# Spectrally condensed turbulence in two dimensions

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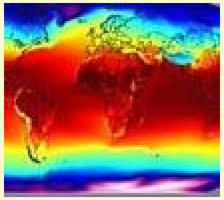


# Motivation

Turbulence often coexists with coherent flow

2D turbulence is capable of generating such flows

spectral condensation, crystallization

Turbulence-condensate interplay – dynamical steady-state

energy transfer from turbulence to flows

effects of shear flows on turbulence

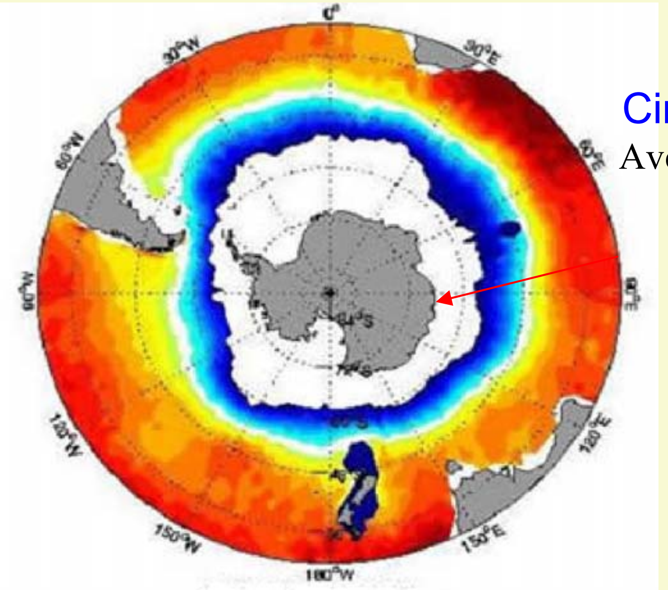
Practical applications

atmospheric and oceanic processes,

magnetically confined plasma, etc.

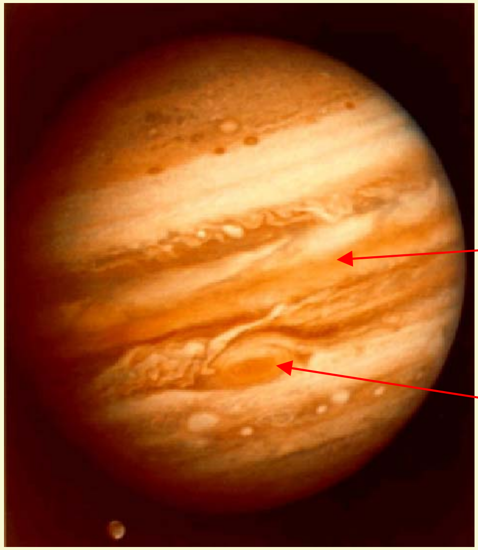
# 2D turbulence and spectral condensation

# Large coherent flows coexist with turbulence



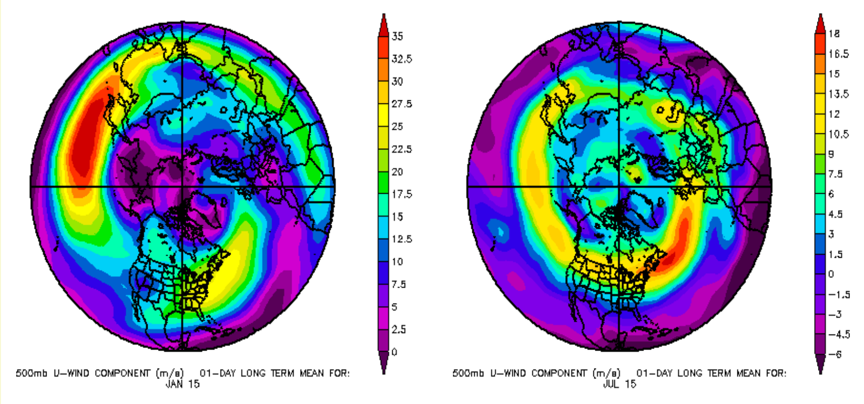
Antarctic
Circumpolar Current

Average volume transport
$\sim 1.5\times10^{8}\ \mathrm{m^3 s^{-1}}$



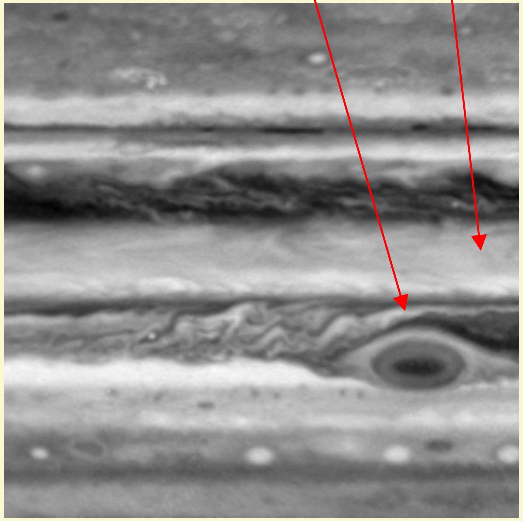
Jupiter's
zonal flows

GRS




Earth: atmospheric zonal winds

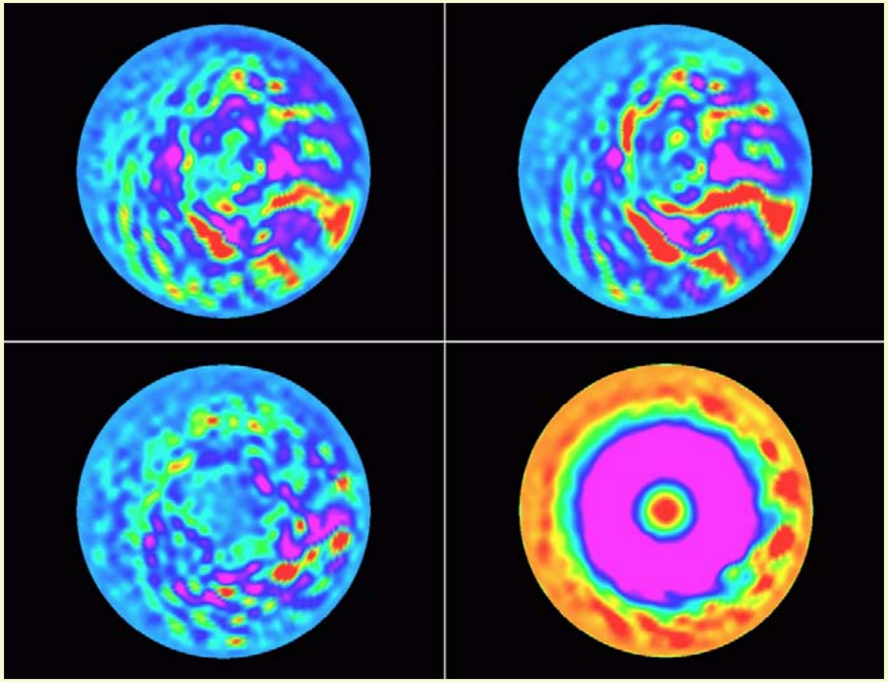
**Planetary atmospheres are dominated by turbulent structures (cyclones, zonal winds, *etc*)**



Cassini spacecraft

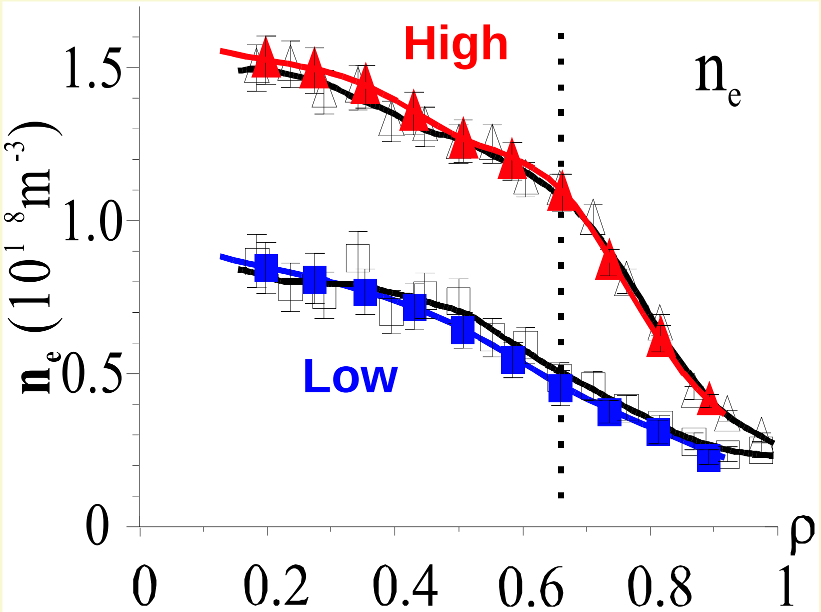
*Courtesy NASA*

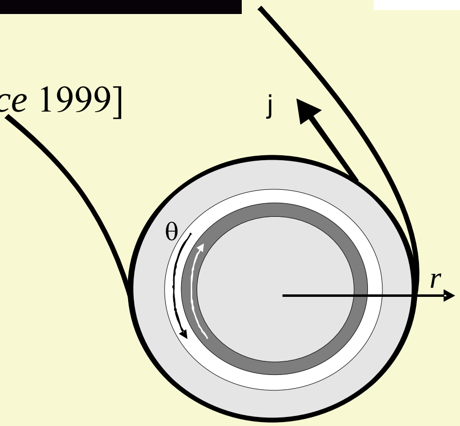
# Turbulence-driven structures in fusion plasma



Zonal flows in plasma

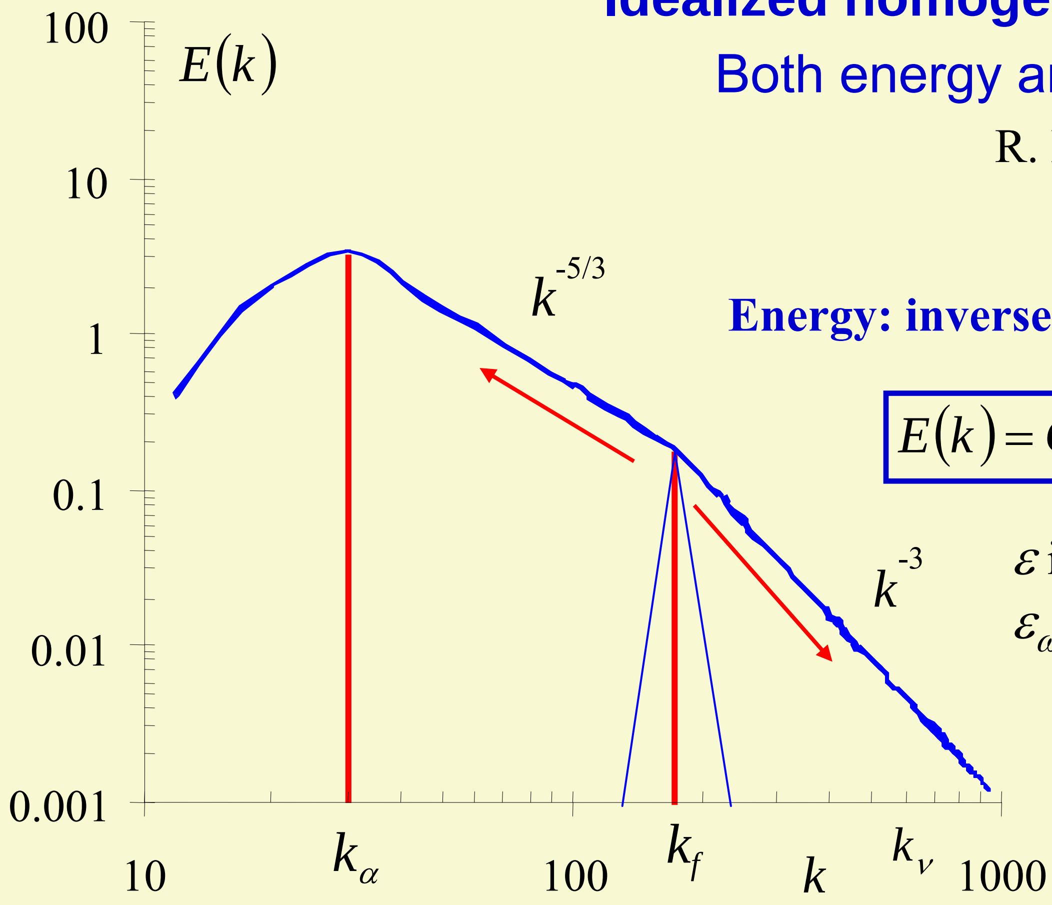
Numerical simulations [Z. Lin, *Science* 1999]




**In magnetically confined plasma, turbulence-driven anisotropic flows develop, which inhibit radial transport of particles and energy**

# 2D turbulence




**Idealized homogeneous isotropic 2D turbulence**

Both energy and enstrophy are conserved

R. Kraichnan (1967)

**Dual cascade:**

**Energy: inverse cascade, Enstrophy: forward cascade**

$$E(k) = C_k \varepsilon^{2/3} k^{-5/3}$$

$$E(k) = C_\omega \varepsilon_\omega^{2/3} k^{-3}$$

$\varepsilon$ is the energy dissipation rate

$\varepsilon_\omega$ is the enstrophy dissipation rate

$$\Omega = \frac{1}{2}\int_V |\boldsymbol{\omega}|^2 dV$$

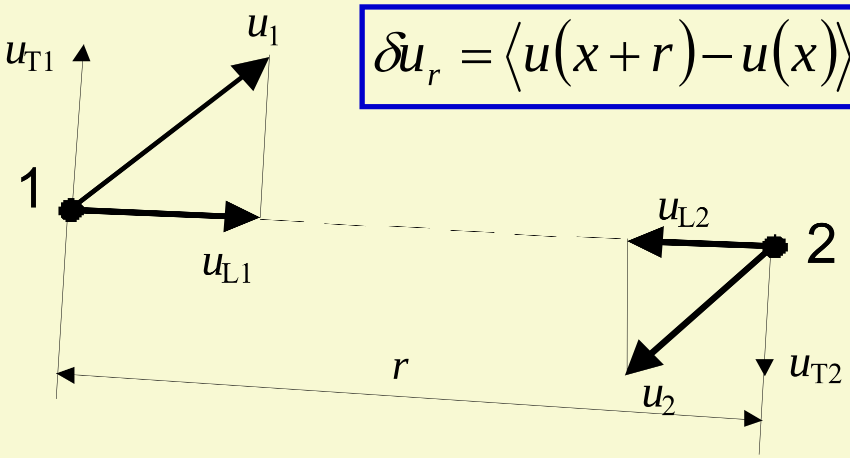
enstrophy

**Opposite to 3D, energy flows from smaller to larger scales**

**Basis for self-organization**

# Structure functions and Kolmogorov law

Label an 'eddy' by a velocity increment $\delta u_l$ across a distance $r$:




$$\delta u_r = \langle u(x+r) - u(x) \rangle$$

Statistical moments of this increment are called *structure functions* of the n[th] order:

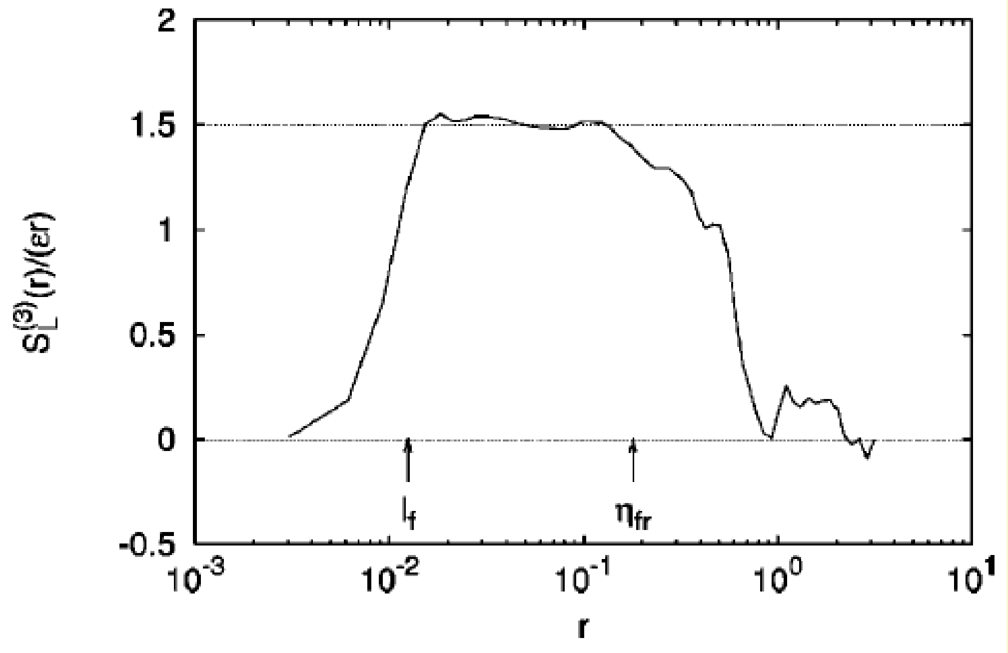
$$S_n(r) = \langle (\delta u_r)^n \rangle = \langle (u(x+r) - u(x))^n \rangle$$

## Kolmogorov law

relates the third-order longitudinal structure function of turbulence to the mean energy dissipation per unit mass $\varepsilon$

**in 2D** (e.g. [Lindborg 1999]):

$$S_{3L}(r) = \langle \delta V_L^3(r) \rangle = \frac{3}{2} \varepsilon r$$




[G. Boffetta, A. Celani, M. Vergassola, 2000]

# Spectral condensation of 2D turbulence

The maximum of the energy spectrum lies in the low-*k* range, at $k_\alpha$, in the absence of the energy dissipation at large scales $k_\alpha$ can not be constant in time since it accumulates spectral energy

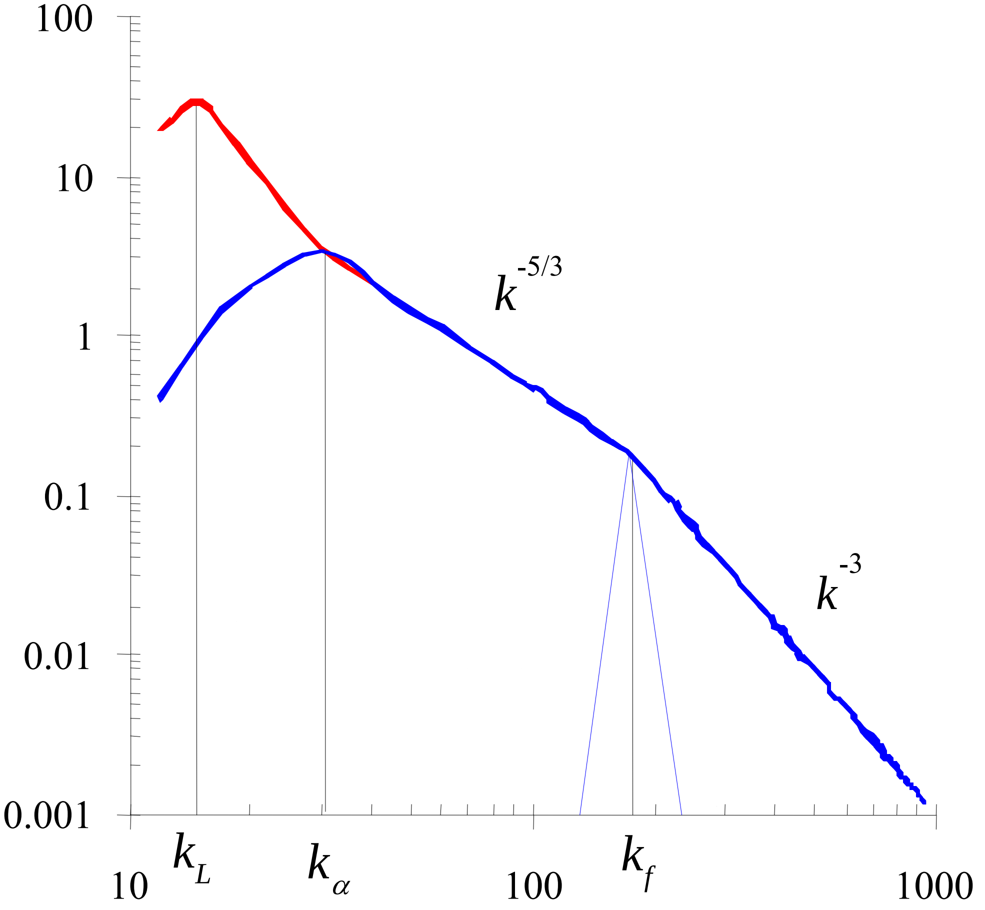
$$k_\alpha = f(\varepsilon, t)$$

Dissipation at large scales (bottom damping) $\alpha$ stabilizes the maximum of the spectrum at the scale

$$k_\alpha \approx \left(\alpha^3 / \varepsilon\right)^{1/2}$$

Kraichnan, 1967: predicted condensate

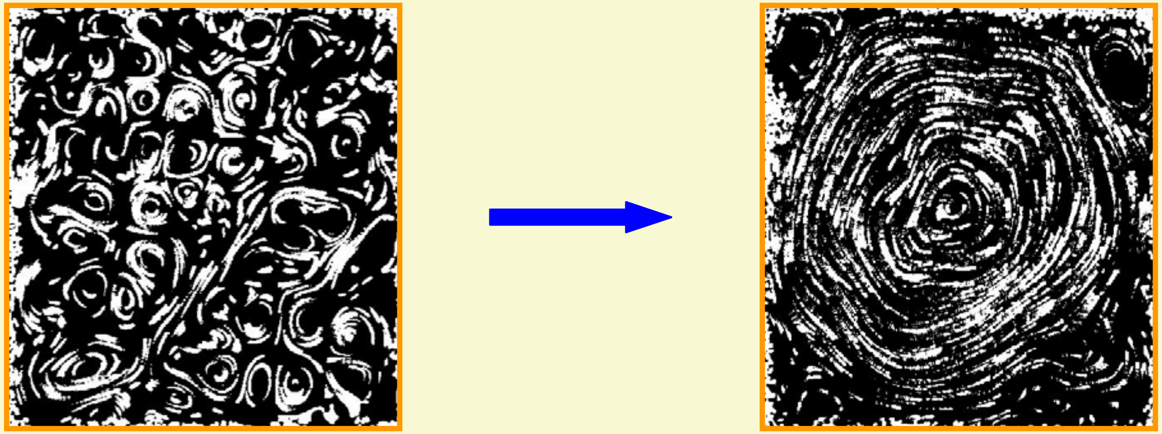
System size < dissipation scale




At low dissipation in a bounded system, at $k_\alpha << k_L$ spectral energy accumulates in a box-size coherent structure

# Spectral condensation of turbulence in thin layers

Experiments: Sommeria (1986), Paret &Tabeling (1998), Shats et al (2005, 2007)



Time evolution to condensed state

[Shats *et al* (2005)]

Numerical simulations of 2D turbulence:

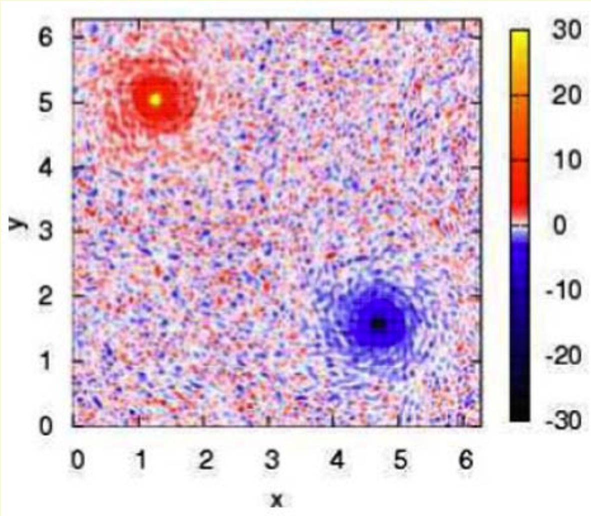
Hossain (1983), Smith &Yakhot (1993)…

van Heijst, Clercx, Molenaar (2004-2006),

Chertkov et al. (2007)

Periodic boundary condition – dipole

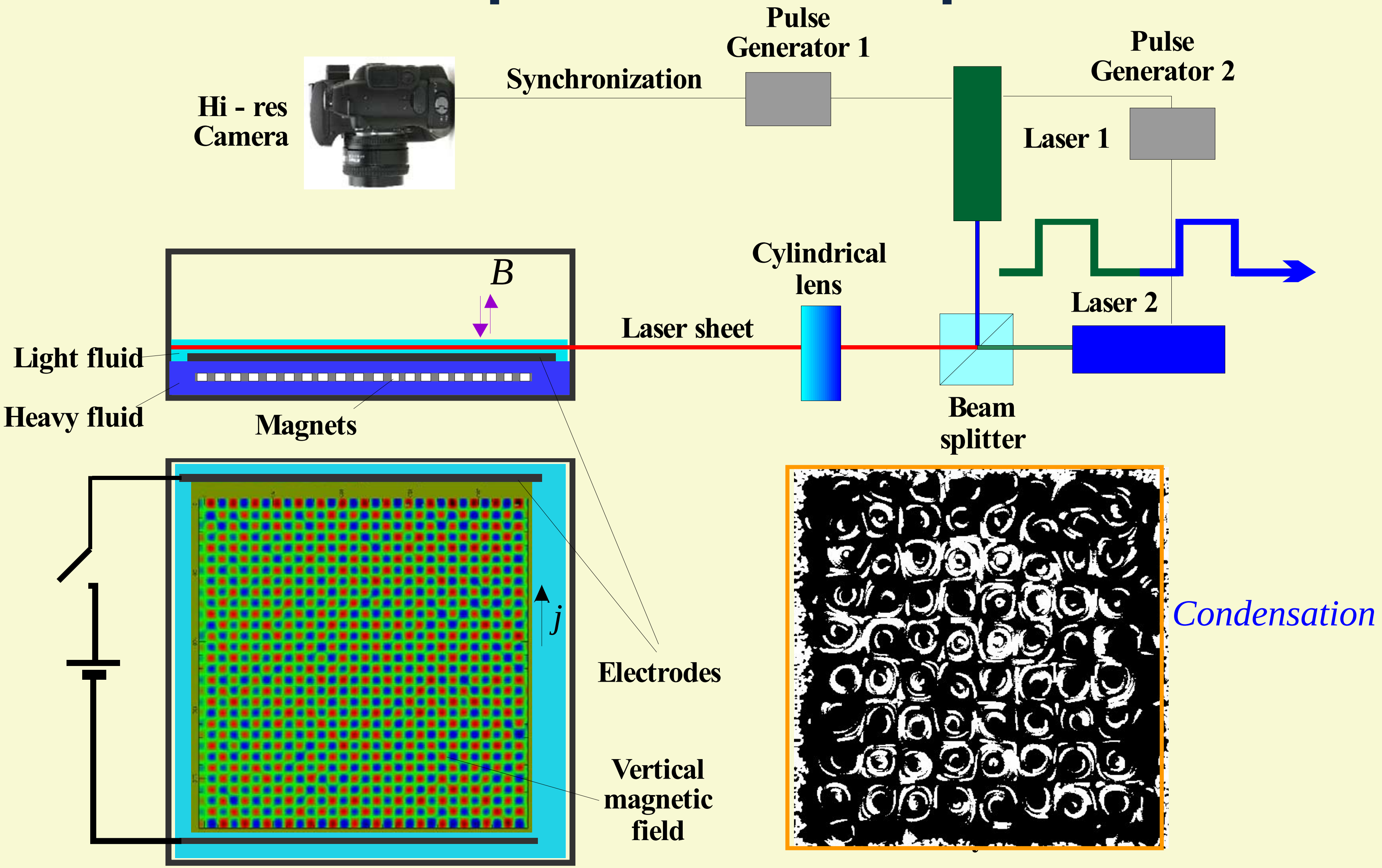
No-slip boundary – single vortex




Vorticity of the condensate

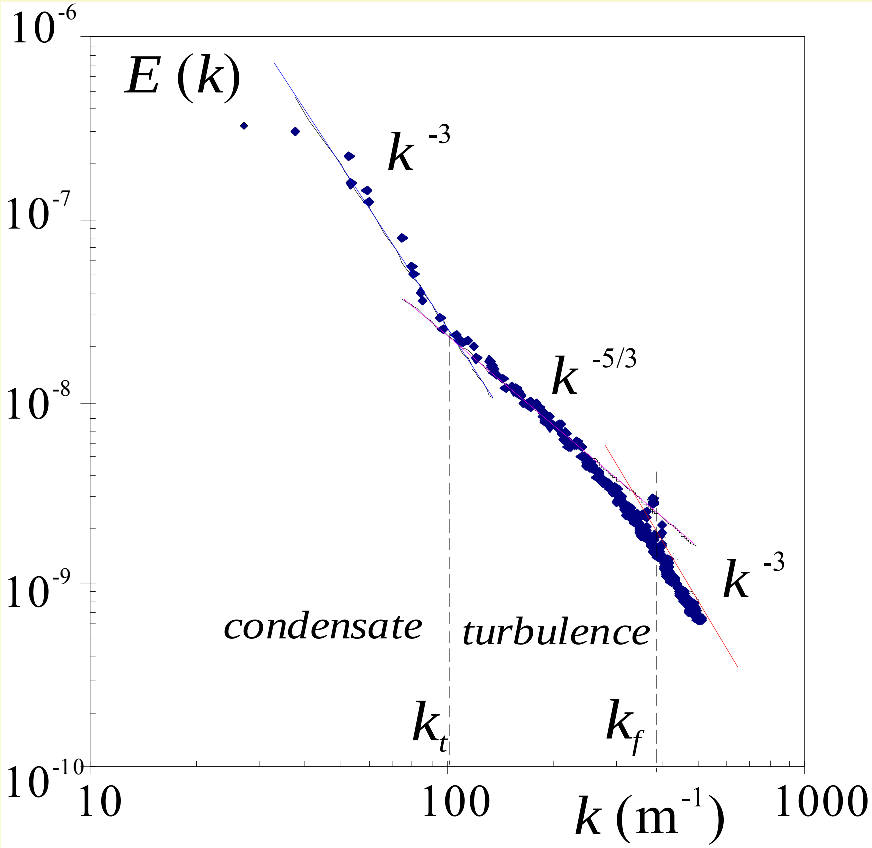
[M. Chertkov et al. (2007)]

# Experimental setup




- Bottom layer: isolator Fluorinert FC-77 (resist. = $2 \times 10^{15}$ Ohm cm; SG = 1.78)
- Top layer: electrolyte NaCl solution (SG = 1.04)

# Condensed turbulence spectrum is robust




$k^{-(3\text{-}4)}$ at large scales

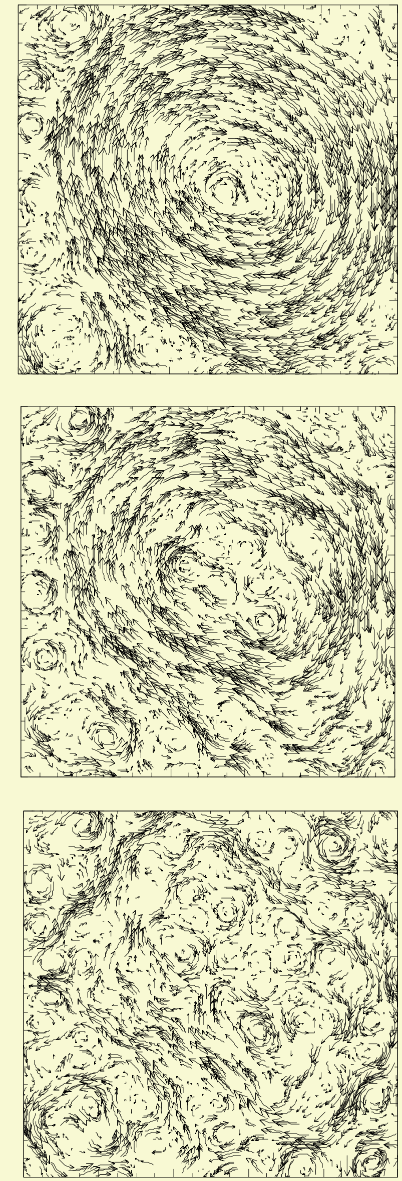
$k^{-5/3}$ at meso-scales

$k^{-(3\text{-}4)}$ at small scales

*-Bottom drag,*
*-boundary size,*
*-forcing*
*affect condensate strength and topology*

$L$ = 0.1 m

Time-averaged



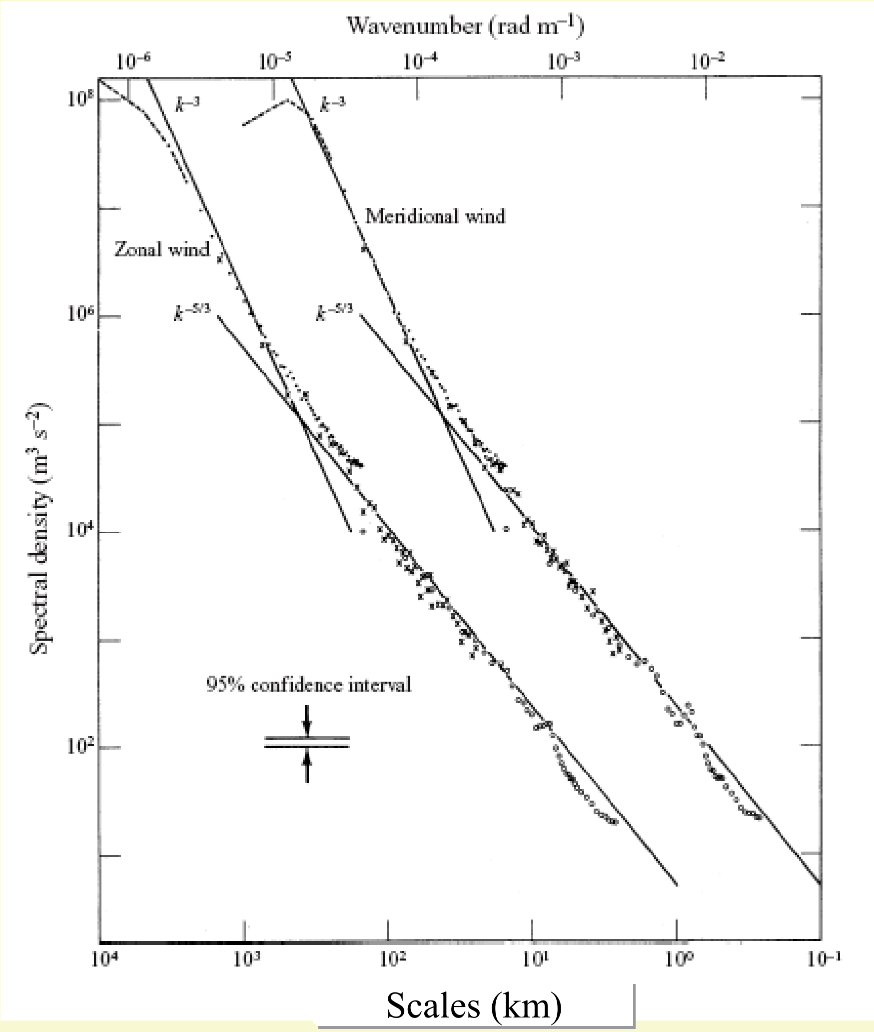
strong

weak

# Nastrom-Gage spectrum of atmospheric winds

**Atmospheric spectrum**

[Nastrom, Gage, Jasperson, Nature (1984) ]




$k^{-3}$ and $k^{-5/3}$ ranges are present but in the reversed order compared to the Kraichnan theory

$$E(k) = C_k \varepsilon^{2/3} k^{-5/3} \quad at\ k < k_f$$

$$E(k) = C_\omega \varepsilon_\omega^{2/3} k^{-3} \quad at\ k > k_f$$

*What is the origin of*

*$k^{-3}$ and $k^{-5/3}$ ranges in atmosphere?*

*Meso-scale $k^{-5/3}$ range can be due to*

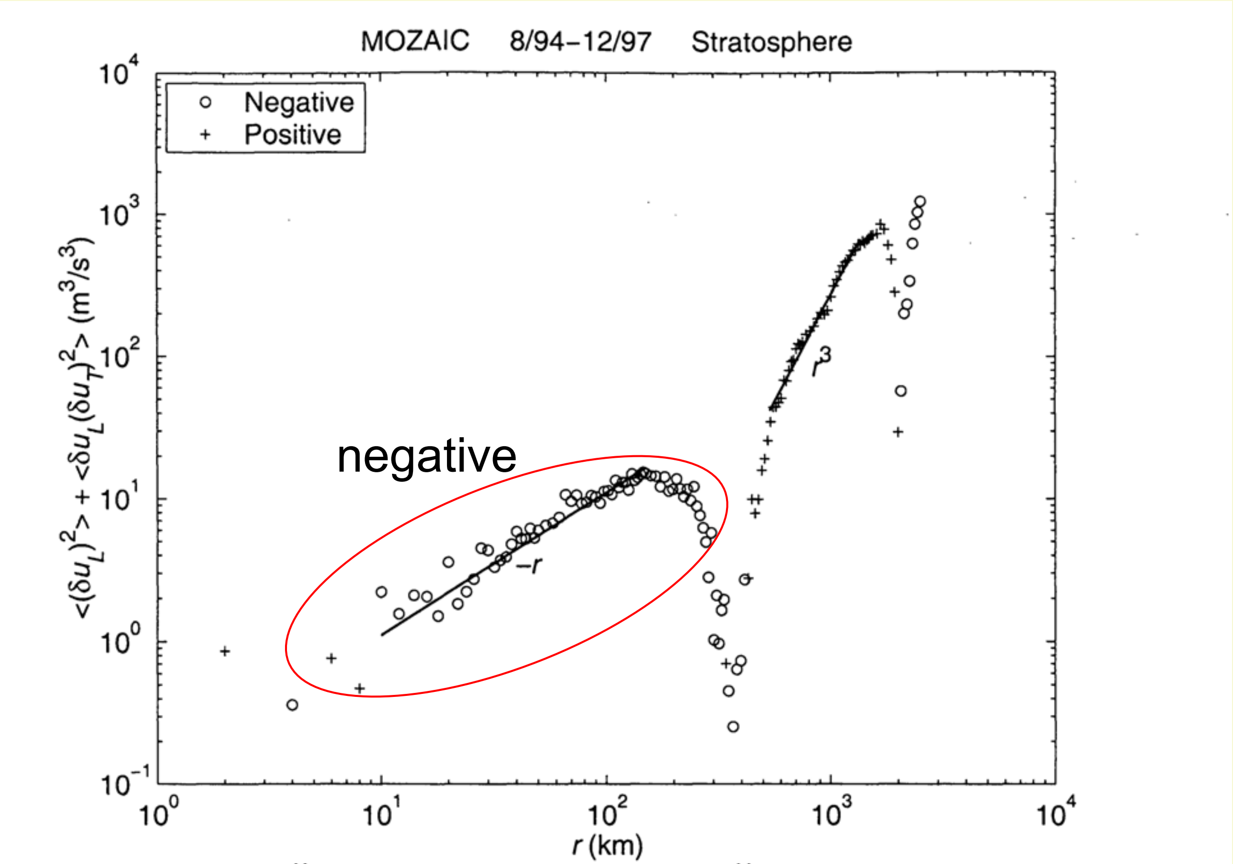
- *3D (downscale) direct energy cascade,*
- *2D inverse (upscale) cascade*

*Large-scale $k^{-3}$ range can be due to*

- *direct enstrophy cascade (large-scale forcing)*
- *spectral condensation*

**Kinetic energy spectrum alone cannot resolve the question of the sources**

# Energy flux in atmospheric turbulence




Circles and crosses indicate negative and positive values

Third-order velocity moment gives the energy flux direction

$$S_3(r) = \frac{3}{2}\varepsilon r$$

Negative $S_3$ at scales up to 500 km interpreted as evidence against inverse energy cascade in the mesoscale range

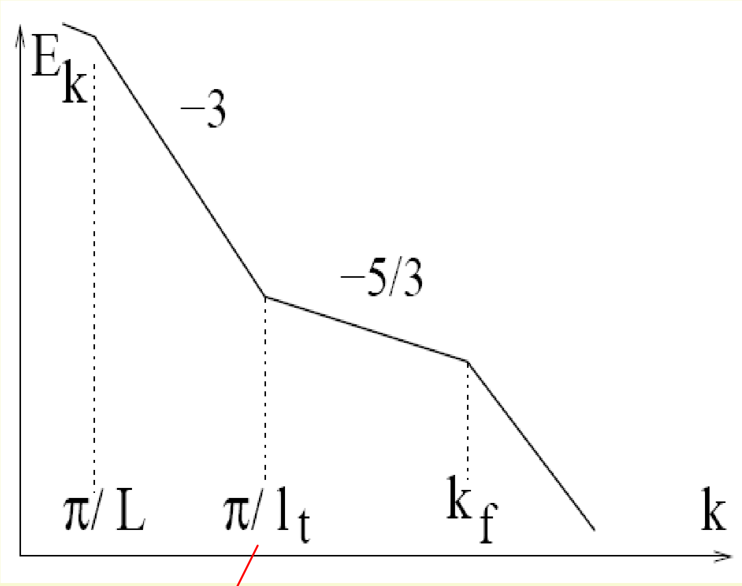
[Cho, Lindborg, J. Geophys. Res. (2001) ]

**Need to understand spectral flux in the presence of large coherent flow, which may affect higher moments**

**Condensate – coherent flow – self-generated by turbulence**

**In the lab can control strength and spectral extent of condensate (?)**

# Model of the spectrum




In the inverse cascade, the turnover time of the eddy of scale $l$ is $t_l = l/\sqrt{S_2} \approx l^{2/3} C^{-1/2} \varepsilon^{-1/3}$

1. Assume that the condensate (vortex) appears when the system size $L$ is such that $t_L\, \alpha < 1$

2. Characterize the condensate amplitude by its mean velocity $V$.

This velocity can be estimated from the energy balance, $\alpha V^2 \cong 2\varepsilon$ which gives $V \cong \sqrt{2\varepsilon/\alpha}$

3. We estimate that the condensate related velocity fluctuation on the scale $l$ as $Vl/L$. Then we expect the knee of the spectrum to be at the scale $l_t$ defined by

$$V l_t / L \cong C^{1/2} (\varepsilon l_t)^{1/3}$$

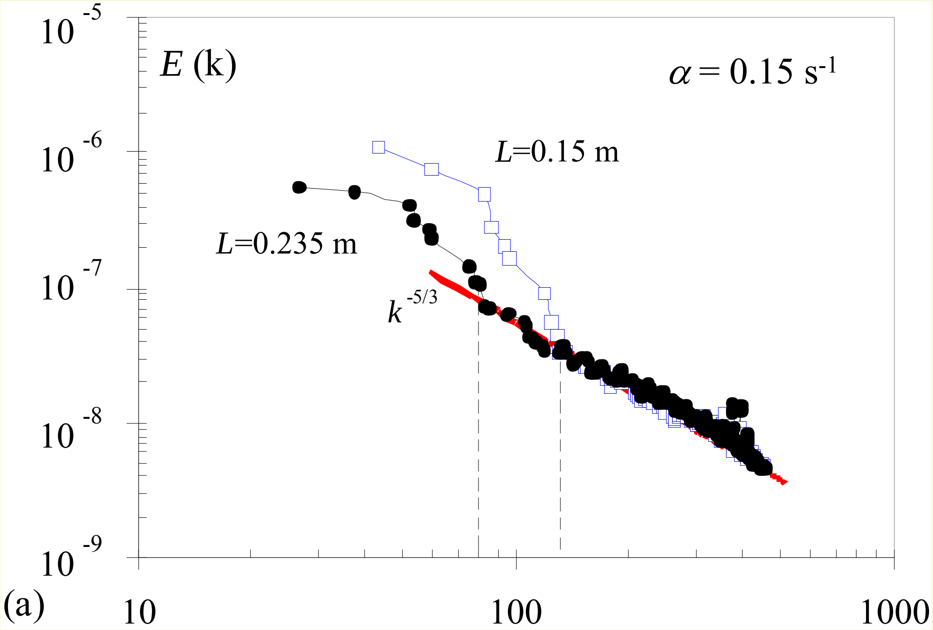
This gives

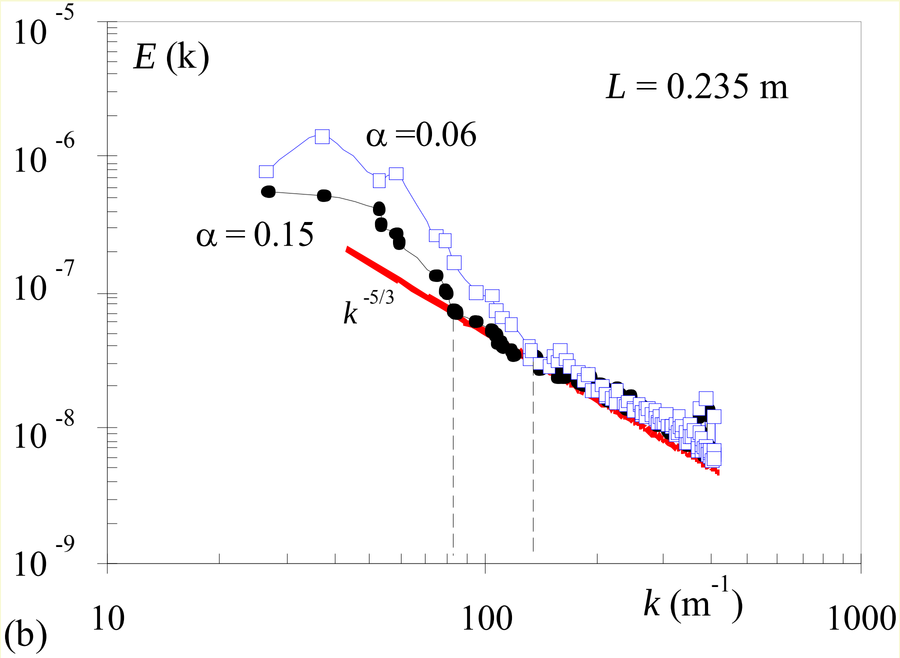
$$l_t \approx L^{3/2} (C\alpha/2)^{3/4} \varepsilon^{-1/4}$$

$$k_t = \frac{\pi}{l_t} \approx \pi L^{-3/2} (C\alpha/2)^{-3/4} \varepsilon^{1/4}$$

# Knee of the spectrum shifts with $\alpha$ and $L$

$$k_t = \frac{\pi}{l_t} \approx \pi L^{-3/2} \left( C\alpha / 2 \right)^{-3/4} \varepsilon^{1/4}$$



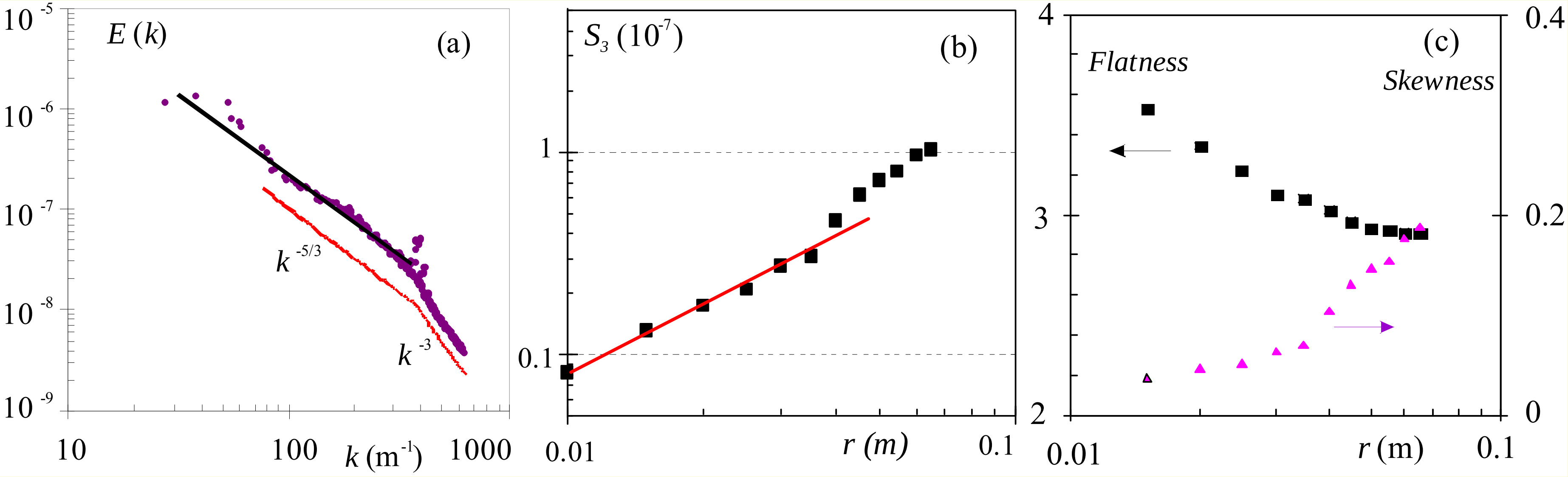

$k_t$ increases with the decrease in the boundary size $L$

$k_t$ increases with the decrease in the damping rate $\alpha$

**In the lab we can control strength, spectral extent of the condensate**

# Case of weak condensate

$$E(k) = C_k \varepsilon^{2/3} k^{-5/3} \qquad S_3(r) = \left(\langle \delta V_L^3 \rangle + \langle \delta V_L \delta V_T^2 \rangle\right)/2 = \varepsilon r \qquad Sk = S_3/(S_2)^{3/2}, \quad F = S_4/S_2^2$$



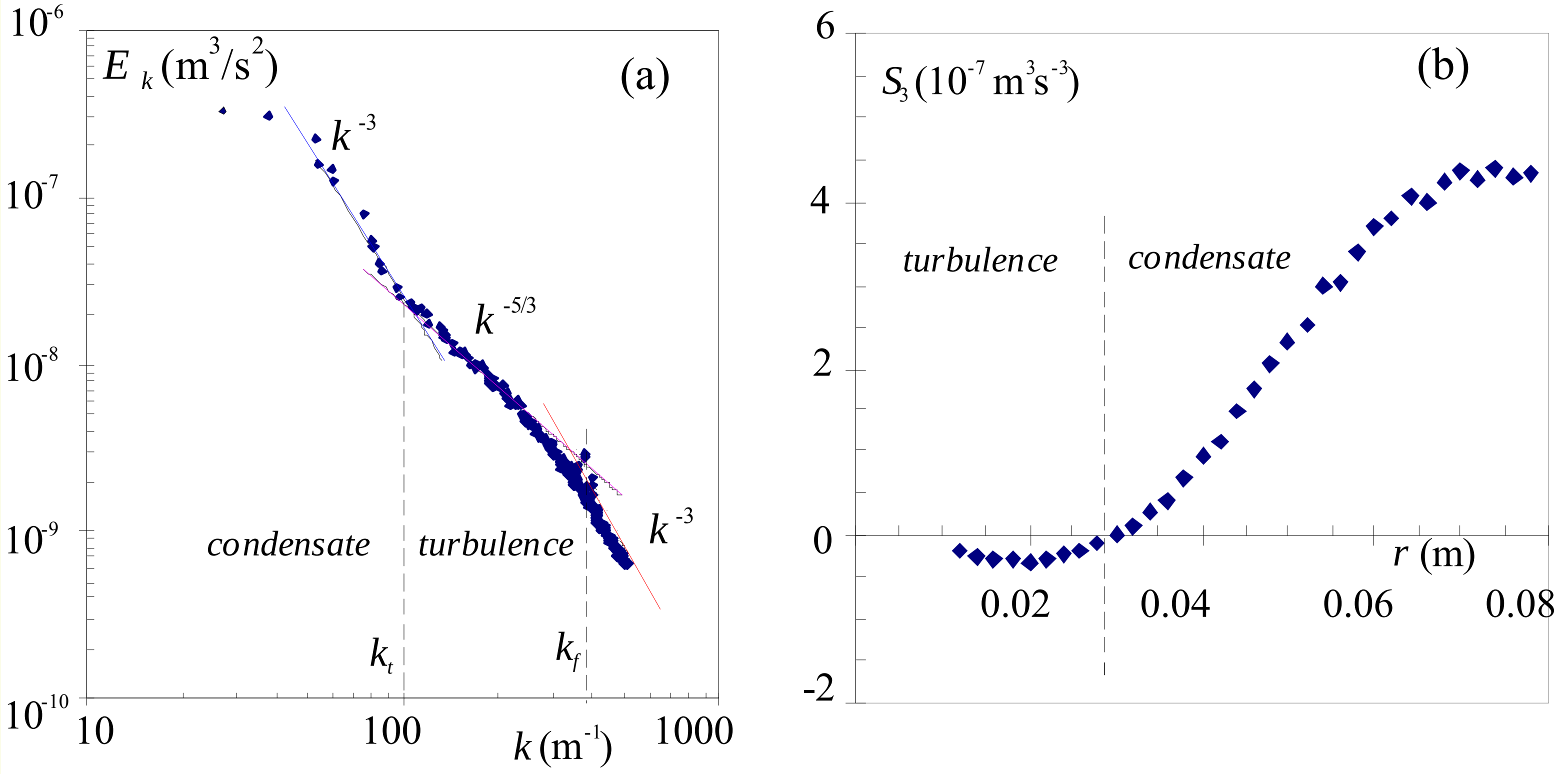

Weak condensate case shows small differences with isotropic 2D turbulence

~ $k^{-5/3}$ spectrum in the energy range

Kolmogorov law – linear $S_3$ (r) dependence; Kolmogorov constant C ≈ 5.6

Skewness and flatness are close to their Gaussian values (*Sk* = 0, *F* = 3)

# Case of stronger condensate




$$\delta V = \delta\bar{V} + \delta\tilde{V}$$

Mean shear flow (condensate) $\delta\bar{V}$

changes all velocity moments:

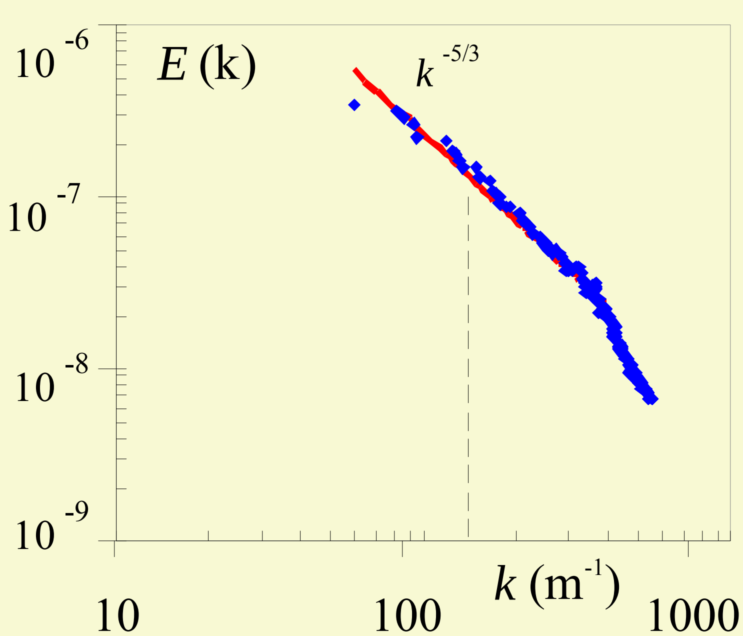
$$\left\langle \delta V^2 \right\rangle = \left\langle \delta\bar{V}^2 + 2\delta\bar{V}\delta\tilde{V} + \delta\tilde{V}^2 \right\rangle$$

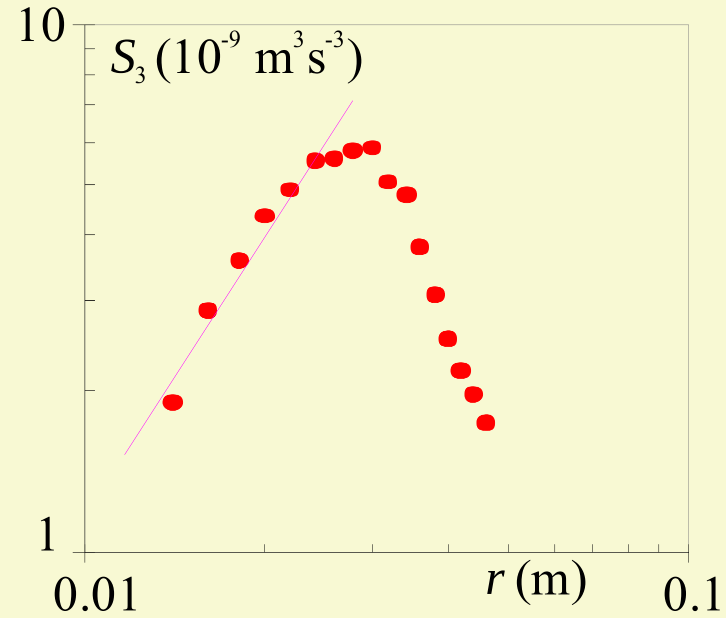
$$\left\langle \delta V^3 \right\rangle = \left\langle \delta\bar{V}^3 - 3\delta\bar{V}^2\delta\tilde{V} + 3\delta\bar{V}\delta\tilde{V}^2 - \delta\tilde{V}^3 \right\rangle$$

# Mean subtraction recovers isotropic turbulence

1.Compute time-average velocity field (*N*=400): $\overline{V}(x,y) = 1/N \sum_{n=1}^{N} V(x,y,t_n)$

2. Subtract $\overline{V}(x,y)$ from *N*=400 instantaneous velocity fields




Recover ~ $k^{-5/3}$ spectrum in the energy range

$S_3$ (r) is positive – recovered inverse energy cascade

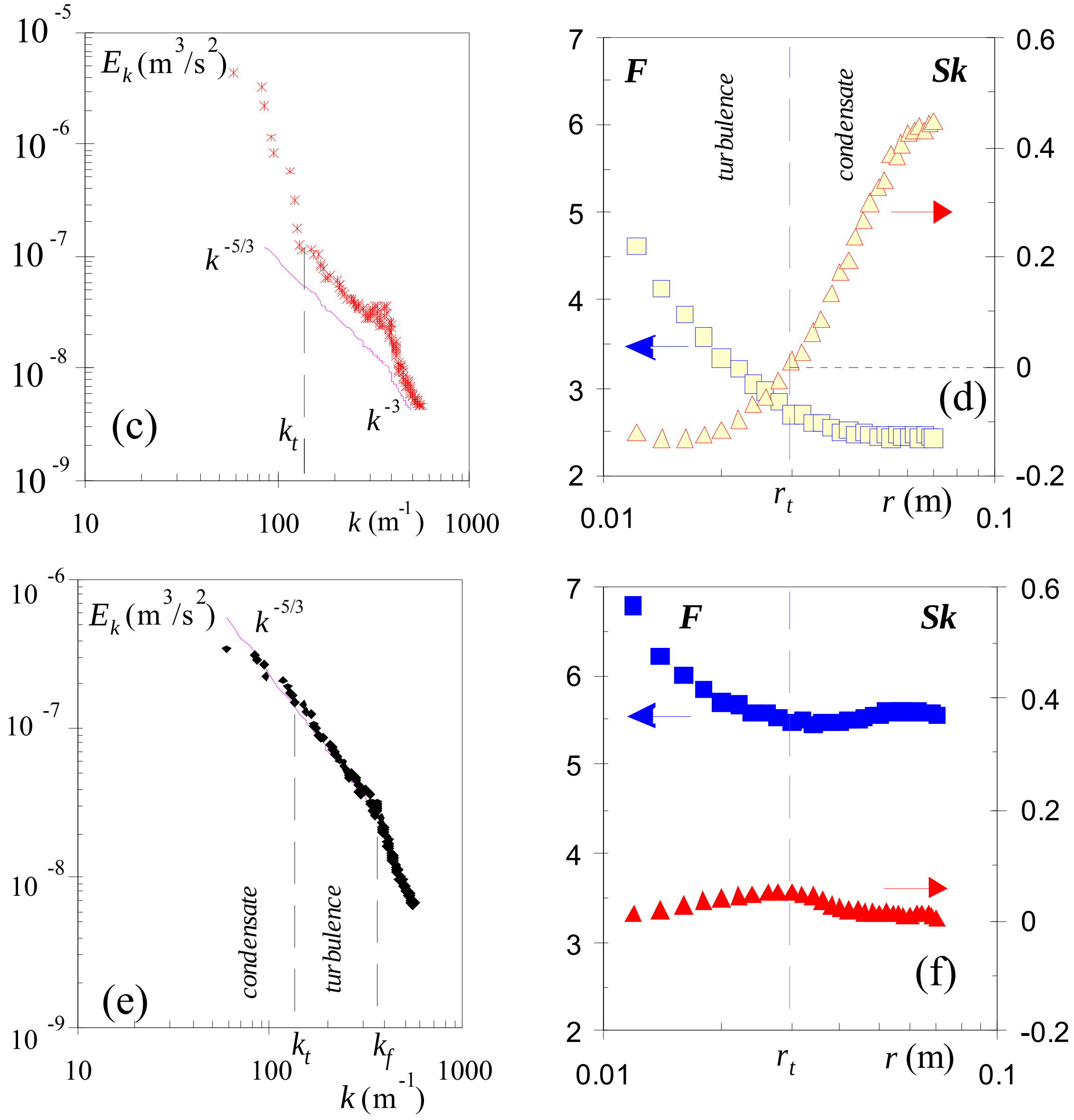
Kolmogorov law – linear $S_3$ (r) dependence in the “turbulence range”;

Kolmogorov constant C ≈ 7

# Strong condensate: effect of mean subtraction

**Before:**

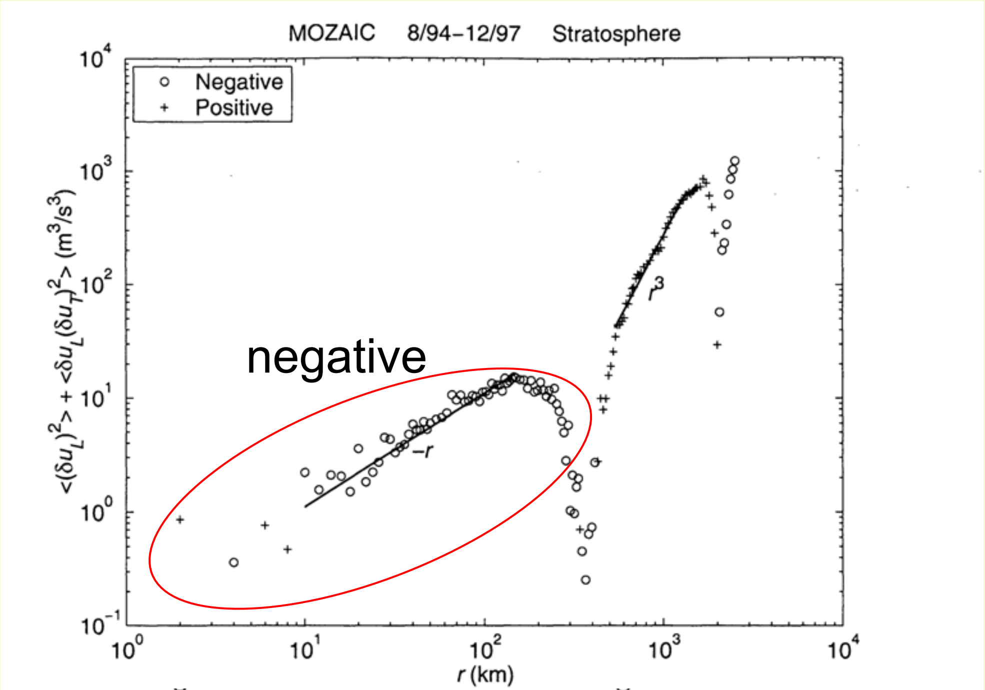
**After:**




**Normalized moments ~ scale-independent**

**Flatness is higher than in isotropic turbulence**

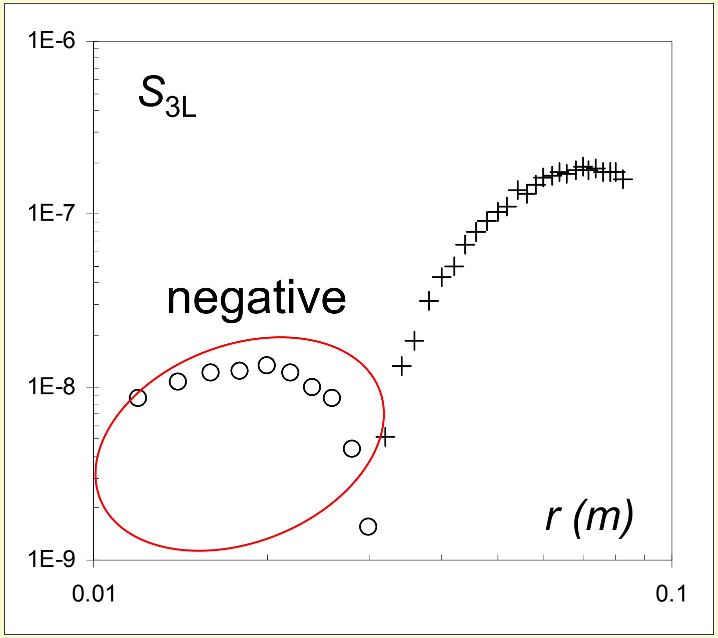
# Similarity with atmospheric turbulence



Circles and crosses indicate negative and positive values

[Cho, Lindborg, J. Geophys. Res. (2001) ]

Laboratory experiment
Stronger condensate, no mean subtraction




Mean shear flows present in the atmosphere affect velocity moments, similarly to laboratory experiments

# Different moments affected in different ranges

## Second moment

Large-scale flow is spatially smooth: $\delta V \approx sr$ , $\left\langle (\delta V)^2 \right\rangle \approx s^2 r^2$

Small-scale velocity fluctuations in turbulence $\left\langle (\delta v)^2 \right\rangle \approx C(\varepsilon r)^{2/3}$

Small-scale fluctuations dominate at scales smaller than $l < l_t \approx C^{3/4} s^{-3/2} \varepsilon^{1/2}$

## Third moment

Large-scale flow: $\left\langle \delta V^3 \right\rangle = \left\langle \delta V \delta v^2 \right\rangle \approx srC(\varepsilon r)^{2/3}$

Small-scale fluctuations: $\left\langle (\delta v)^3 \right\rangle \approx \varepsilon r$

Large-scale flow dominates 3rd moment in a range to much smaller scales:

$l_* \approx C^{-3/2} s^{-3/2} \varepsilon^{1/2}$ , since C > 1

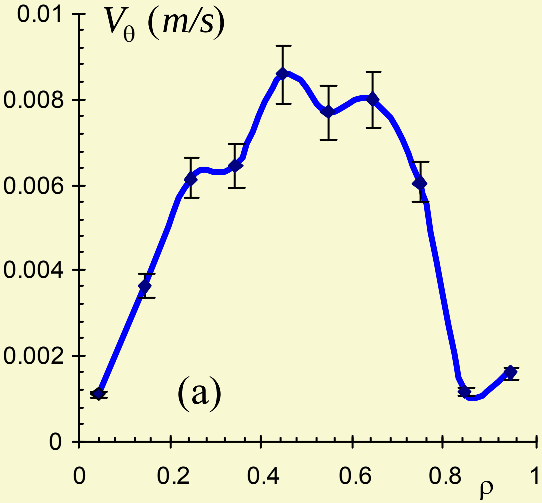
**Condensate imposes different scales on different moments**

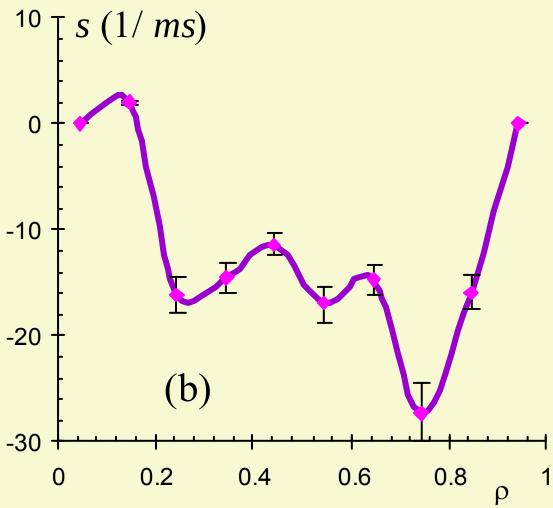
# The strongest condensate can suppress turbulence

(a) Self-generated condensate

(b) Externally imposed flow

# Mean flow and its shear



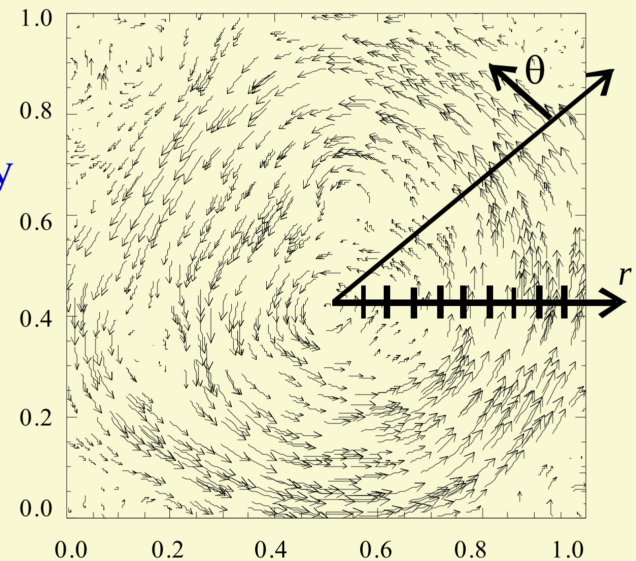




Angular velocity

$$\Omega = V_\theta / r$$

Shear

$$s = \frac{d\Omega}{dr}$$

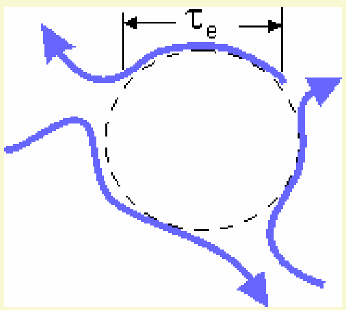


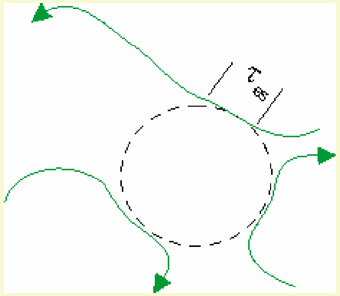

Eddy lifetime

$$\tau_e \approx \frac{l}{\sqrt{S_2(l)}}$$

The condition for the turbulence suppression

$$\boxed{\omega_s \tau_e > 1}$$



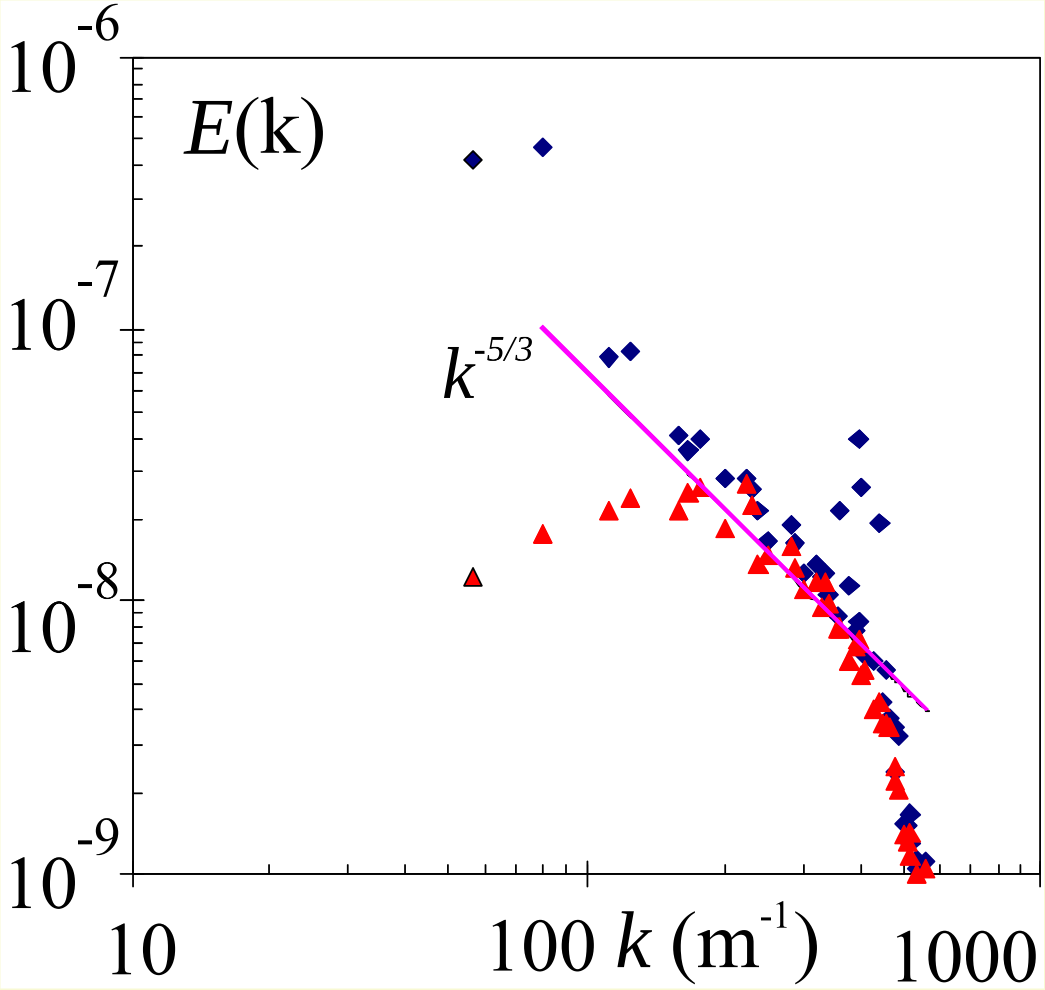

Shearing rate

$$\omega_s = l\frac{d\Omega}{dr} = l\left[\frac{1}{r}\frac{dV_\theta}{dr} - \frac{V_\theta}{r^2}\right] = sl$$

**Suppression means reduction in the eddy lifetime**

[M.G. Shats, H. Xia, H. Punzmann and G. Falkovich (2007)]

# Condensate “shears” large scales first




**Shearing acts more efficiently on large scales:**

$$s = \omega_s \tau_e \sim l^{5/3}$$

Suppression criterion s > 1
is satisfied for scales

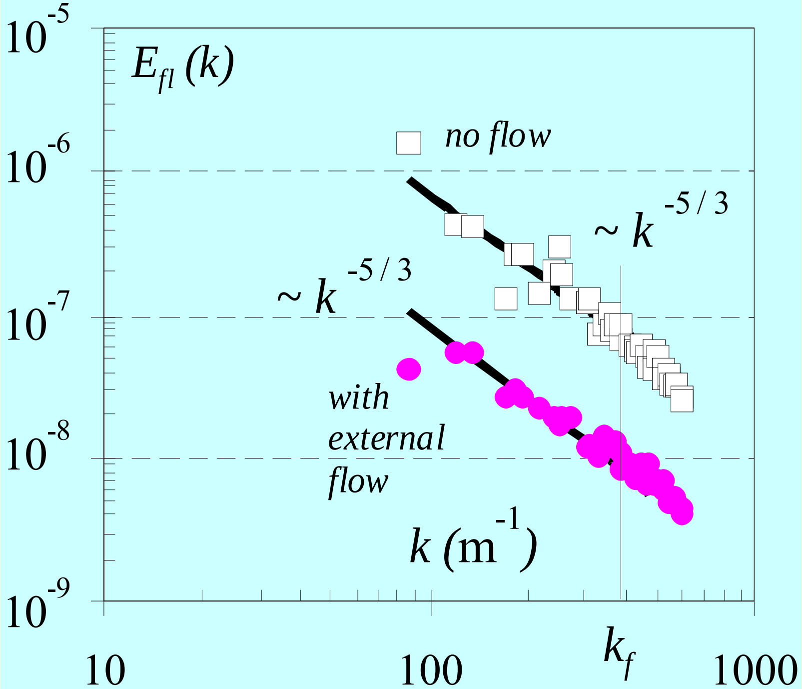
$$l > 0.022 \text{ m}$$

$k < 145$ m$^{-1}$ suppressed

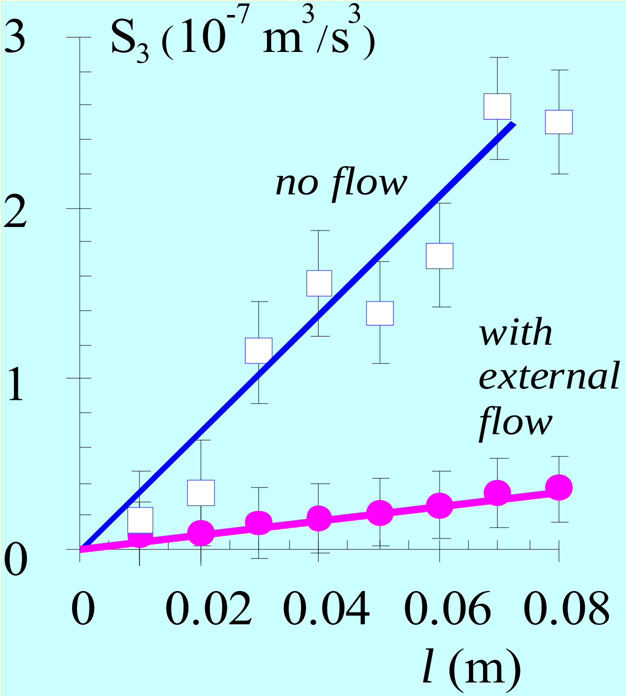
# Imposed flow reduces turbulence

**Externally imposed flow leads to turbulence reduction but retains $k^{-5/3}$ scaling**

**Energy spectra**




**3rd order structure function**




The energy flux through inertial range $\varepsilon = -2/3\, S_3(l)/l$

$\varepsilon$ is constant for all scales $l$

$\varepsilon$ is reduced by a factor of ~10 in the presence of the strong flow

**Mean flow reduces energy injected into turbulent cascade $\varepsilon$, leads to $E(k)$ drop:**

$$E(k) = C_k \varepsilon^{2/3} k^{-5/3}$$

# Sweeping due to mean flow

**Imposed flow affects scales down to the forcing scale**

**Mean flow sweeps forcing scale vortices relative to magnets**

**Sweeping**

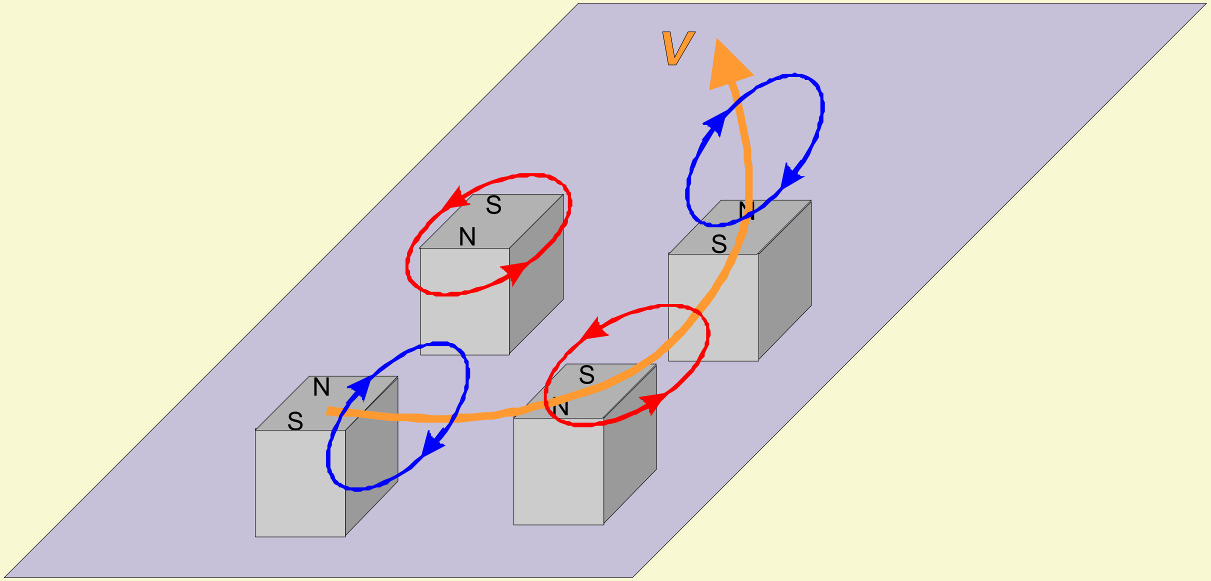
becomes important when

sweeping parameter

$$sw = \omega_{sw}\tau_e = \frac{V_\theta}{\sqrt{S_2}} > 1$$

Eddy life time

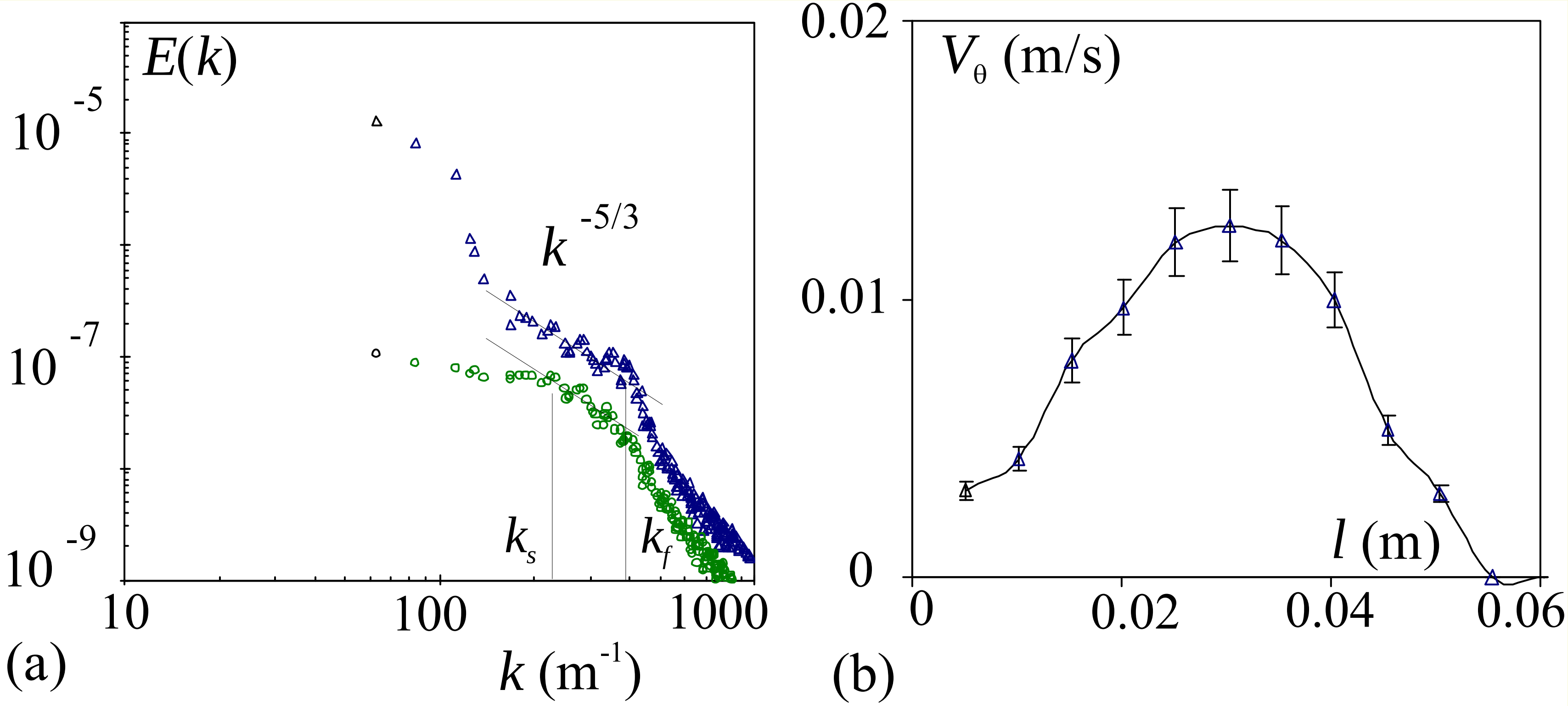
$$\tau_e = l / \sqrt{\langle |\delta u^2(l)| \rangle}$$




**Sweeping is more efficient on small scales:**

$$sw = \omega_{sw}\tau_e = \frac{V_\theta}{S_1} \propto l^{-1/3}$$

# Shearing and sweeping



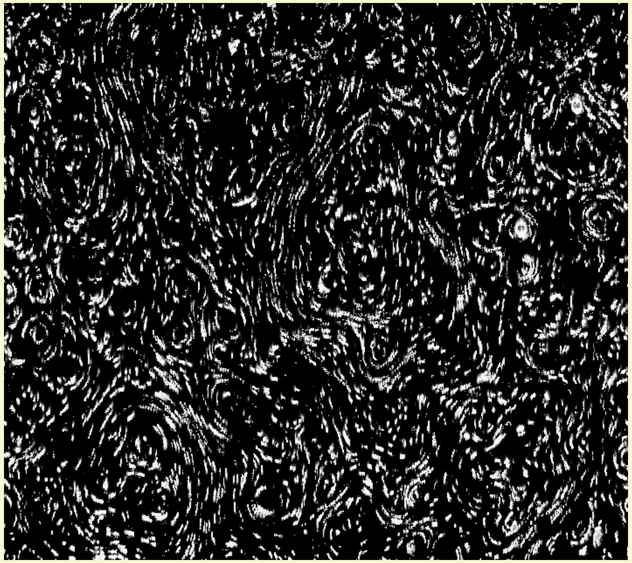

**Strongest condensate causes both shearing and sweeping – reduces turbulence that feeds it**

# Summary

- Spectral condensation leads to the generation of mean flow coherent across the system size
- Spectrum of condensed turbulence shows 3 power laws: $\sim k^{-3}$ at large scales; $k^{-5/3}$ in the meso-scales, $k^{-(3\text{-}4)}$ at small scales
- Condensate modifies statistical moments of velocity fluctuations
- Different moments are affected in different ranges of scales
- Velocity moments of condensed turbulence similar to those in the atmosphere
- Coherent shear flow suppresses turbulence which generates it via shearing and sweeping

# Flow externally imposed on turbulence

Particle streak photo
of turbulent flow



Turbulence with externally
imposed mean flow