separated, we can observe unusual effects be probing both simultaneously.

A quantum dot in tunneling contact with one vortex will affect the STM spectrum at the other



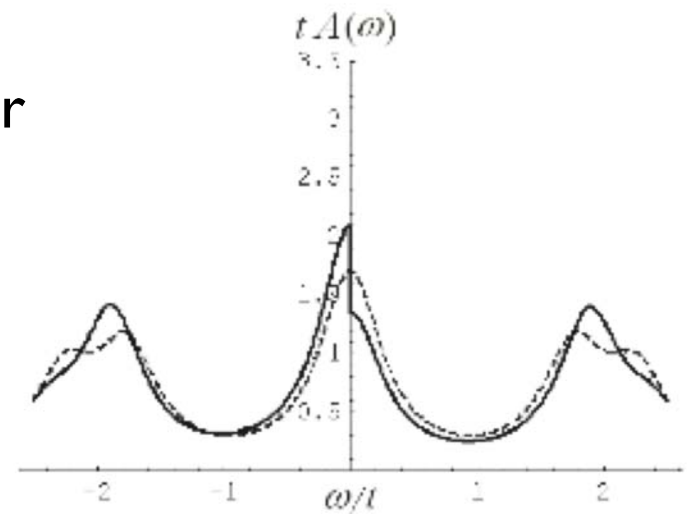
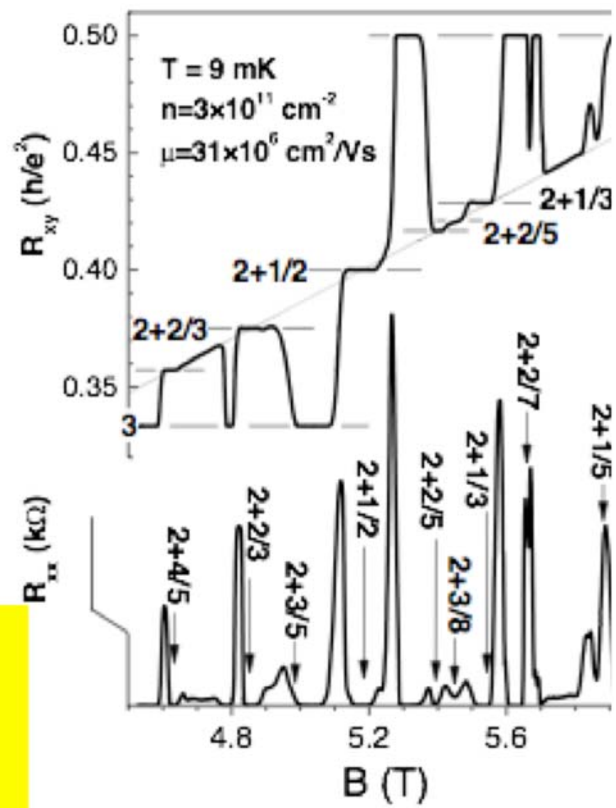


FIG. 2: The tunneling curves for $J = 0.5t$. Dashed curve

Quantum Hall Analog: 5/2 Plateau

- There is evidence from numerics that the FQH state observed at 5/2 is in the same universality class as:



$$\Psi(z_j) \;=\; \prod_{j<k}(z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \cdot \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right) \;.$$

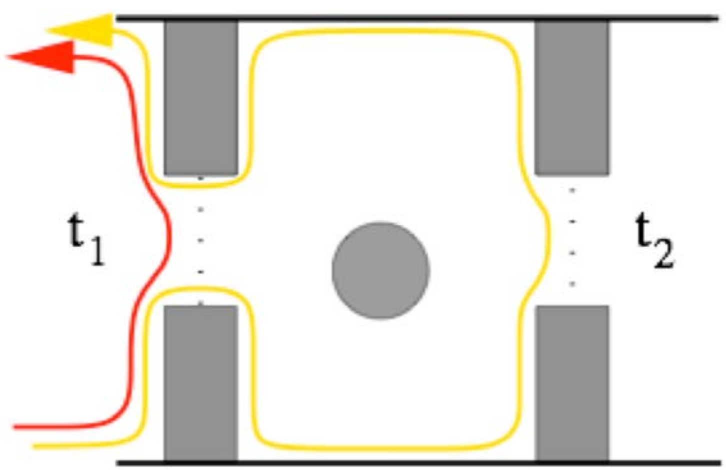
Moore and Read ’91

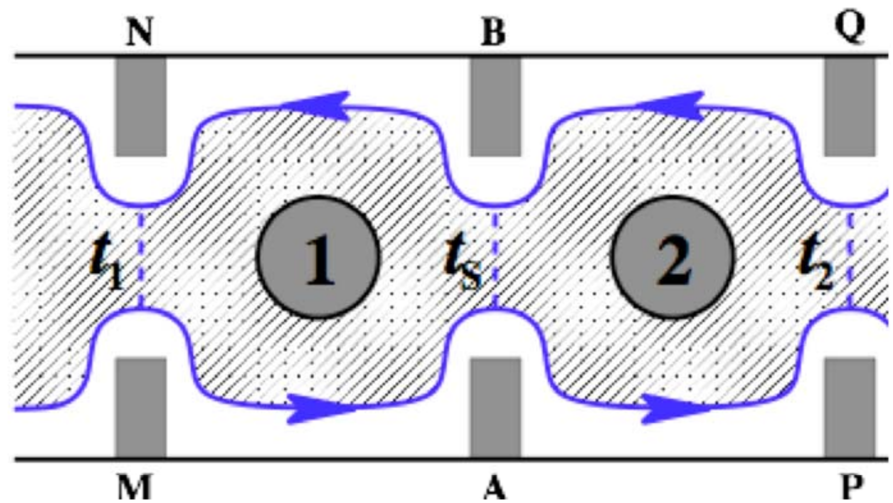
where $\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \mathcal{A}\left(\frac{1}{z_1 - z_2}\,\frac{1}{z_3 - z_4}\,\ldots\right)$

Greiter *et al.* ’92: reminiscent of: $\Psi_{BCS} = \mathcal{A}\,(g(r_1 - r_2)\,g(r_3 - r_4$

- If this conjecture is correct, the 5/2 quantum H
state has the same topological properties as a spinless chiral p-wave SC.

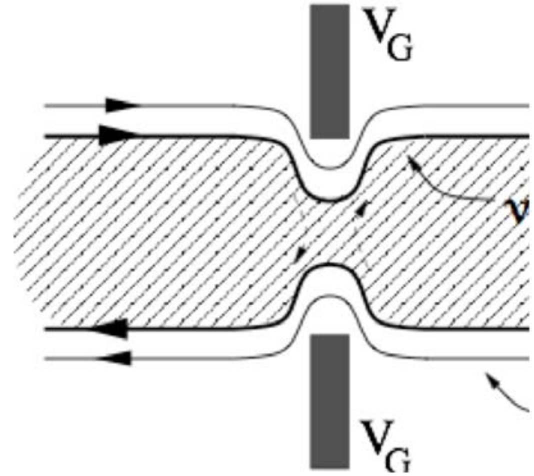
- Interferometry experiments have been propose
to measure these properties, as A. Stern will discuss.





Fradkin *et al.* '98; Das Sarma, Freedman, and Nayak '05;

Edge Excitations



- Quantum Hall states have current carrying edge excitations. Tunneling between these edge modes can be used to probe the state. e.g. $R_{xx} \sim \lambda_{1/4}^2 T^{-3/2}$

Fendley, Fisher, Nayak PRL, '06.

- Chiral SCs also have edge excitations, but they car heat, not electrical charge.

However, when a vortex tunnels across a SC, a voltage is generated. The temp. dependence of vortex tunneling is governed by the edge theory, as will be described in M.P.A. Fisher's talk

Is SrRuO a chiral p-wave SC?

- In this meeting, we have heard about evidence (e.g. Kerr effect, Josephson junctions) that SRO is chiral.

  We have also heard some evidence that it is not chiral (absence of measurable edge currents).

- Settling this issue is clearly a prerequisite for any discussion about the topological properties of SRO and possible applications to quantum computing.

Stabilization of Half Quantum Vortices

- Even if SRO does prove to be a chiral p-wave SC, it is not likely to be in the same universality class as a fully polarized chiral p-wave SC. (i.e. the A_I phase)

- The topologically interesting excitations in the A phase are half-quantum vortices -- essentially vortices for only one of the spins.

- Spin-orbit interaction would confine them linearly, but this can be neutralized by tipping the d-vector into the plane.

- Half-quantum vortex:

$$\hat{d} = \hat{x}\cos(\theta/2) + \hat{y}\sin(\theta/2), \quad \phi = \pm\theta/2,$$
$$\hat{l} = \text{const.}$$

This pays an energy cost $E_{\rm so} = -\Omega_{\rm so}(\hat{d}\cdot\hat{l})^2$
unless we force d into the plane with a B-fielc
(perhaps as little as 200G). Das Sarma, Nayak, Tew

- However, there will still be logarithmic interactions between vortices.
Last week's talks by Kim and Chung discussed how these might be overcome.

- Also, need to worry about even/odd number of layers

Summary

- Topological quantum computation is an exciting and promising approach to defeating decoherence and other errors which could doom quantum computation.
- It depends on the existence in nature of non-Abelian topological phases.
- The 5/2 quantum Hall state and SRO are the two leading candidates.
- Still would have an issue of stabilizing and isola half-quantum vortices.