Nonequilibrium Statistical Mechanics of Climate Variability

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Outline

• what do I mean by climate variability?
• stochastic models of climate fluctuations
• detailed balance and climate (Weiss, 2003)
• nonequilibrium statistical mechanics
• structure of climate variability (Weiss, 2007)
Climate Variability

• Climate system has many identifiable variations on many timescales

• El-Niño
  ▪ coupled tropical atmosphere-ocean
  ▪ timescales: 9 months, 4-5 years

• Gulf Stream
  ▪ internal ocean variability
  ▪ timescales: months to very long
• Storm Tracks
  - variations in weather due to SST
  - timescales: seasonal to decadal
• North Atlantic Oscillation
  - atmospheric mass variations across North Atlantic
  - timescales: years to decades
• Fluctuations have large impact and are difficult to predict under climate change
• Will longer records reveal new phenomena with longer timescales?
El-Niño

- **El-Niño theme page:**
  [http://www.pmel.noaa.gov/tao/elnino/nino-home.html](http://www.pmel.noaa.gov/tao/elnino/nino-home.html)
Stochastic Model of El-Niño

- SST evolves on slow ocean timescale
  - months
- forced by chaotic weather
  - Lyapunov time of a few days
- on monthly timescale, weather is random noise
Stochastic Climate Models

• Stochastic prediction models go back decades
  (Epstein, 1969; Fleming, 1971; Leith, 1974; Hasselman, 1976; North and Cahalan 1981)

• Dynamical GCMs and chaos intervened

• Renewed interest in last decade
  (e.g. Penland and Magorian, 1993; Farrell and Ioannou, 1993; Moore and Farrell, 1993)
Stochastic Dynamics

• Truncate state to finite state vector $\vec{x}$
  ▪ e.g. put SST $T(x,y)$ on a grid
• state evolves via nonlinear deterministic operator
  \[
  \frac{d\vec{x}}{dt} = \vec{\mathcal{N}}(\vec{x})
  \]

• linearize about some mean state \( \vec{x}_0 \)
• parameterize everything ignored as random noise
  \[
  \frac{d\vec{x}}{dt} = A\vec{x} + \text{noise}
  \]

• \( \vec{x} \) is an N dim vector, departure from mean
• \( A \) is an N x N matrix
• simplest noise process: additive Gaussian white noise

\[
\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + F\vec{\xi}(t)
\]

\[
\left\langle \vec{\xi}(t) \right\rangle = 0
\]

\[
\left\langle \vec{\xi}(t)\vec{\xi}^T(s) \right\rangle = I\delta(t - s)
\]

• N x N diffusion matrix

\[
D = FF^T/2
\]
• For this to make sense, deterministic part must be a stable fixed point

• Adding noise results in a statistically steady distribution governed by
  ▪ deterministic dynamics $A$
  ▪ diffusion $D$
• Traditionally in statistical mechanics, noise parameterizes fast molecular motion
• Here, separation of timescales assumed
  ▪ slow motion regarded as deterministic
  ▪ fast motion parameterized as noise
• Some rigorous results, but work still needed (e.g. Majda, et al.)
Constructing stochastic models

Two common approaches:

1. Start with known nonlinear equations
   - linearize about fixed point
   - add unknown random noise
   - typically assume $D = I$ from ignorance
   - rigorous reduction would give noise

2. Create matrices from data
   - fit time average correlations
Non-normality

• Many papers focus on non-normality of deterministic dynamics
  • \( A A^T - A^T A \neq 0 \)
  • eigenvectors not orthogonal

• first sign: multivariate linear system has complexity due to matrix non-commutativity
Transient amplification

- consider deterministic trajectories
- stable fixed point:
  - all perturbations eventually decay to zero
- non-orthogonal eigenvectors:
  - interference
  - transient growth
Two views of climate fluctuations

• Instabilities and nonlinear dynamics
• Transient amplification of small noise
• lots of fireworks
• reconciliation, at least for El-Niño:
  ▪ near a bifurcation, indistinguishable
  ▪ bifurcation parameter: Atmosphere-ocean coupling
• mystery:
  ▪ why is El-Niño near a Hopf bifurcation?
Model evaluation

- simple stochastic models do surprisingly well

SST Prediction
(Saha, et al 2006)
250 hPa pattern
theoretical objection

• appropriate coordinate transformation
  ▪ renders eigenvectors orthogonal
  ▪ removes transient amplification

• real physical phenomena should not depend on arbitrary coordinate choice

• climate fluctuations do not “look” random
  ▪ have well-defined life-cycles
Detailed Balance and Climate

(Weiss, 2003)

- non-normality not a fundamental property
- fundamental coordinate-invariant property is detailed balance
- two kinds of steady states

thermodynamic equilibrium

thermodynamic non-equilibrium

detailed balance satisfied
no current

detailed balance violated
nonzero current
• For systems

\[
\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + F\vec{\xi}(t)
\]

• equivalent statements:
  ▪ detailed balance is violated
  ▪ \(AD - DA^T \neq 0\)
  ▪ cannot mutually diagonalize \(A\) and \(D\)
    • essential multi-dimensional character
  ▪ \textit{nonequilibrium} steady-state
Probability Current

• nonequilibrium steady-state maintained by a probability current

\[ \frac{\partial p}{\partial t} + \nabla \cdot (\vec{u} p) = 0 \]

• phase space velocity

\[ \vec{u} = \Omega \vec{x} \]

• \( \Omega \) = matrix of frequencies
  ▪ reminiscent of Landau’s theory of turbulence
Example

\[ A = \begin{pmatrix} -1 & -\cot \theta \\ 0 & -2 \end{pmatrix}, \quad D = I/2 \]

- detailed balance: \( \theta = \pi/2 \)

steady-state pdf and probability current
• equilibrium vs. nonequilibrium
• identical steady-state pdf’s:
  ▪ isotropic Gaussian; identity covariance
Nonequilibrium Statistical Mechanics

• stat mech has made significant progress in past decade
  ▪ review articles: Evans and Searles, 2002; Seifert, 2007; Gallavotti, 2007

• Fluctuation Theorems:
  ▪ many kinds related in various ways
  ▪ many scenarios
    • steady-state, transient, forced, …
  ▪ different classes of dynamics
    • Langevin, discrete processes, chaotic nonlinear, Hamiltonian, quantum, …
  ▪ confused and confusing literature
Steady-state Fluctuation Thm

- FT describe probability of finding “violations” of 2nd Law of Thermodynamics
- non-equilibrium steady state maintained by dissipating heat => entropy production
- consider trajectory segment with time $t$
- trajectory has entropy change $\Delta s$
- $p(\Delta s)$ prob of finding trajectory with $\Delta s$

$$\frac{p(-\Delta s)}{p(\Delta s)} = \exp(-\Delta s)$$
• previously applied to microscopic systems
  ▪ molecular motors
  ▪ chemical reactions
  ▪ molecular dynamics of fluids

• numerous experimental verifications

![Graphs](image.png)

Colloid in optical trap
(Carberry, et al, 2004)
Surprising Conclusion

- stochastic climate models are a good approximation to reality
- stochastic climate models are in class of models covered by noneq. stat mech.
- recent advances in nonequilibrium stat mech apply to global scale climate phenomena
Structure of climate fluctuations

(Weiss, 2007)

• Fluctuation Theorems provide constraint on probability distributions
• For microscopic systems, interested in averages over many fluctuations
• For climate, care about individual fluctuations as well as averages
• Look at properties not constrained by Fluctuation Theorem
Irreversibility

- trajectory probability $p(x_0, x_1, t)$
  - different than traditional probabilities $p(x)$, $p(x_1, t \mid x_0)$
- straightforward operational definition based on a timeseries
- irreversibility

$$r(x_1, x_0, t) = \ln \left( \frac{p(x_0, x_1, t)}{p(x_1, x_0, t)} \right)$$
irreversibility pdf $p(r)$

- Fluctuation Theorem

\[
\frac{p(-r)}{p(r)} = \exp(-r)
\]

- want details of $p(r)$ as a function of parameters of stochastic model: $A$, $D$

- linear model:
  - can write solutions formally
  - multi-variate => must solve matrix equations numerically
results

• despite linearity, $p(r)$ is non-Gaussian
• average irreversibility $\langle r \rangle \geq 0$
  for small $t$: $\langle r \rangle (t) \sim \sigma t$
  large $t$: $\langle r \rangle (t) \rightarrow 0$
• hence $\langle r \rangle$ has a max at some time
  ▪ defines a timescale for fluctuations
• $\sigma$ is the average entropy production, and is related to the noise amplification
• dimensional reduction:
  ▪ fast modes do not affect $p(r)$
work in progress

- analyze irreversibility of El-Niño data
- data is three-month averages of SST
- entropy production gives correct Lyapunov timescale of chaotic atmosphere: few days
- quantitative comparisons in progress
- analyze other climate fluctuations next
- model climate change as forced change in steady state
Broad Conclusions

• Climate variability poorly understood but crucial for current and future climate impacts
• Can be modeled as fluctuations about a nonequilibrium steady-state
• Recent and future progress in statistical mechanics has implications for climate
• Stat mech may have something to say about climate change