On the relation between the magnitude and exponent of OTOCs

(Based on work with Alexei Kitaev)

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Outline

- Introduction/conventions;
- A ladder identity, branching time, sketch of the derivation;
- Applications of the ladder identity:
  1. computational shortcuts;
  2. exact maximal chaos, e.g. in a 1D model.

Introduction/conventions

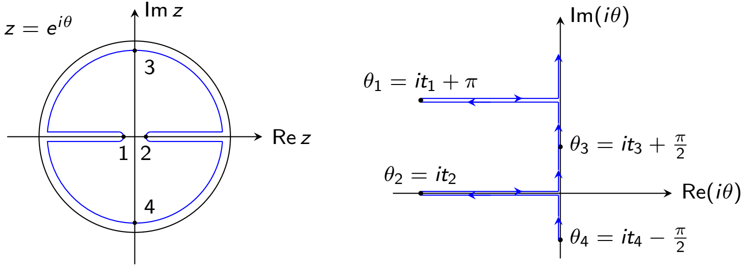
- Chaos: out-of-time-order correlators (Larkin-Ovchinnikov 1969):

$$\langle X(\theta_1) Y(\theta_3) X(\theta_2) Y(\theta_4) \rangle, \quad \theta = it + \tau$$

For convenience we set $\beta = 2\pi$, evenly spaced.

$$\theta_1 = it_1 + \pi\,, \quad \theta_2 = it_2\,, \quad \theta_3 = it_3 + \frac{\pi}{2}\,, \quad \theta_4 = it_4 - \frac{\pi}{2}\,.$$

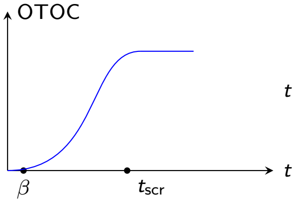
$$t_1 \approx t_2 \gg t_3 \approx t_4$$



Early time regime

- SYK. $X = \chi_j$, $Y = \chi_k$. Average over j, k:

$$\mathrm{OTOC}(t_1, t_2, t_3, t_4) := \frac{1}{N^2} \sum_{jk} \langle \chi_j(\theta_1)\chi_k(\theta_3)\chi_j(\theta_2)\chi_k(\theta_4)\rangle + G(\theta_{12})G(\theta_{34})$$

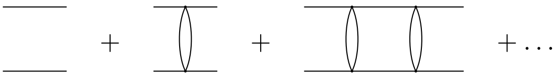


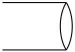
$t := \frac{t_1+t_2-t_3-t_4}{2}$

- Early time regime: initial growth $1 \ll t \ll t_{scr}$.
- Lyapunov exponent $0 < \varkappa \leqslant 1$.

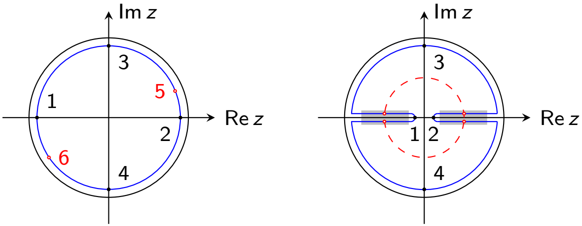
Kinetic equation

- Leading contribution: ladder diagrams (Kitaev, Polchinski-Rosenhaus, Maldacena-Stanford ...):

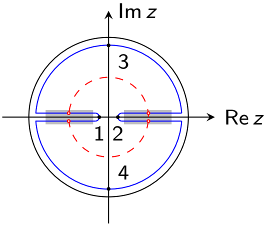


$\mathcal{F} = \sum_n \mathcal{F}_n$, $\quad \mathcal{F}_n =$  $\cdot\, \mathcal{F}_{n-1} \;\Rightarrow\; \mathcal{F} = \mathcal{F}_0 + K\mathcal{F}$

- For OTOC: deform the contour to double Keldysh.

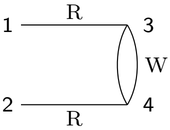


Retarded kernel



- Main contribution comes from the real time folds (gray). Sum of four points with same real time $\Rightarrow G^R$. With exponentially small error, discard $\mathcal{F}_0$ and change the integral contour to real line.
- Kinetic equation (Kitaev 2015, Murugan-Stanford-Witten 2017):

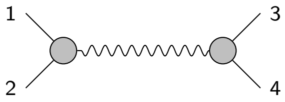
$$F \approx K^R \cdot F, \quad K^{\mathrm{R}}(t_1, t_2, t_3, t_4) :$$



Single-mode ansatz

- Single-mode ansatz for early time regime (Kitaev-Suh, 2017)

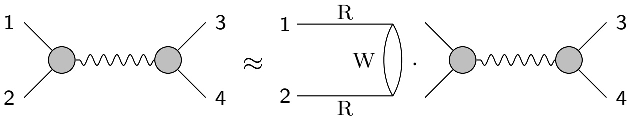
$$\mathrm{OTOC}(t_1, t_2, t_3, t_4) \approx \frac{e^{\varkappa(t_1+t_2-t_3-t_4)/2}}{C} \Upsilon^R(t_{12}) \Upsilon^A(t_{34})$$



- C is large, in SYK $C \sim \frac{N}{\beta J}$.

Eigenvalue $k_R(\alpha)$

- Single-mode ansatz:



$$\Upsilon^{\mathrm{R}}(t_{12})e^{\varkappa(t_1+t_2)/2} = \int dt_5 dt_6 K^R(t_1, t_2, t_5, t_6)e^{\varkappa(t_5+t_6)/2}\Upsilon^{\mathrm{R}}(t_{56})$$

- Define a variant of kernel for parameter $\alpha < 0$:

$$K_\alpha^{\mathrm{R}}(t, t') = \int K^{\mathrm{R}}\left(s + \frac{t}{2}, s - \frac{t}{2}, \frac{t'}{2}, -\frac{t'}{2}\right) e^{\alpha s}\, ds\,.$$

- Kinetic equation: $K^R_{-\varkappa}\Upsilon^{\mathrm{R}} = \Upsilon^{\mathrm{R}}$.

- For general $\alpha < 0$, we denote its largest eigenvalue by $k_{\mathrm{R}}(\alpha)$:

Lyapunov exponent $\varkappa$: $k_{\mathrm{R}}(-\varkappa) = 1$

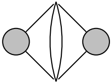
A ladder identity

Kinetic equation is useful: find $\varkappa$, $\Upsilon^{\mathrm{R(A)}}$, but can not determine C.

- An identity in SYK (Kitaev 2017, talk@IAS)

$$N \cdot \frac{2\cos\frac{\varkappa\pi}{2}}{C} \cdot k_{\mathrm{R}}'(-\varkappa) \cdot \left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}}\right) = 1\,.$$

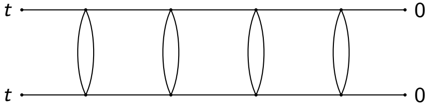
- $\left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}}\right)$: inner product of vertex functions:

$\left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}}\right) =$  $= (q-1)J^2 \int dt\, \Upsilon^{\mathrm{A}}(t)\left(G^{\mathrm{W}}(t)\right)^{q-2}\Upsilon^{\mathrm{R}}(t)\,.$

- Branching time

$$t_B := k_{\mathrm{R}}'(-\varkappa)\,.$$

Branching time



- Average distance between rungs $= t/\langle n \rangle$;
- Count number of rungs:

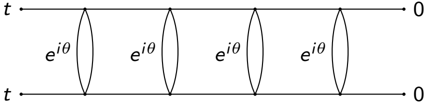
$$F(t) = \sum_n F_n(t)\,, \quad \langle n \rangle = \frac{\sum_n nF_n(t)}{\sum F_n(t)}\,.$$

- Introduce an auxiliary (generating) function:

$$F(\theta, t) := \sum_n F_n(t) e^{in\theta}$$

Branching time

$$F(\theta, t) = \sum_n$$



- Idea: around $\theta = 0$, find $F(\theta, t)$ using kinetic equation;
- $F(\theta, t) \sim \frac{e^{\varkappa(\theta)t}}{C}$, the Lyapunov exponent $\varkappa(\theta)$ satisfies:

$$e^{i\theta} k_R(-\varkappa(\theta)) = 1$$

- Thus,

$$\langle n \rangle = -i\partial_\theta \log F(\theta, t)|_{\theta=0} \approx -i\varkappa'(0)t + (\text{non-growing})$$

$$\varkappa'(0) = ik_R'(-\varkappa)^{-1} = it_B^{-1} \quad \Rightarrow \quad \langle n \rangle \approx \frac{t}{t_B}$$

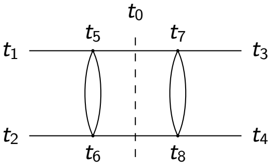
Derivation of the identity

Next, we sketch the derivation of

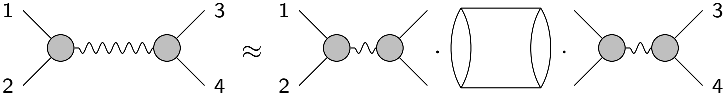
$$N \cdot \frac{2\cos\frac{\varkappa\pi}{2}}{C} \cdot k'_{\mathrm{R}}(-\varkappa) \cdot \left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}}\right) = 1\,.$$

Idea: cut a long ladder into pieces and find a consistency condition.

- Cut. Fix t_0, find adjacent $\frac{t_5+t_6}{2} < t_0 < \frac{t_7+t_8}{2}$



- Consistency condition

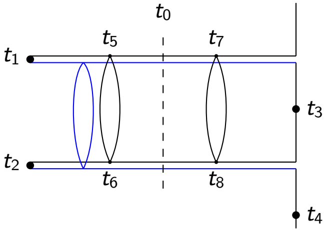


Derivation: $\cos\frac{\varkappa\pi}{2}$ factor

- Naively, we would have a formula:

$$\mathrm{OTOC} \approx \mathrm{OTOC}_L \cdot \mathrm{BOX} \cdot \mathrm{OTOC}_R$$

- Subtlety: multiple choices on the double Keldysh contour

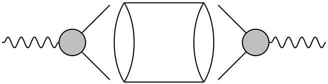
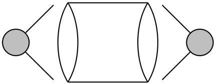


- Sum of two choices:

$$\mathrm{OTOC} \approx \left(e^{i\frac{\varkappa\pi}{2}} + e^{-i\frac{\varkappa\pi}{2}}\right) \mathrm{OTOC} \cdot \mathrm{BOX} \cdot \mathrm{OTOC}$$

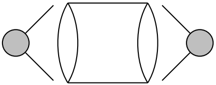
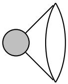
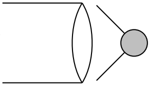
Derivation: box

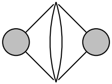
- Next, integral over four time t_5, t_6, t_7, t_8

 $: \int ds dt_* e^{-\varkappa s}$ 

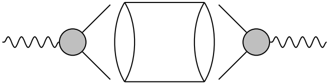
$s = \frac{t_5+t_6-t_7-t_8}{2}$: size of the box; t_*: center of mass time $\int dt_* = s$

- How does this term related to $k_R'(-\varkappa)$?

$\int ds e^{-\varkappa s}$  $=$  $\int ds e^{-\varkappa s}$ 

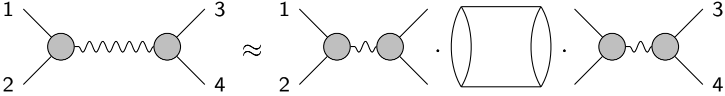
$= k_R(-\varkappa)$ 

- Take $\varkappa$ derivative:

$t_B\left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}}\right) =$ 

Derivation: summary

- To summarize, we start with



- Compare two sides, find

$$\frac{1}{C} = N \cdot \frac{2\cos\frac{\varkappa\pi}{2}}{C^2} \cdot t_B \cdot (\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}})$$

Applications

The ladder identity:

$$N \cdot \frac{2\cos\frac{\varkappa\pi}{2}}{C} \cdot t_B \cdot \left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}}\right) = 1\,.$$

Next:

- Computational shortcuts: $C \Leftrightarrow \varkappa$;
- In a 1D model, find exact maximal chaos using the identity.

Computational shortcuts I: near maximal chaos

SYK at strong coupling $J \gg 1$, near maximal chaos $\varkappa = 1 - \delta\varkappa$.

- Schwarzian action:

$$I_{\mathrm{Sch}}[\varphi] = -\frac{N\alpha_S}{J}\int_0^{2\pi} \mathrm{Sch}\left(e^{i\varphi(\tau)}, \tau\right) d\tau$$

- Use Schwarzian action:

$$\mathrm{OTOC} \approx \frac{Je^{(t_1+t_2-t_3-t_4)/2}}{2N\alpha_s} \cdot \frac{2\Delta b^\Delta J^{-2\Delta}}{\left(2\cosh\frac{t_{12}}{2}\right)^{2\Delta+1}} \cdot \frac{2\Delta b^\Delta J^{-2\Delta}}{\left(2\cosh\frac{t_{34}}{2}\right)^{2\Delta+1}}$$

- Find correction $\delta\varkappa \approx 2\cos\frac{(1-\delta\varkappa)\pi}{2}/\pi$ (Maldacena-Stanford, 2016)

$$\delta\varkappa \approx \frac{C}{\pi N t_B(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}})} = \frac{6\alpha_S}{Jk_{\mathrm{R}}'(-1)\Delta(1-\Delta)(1-2\Delta)\tan(\pi\Delta)}\,.$$

Computational shortcuts II: prefactor

- Large q SYK, fix $\mathcal{J} = \sqrt{2^{1-q} q}\, J$ (Maldacena-Stanford, 2016):

$$\frac{v}{2\cos\frac{\pi v}{2}} = \mathcal{J}, \quad 0 < v < 1\,.$$

- Exact correlation function at all coupling v;

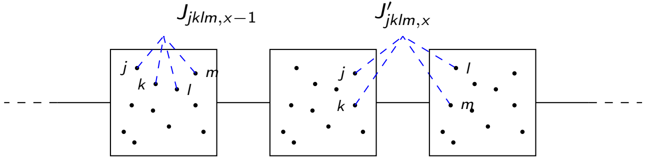
$$K^{\mathrm{R}} = \theta(t_{13})\theta(t_{24}) \frac{v^2}{2\cosh^2\frac{v t_{34}}{2}}, \quad \varkappa = v$$

- Use the identity to find the prefactor (Qi-Streicher, 2018):

$$\mathrm{OTOC}(t_1, t_2; t_3, t_4) \approx \frac{1}{N \cos\frac{v\pi}{2}} \frac{e^{v(t_1+t_2-t_3-t_4)/2}}{\left(2\cosh\frac{v t_{12}}{2}\right)\left(2\cosh\frac{v t_{34}}{2}\right)}\,.$$

Maximal chaos in a 1D model

- Regard $\varkappa$ and C as analytic functions of some parameter, then the analytical properties of $\varkappa$ and C are locked by the ladder identity.
- A concrete example: SYK chain (YG-Qi-Stanford, 2016)



Operators at two different locations

Operators at two different locations:

$$\mathrm{OTOC}_{x,0}(t_1,t_2,t_3,t_4) := \frac{1}{N^2}\sum_{j,k}\langle \chi_{j,x}(t_1)\chi_{k,0}(t_3)\chi_{j,x}(t_2)\chi_{k,0}(t_4)\rangle + G_{12}G_{34}$$

- Fourier transform:

$$\mathrm{OTOC}_{x,0}(t_1,t_2,t_3,t_4) = \int \frac{dp}{2\pi} e^{ipx}\,\mathrm{OTOC}_p(t_1,t_2,t_3,t_4)$$

- Each OTOC_p: ladder diagrams dominate. Retarded kernel factorizes:

$$K^{\mathrm{R}}(p) = s(p)\,K^{\mathrm{R}}, \quad s(p) = 1 - 2a(1-\cos p) \approx 1 - ap^2\,.$$

$s(p)$: "band structure" of the bilocal fields $a = \frac{J_1^2}{3J^2} \in (0, 1/3)$.

Fourier Transform

- The ladder identity holds for each OTOC_p:

$$C(p) = N \cdot 2\cos\frac{\varkappa(p)\pi}{2} \cdot t_B \cdot (\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}}),$$

- The dependence of t_B and $(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}})$ on p is not important (analytic and do not vanish in the domain of interest).

$$\mathrm{OTOC}_{x,0}(t_1, t_2, t_3, t_4) \sim \frac{1}{N} \underbrace{\int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{e^{\varkappa(p)t + ipx}}{2\cos\frac{\pi\varkappa(p)}{2}}}_{u(x,t)} \cdot \frac{\Upsilon^{\mathrm{R}}(t_{12})\Upsilon^{\mathrm{A}}(t_{34})}{t_B(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}})}.$$

$t = \frac{t_1 + t_2 - t_3 - t_4}{2}$. $\varkappa(p) \approx \varkappa(0) - t_B^{-1} a p^2$ when $|p| \ll 1$.

Butterfly wavefront

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{e^{\varkappa(p)t+ipx}}{2\cos\frac{\varkappa(p)}{2}}, \quad \varkappa(p) \approx \varkappa(0) - t_B^{-1} a p^2$$

- Butterfly wavefront: $u(x,t) \sim 1$.
- For large $x > 0$ and t, we can estimate by saddle point of the exponent:

$$\varkappa'(p)t + ix = 0, \quad p = i|p|.$$

- Find a butterfly velocity

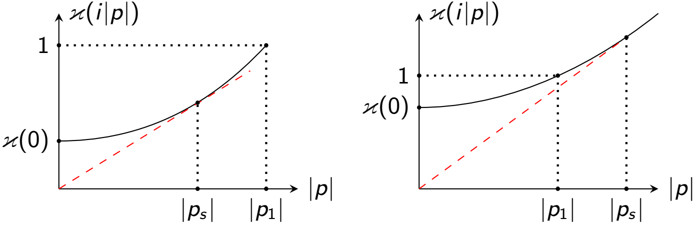
$$v_s = \frac{i\varkappa(p_s)}{p_s} = i\varkappa'(p_s)$$

Graphic solutions: two scenarios

- The relevant saddle point $p_s = i|p_s|$ is purely imaginary. Deform the integral contour to pass, might cross the pole:

$$\cos\frac{\varkappa(p_1)\pi}{2} = 0, \quad \varkappa(p_1) = 1, \quad p_1 = i|p_1|\,.$$

- Two scenarios:

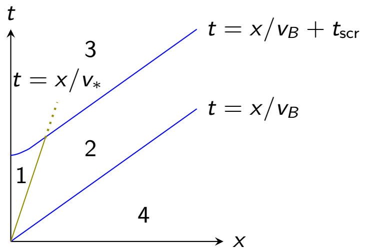


Pole contribution: maximal chaos

- In the second scenario, the pole dominates:

$$u_1(x,t) \approx \frac{e^{t-|p_1||x|}}{\pi i \varkappa'(p_1)}, \quad v_1 = \frac{1}{|p_1|}$$

- In SYK at $J \gg 1$, $\delta\varkappa = 1 - \varkappa(0) \ll 1$, $|p_1| \approx \sqrt{t_B \delta\varkappa / a} \ll |p_s|$, pole dominates. $v_B = v_1$.



Summary and discussion

- An identity relates C and $\varkappa$;

$$N \cdot \frac{2\cos\frac{\varkappa\pi}{2}}{C} \cdot t_B \cdot \left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}}\right) = 1\,, \quad t_B = k'_{\mathrm{R}}(-\varkappa)$$

  The derivation seems general, may work for other models.

- Applications:
  - computational shortcuts, $\delta\varkappa \propto t_B^{-1}$
  - maximal chaos
    Why?