

Slow equilibration and Fluctuation Relations

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Quasi-integrable systems: four famous examples

- **Planetary systems**
- **Weakly nonlinear waves**
- **The Fermi-Pasta-Ulam chain**
- **Quantum chains**

1. Planetary systems



→ Kepler (Laplace-Lagrange) (integrable) + perturbation see Laskar

Very short Lyapunov times. Unstable? Moser

Mercury	1.4M
Venus	7.2 M
Earth	4.8M
Mars	4.5M
Jupiter	8.4 M
Saturn	6.4M
Uranus	7.5M
Neptune	6.7M

with some grains of salt (what is an individual Lyapunov?)

Diffusion of the eccentricity of Mercury, slightly different runs

Laskar

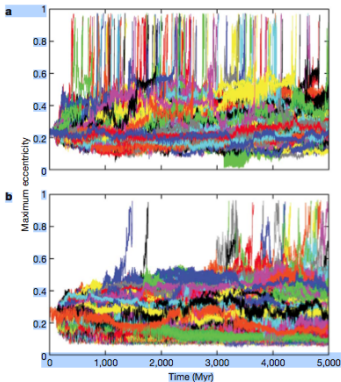
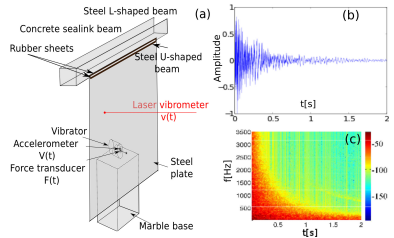


Figure 1 | Mercury's eccentricity over 5 Gyr. Evolution of the maximum

n.b. a thousand Lyapunov times

2. Weakly interacting waves

Zakharov, Nazarenko...



Düring, Rica, Josserand,...

→ Plane waves (integrable) + perturbation

3. The Fermi-Pasta-Ulam chain

$$H(p, q) = \frac{1}{2} \sum_{i=1}^N V(q_{i+1} - q_i)$$

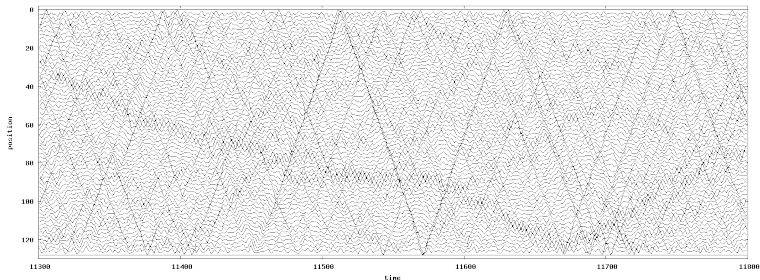
$$V(r) = \frac{r^2}{2} + \alpha \frac{r^3}{3} + \beta \frac{r^4}{4}$$

$$V(r) = \frac{r^2}{2} + \alpha \frac{r^3}{3} + \beta \frac{r^4}{4} = \underbrace{V_0(e^{\lambda r} - 1 - \lambda r)} + \text{'small'}$$

→ Toda (integrable) + perturbation (Benettin and Ponno)

The Fermi-Pasta-Ulam chain

Trajectories



4. Quantum chains

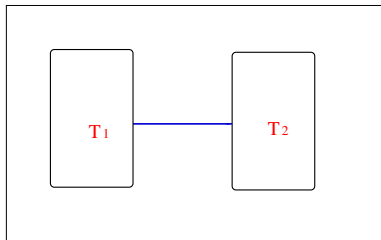
$$H(J_2, \delta, U) = -J_1 \sum_{l=1}^L [1 + (-1)^l \delta] (c_l^\dagger c_{l+1} + \text{H.c.}) \\ - J_2 \sum_{l=1}^L [c_l^\dagger c_{l+2} + \text{H.c.}] + U \sum_{l=1}^L n_l n_{l+1}.$$

Bertini, Essler, Groha, Robinson; DeLuca and Mussardo; ...



Various integrable limits

Plus the simplest example:



$$P \propto e^{-\beta_1 E_1 - \beta_2 E_2} = e^{-\beta_+ E_+ - \beta_- E_-}$$

E_+ is conserved, E_- is slowly evolving

$$2\beta_+ = 1/T_1 + 1/T_2 \quad \mathbf{and} \quad 2\beta_- = 1/T_1 - 1/T_2$$

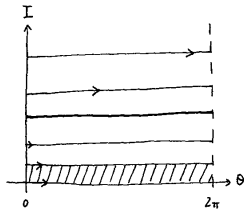
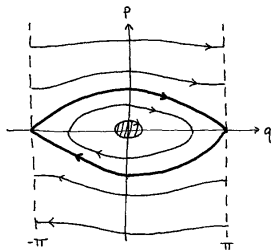
clearly, temperatures associated with nonconserved quantities go to infinity

Integrable systems

N independent constants of motion

Action (I_1, \dots, I_N) **and Angle** $(\theta_1, \dots, \theta_N)$ **variables**

Flow is *laminar*, restricted to tori $I_i = \text{const}, \theta_i = \omega_i t$



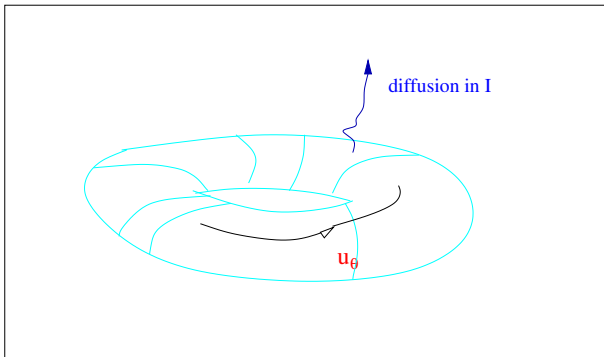
Action (I_1, \dots, I_N) and Angle ($\theta_1, \dots, \theta_N$) variables

Action-angle representation

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}\end{aligned}$$

$$\begin{aligned}\dot{I}_i &= -\frac{\partial H}{\partial \theta_i} \\ \dot{\theta}_i &= \frac{\partial H}{\partial I_i} = \omega_i(\mathbf{I})\end{aligned}$$

Small perturbation:



'stability' = lack of ergodicity

We perturb with weak, additive noise

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + \varepsilon^{\frac{1}{2}} \xi_i(t)\end{aligned}$$

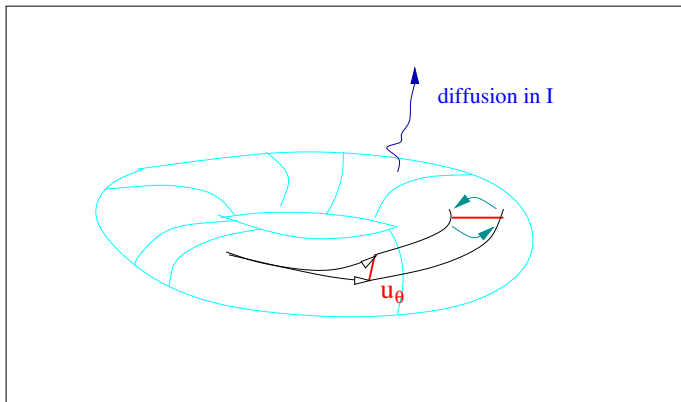
consider the simple case in which the $\xi(t)$ are white noises:

$$\langle \xi(t) \rangle = 0, \quad \text{and} \quad \langle \xi(t) \xi(t') \rangle = 2\delta(t - t').$$

In the action-angle variables, the noise is no longer additive, and reads:

$$\dot{I}_i = \varepsilon^{\frac{1}{2}} \sum_k \{I_i, q_k\} \xi_k(t)$$
$$\dot{\theta}_i = \omega_i + \varepsilon^{\frac{1}{2}} \sum_k \{\theta_i, q_k\} \xi_k(t)$$

Lyapunov time is puzzlingly short $\sim \epsilon^{-\frac{1}{3}}$



No KAM threshold! mostly chaos on the torus

**We concentrate for a
while on FPU**

Constants of motion of FPU: differences of eigenvalues of

$$L = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots & a_N \\ a_1 & b_2 & a_2 & 0 & & \\ 0 & a_2 & b_3 & a_3 & & \\ 0 & 0 & a_3 & b_4 & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ a_N & & & & & b_N \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & a_1 & 0 & 0 & \dots & -a_N \\ -a_1 & 0 & a_2 & 0 & \dots & 0 \\ 0 & -a_2 & 0 & a_3 & \dots & 0 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & \cdot \\ a_N & & & & & 0 \end{pmatrix}.$$

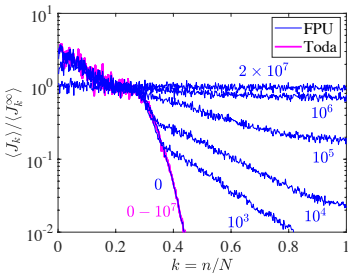
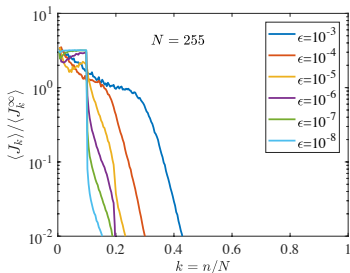
A simple computation shows that

$$\dot{L} = [B, L] = BL - LB.$$

$$a_n = \frac{1}{2} e^{-(Q_n - Q_{n-1})/2},$$

$$b_n = -\frac{1}{2} P_{n-1}.$$

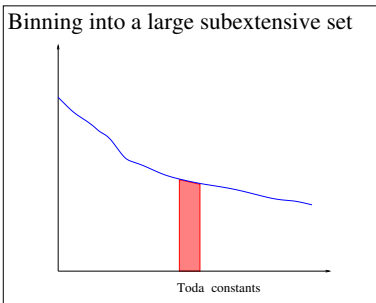
$$H = \sum_{n=1}^n \left(\frac{1}{2} P_n^2 + e^{-(Q_n - Q_{n-1})} \right),$$



Thermalization of FPU analogous to Benettin and Ponno

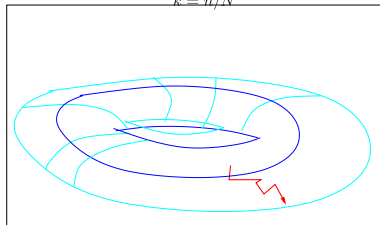
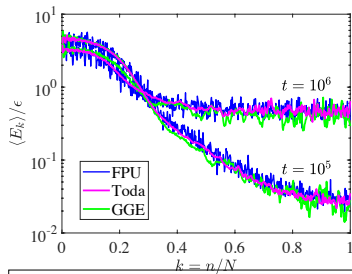
Evolution of Toda constants of motion J_1, \dots, J_N

$$P \propto e^{-\sum \beta_q J_q}$$



analogous to hydrodynamic limit

Coarse-graining is an optional, but the only way to get non-fluctuating quantities, self-averaging and equivalence of ensembles



From Toda torus to Toda torus

Evolution of Toda constants of motion J_1, \dots, J_N

An algorithm:

- Compute time derivative of binned quasi-constants of motion ‘within a torus’
- Evolve the quasi-constants linearly
- Find a point with new constants (GGE or ‘generalized microcanonical’) and iterate

Works with FPU.

What about planets? And quantum?

A numerical version of wave turbulence...

Diffusion of the eccentricity of Mercury, slightly different runs

Laskar

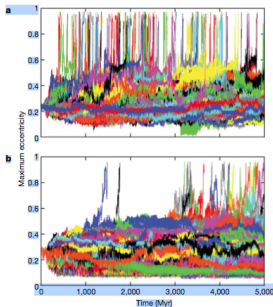


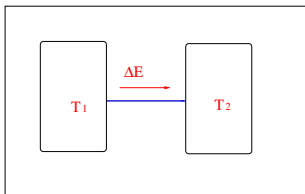
Figure 1 | Mercury's eccentricity over 5 Gyr. Evolution of the maximum

Is it possible to apply the numerical strategy for this case, computing at each level rate of change and diffusion constants of orbital parameters?

Here we do not have determinism!

A fluctuation theorem

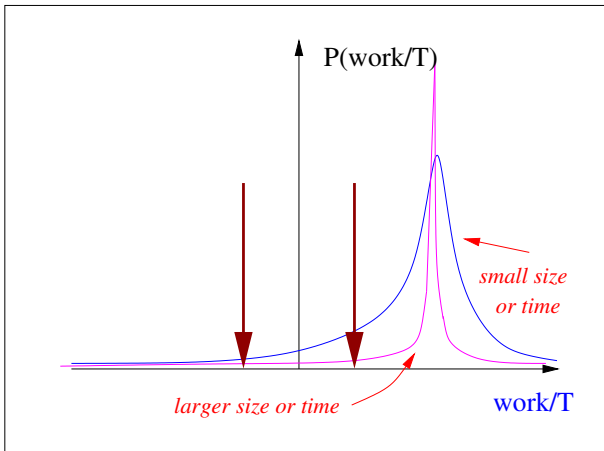
A conduction problem, we expect a fluctuation theorem



of the type $P(\text{heat})/P(-\text{heat}) = e^{(\beta_1 - \beta_2) \text{heat}}$

Indeed: see Jarzynski and Wojcik

$$\frac{P(\Delta E_-)}{P(-\Delta E_-)} = e^{\beta_- \Delta E_-}$$



With the analogy with quasi-constants of motion, we are led to the quantum and classical more general result

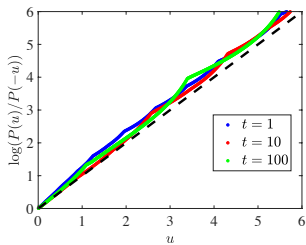
T Goldfriend and JK, see Hickey and Genway also J. Mur-Petit, A. Relao, R. A. Molina, D. Jaksch

$$\frac{P(\Delta Q)}{P(-\Delta Q)} = e^Q$$

$$Q = \sum_r \beta_r J_r$$

which describes change of **charges**

and violations of irreversibility as the system thermalizes (the β_r
going to infinity).



FPU chain, Toda constants, $N = 15$

Usually, fluctuation theorems are limited to very small systems and/or very short times.

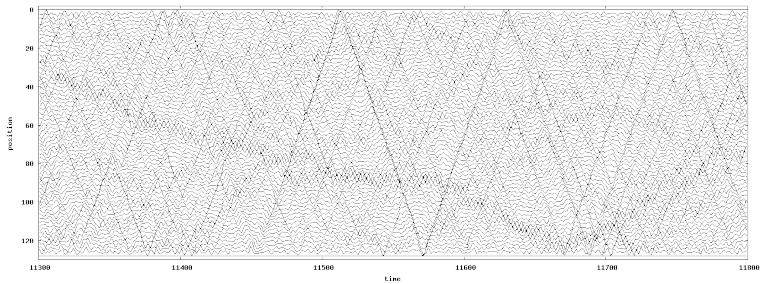
Here the situation is different: the 'size' parameter is the integrability violation, so that even a macroscopic systems may exhibit 'reversals'

For example, it applies to a large FPU chain with a small number of decaying solitons...

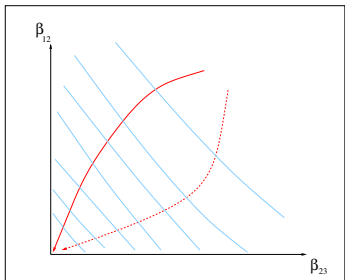
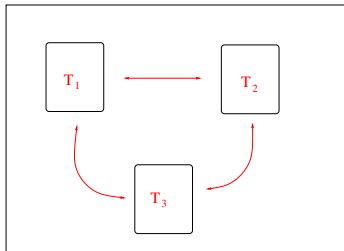
What is the connection with the dynamical effective temperatures? Cugliandolo, Gambassi, Foini

and Konik

Trajectories

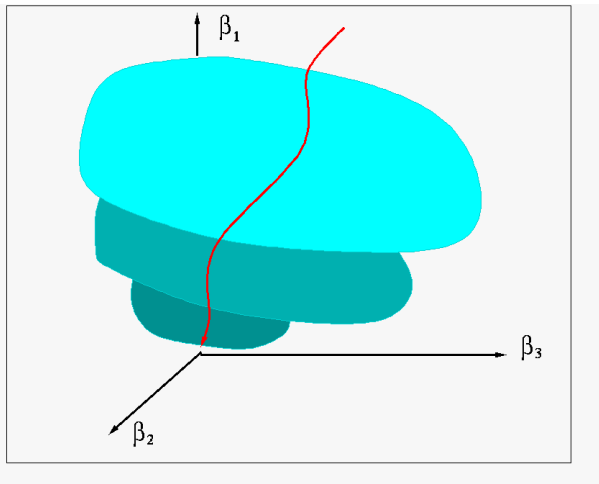


The next-to-simplest GGE:



downhill in constrained entropy (free energy) but (unfortunately) not necessarily along a gradient

Skiing in β space downhill in $-S$:



(Prob downhill) / (Prob uphill) \rightarrow fluctuation theorem