Open Strings On The Rindler Horizon

Edward Witten

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First let us remember the replica trick in field theory.

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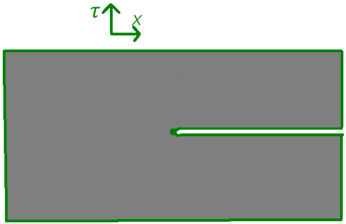

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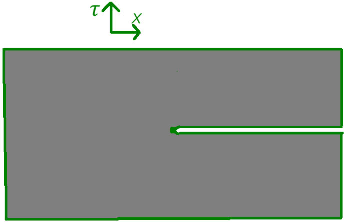

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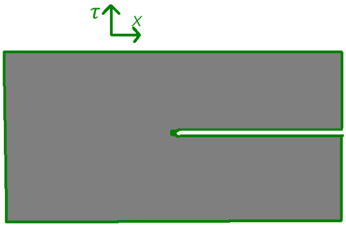

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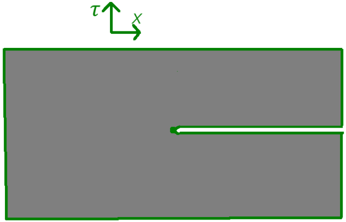

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where ϕ_r, ϕ_r' are field variables in the right half space just above or below the cut. Such a function can be viewed as an operator acting on one set of field variables ϕ_r. This is the density matrix ρ of Rindler space.

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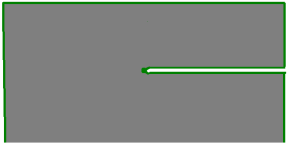

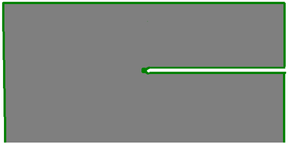

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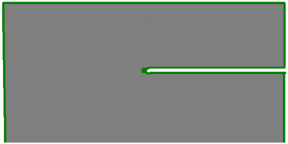

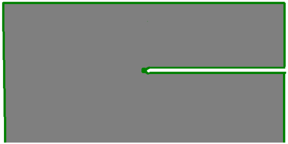

and glue the top of the cut in one to the bottom of the cut in the other, and vice-versa.

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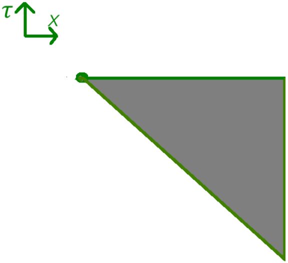
In this particular example of quantum fields in a background spacetime, one can directly access non-integer values of $\mathcal{N}$ by replacing the $\mathcal{N}$-fold cover with a cone of opening angle $2\pi\mathcal{N}$:



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$$\log \mathrm{Tr}\,\rho^{1/N}\Big|_1 = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T}\, Z_N(T),$$

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where $Z_N(T)$ is the partition function of the orbifold on an annulus of modular parameter T. $Z_N(T)$ is well defined although the integral over T is divergent (for integer N) so we can try to analytically continue $Z_N(T)$ and worry about integrating over T later.

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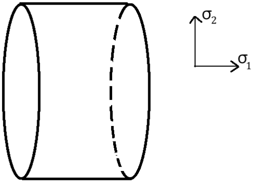

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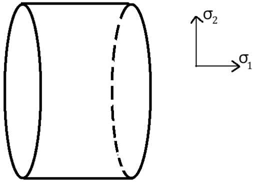

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where $H = L_0$ is the Hamiltonian and the projection operator from $\mathcal{H}$ to $\mathcal{H}_N$ is

$$P = \frac{1}{N}\sum_{k=0}^{N-1} U^k,$$

U being a generator of $\mathbb{Z}_N$.

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The existence of such an J is a restatement of standard facts and will be discussed shortly.

Given the existence and properties of J, the orbifold partition function on the annulus is

$$Z_N(\tau) = \frac{1}{N} \sum_{k=1}^{N-1} J(k/N, \tau).$$

Consider the function

$$K(z,N) = \sum_{k=1}^{N-1} \frac{\pi \sin \pi z}{\sin(\pi k/N) \sin \pi(z - k/N)}.$$

It is a periodic function, $K(z+1,N) = K(z,N)$, and bounded for $\mathrm{Im}\, z \to \pm\infty$. The poles of $K(z,N)$ in the strip $0 \leq \mathrm{Re}\, z \leq 1$ are simple poles of residue 1 at $z = k/N$, $k = 1, \cdots, N-1$.

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But the sum of all residues of KJ on the cylinder vanishes. So we get another formula as the sum over residues at the set $\mathcal{S}$ of poles of J:

$$Z_N(T) = -\sum_{z_0 \in \mathcal{S}} \mathrm{Res}_{z_0}(K(z, N)J(z, \tau)).$$

If the poles of $J(z, \tau)$ are all simple poles, the formula simplifies to

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(This is not quite true as $J(z,\tau)$ has a double pole at $z = 1/2$, so that formula needs to be slightly modified.)

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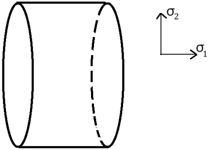

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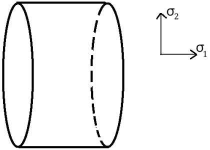

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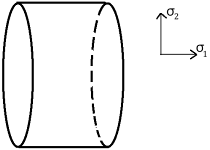

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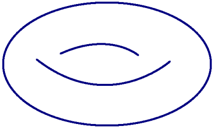

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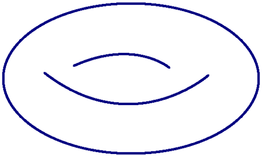

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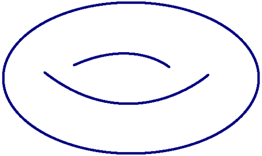

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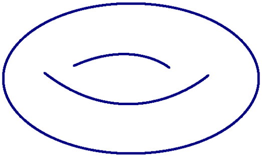

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where D_F and D_B are the fermionic and bosonic kinetic energies. A pole of $J(z, \tau)$ comes from a zero-mode of D_B, and it is straightforward to find the values of z at which these occur.

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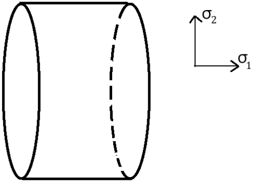

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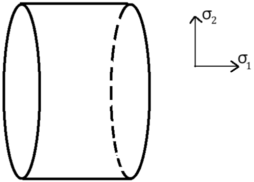

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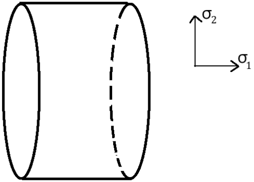

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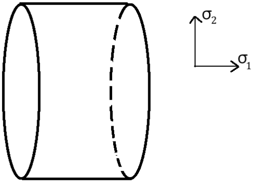

describes an open string of width π propagating for a Euclidean proper time $2\pi T$. Equivalently in the crossed channel, it describes a closed string of circumference $2\pi T$ propagating "sideways" for proper time π. Rescaling lengths so that the closed string has standard circumference 2π, the proper time in the closed-string channel is π/T, which is $2\pi\widetilde{T}$ with

$$\widetilde{T} = \frac{1}{2T}.$$

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where in the second line I rewrite the formula using what we learned about $J(z, \tau)$. $\mathcal{S}$ is the set of all poles of G; we write $\mathcal{S}_1$ for the poles at $\mathrm{Re}\, z = 0$ and $\mathcal{S}_2$ for the poles at $\mathrm{Re}\, z = 1/2$.

For analytic continuation, we have to use

$$K_1 = \pi N \cot \pi N z - \pi \cot \pi z$$

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$$Z_N(T) = -C \sum_{z_0 \in \mathcal{S}_1} \frac{K_1(z_0, N)}{\sin 2\pi z_0} \mathrm{Res}_{z_0} G(z, \tau)$$

$$- C \sum_{z_0 \in \mathcal{S}_2} \frac{K_2(z_0, N)}{\sin 2\pi z_0} \mathrm{Res}_{z_0} G(z, \tau).$$

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$$Z_N(T) \overset{T \to 0}{\sim} \exp(-4\pi \widetilde{T} h)$$

(times a phase space factor $1/\widetilde{T}^{(p-1)/2}$).

So in particular if N is an integer, a tachyon with $h = -k/N$ will give

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$$L_0 = hI_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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can have lead to the observed $\widetilde{T}\exp(-4\pi\widetilde{T}h)$ behavior, since

$$\exp(-4\pi\widetilde{T}L_0) = \exp(-4\pi\widetilde{T}h)\begin{pmatrix} 1 & -4\pi\widetilde{T} \\ 0 & 1 \end{pmatrix}.$$

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This massless scalar – analogous to a twisted sector mode when N is an integer – propagates only on the Rindler horizon. Its existence leads to a mysterious IR divergence in the entanglement entropy or in $\mathrm{Tr}\,\rho^{\mathcal{N}}$.