Strongly interacting fermions in optical lattices: from few to many particles

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In collaboration with
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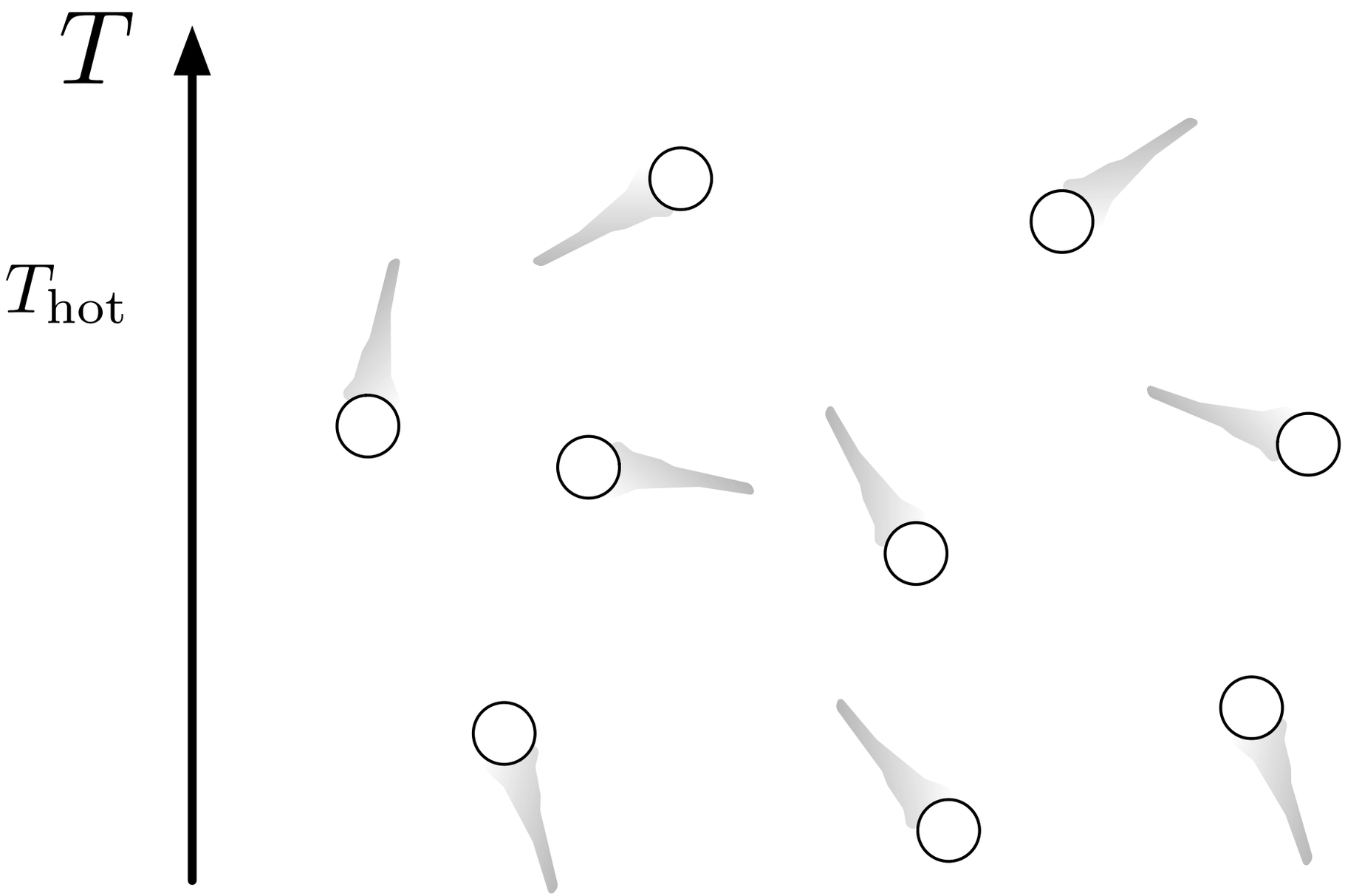
Dilute non-interacting gases



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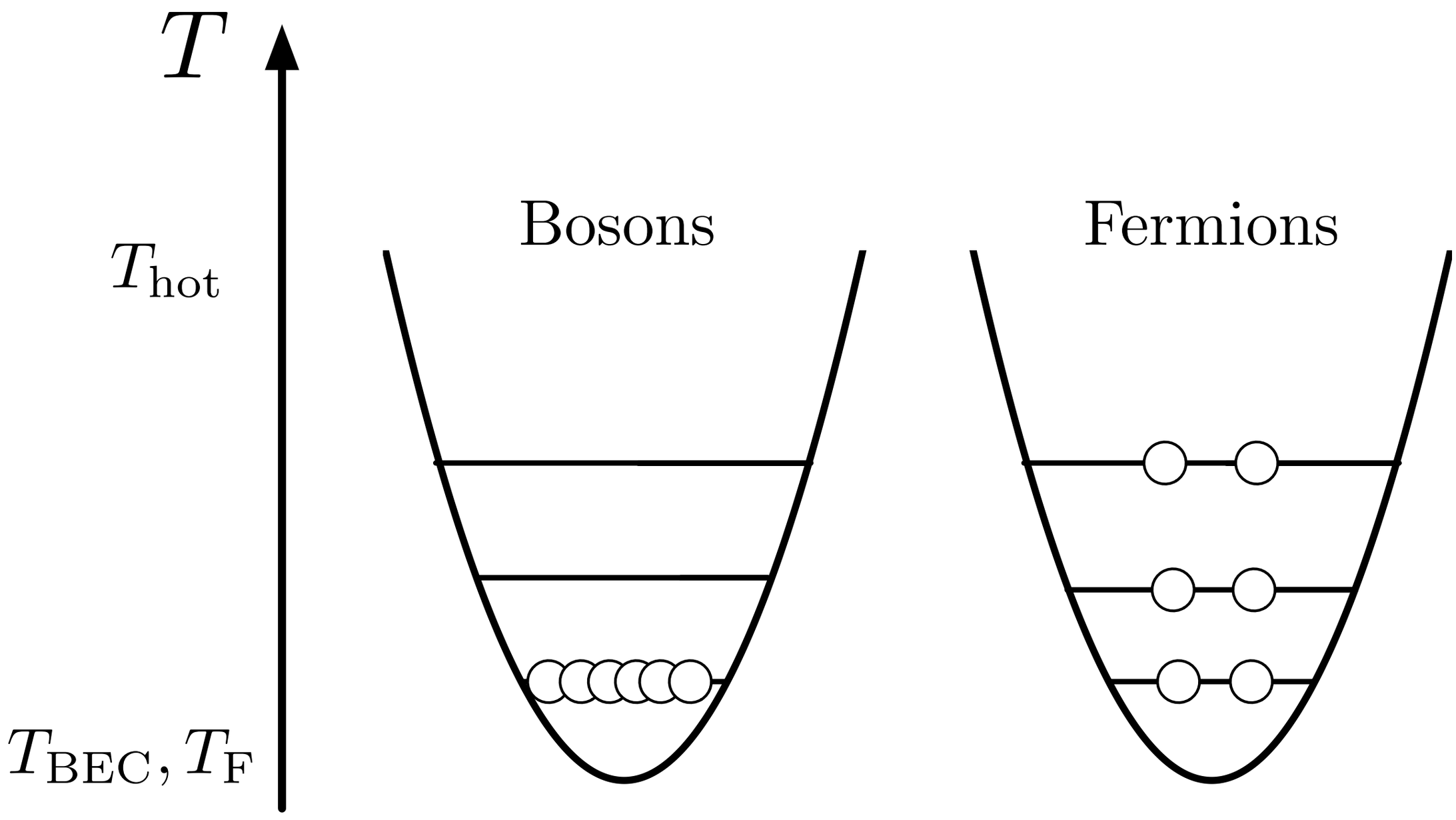

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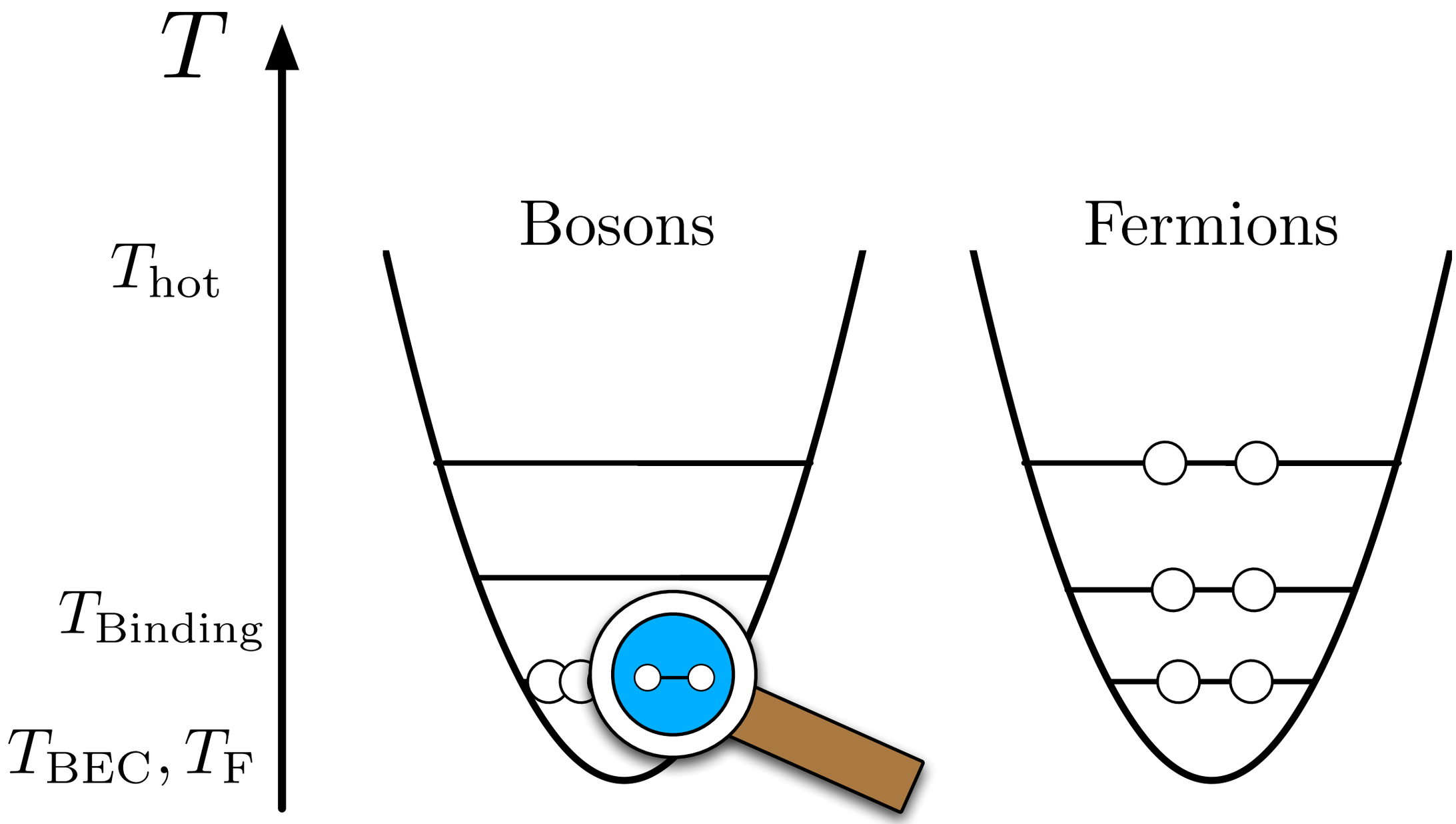
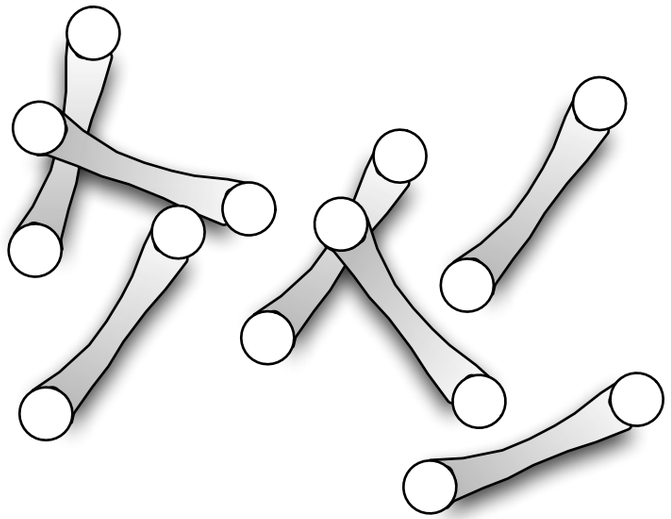


Cooper Pairing

For attractive interaction, Many-Body instability

Binding energy, $T_C \sim e^{-\pi/2k_F|a_s|}$

Radius $\sim e^{\pi/2k_F|a_s|}$



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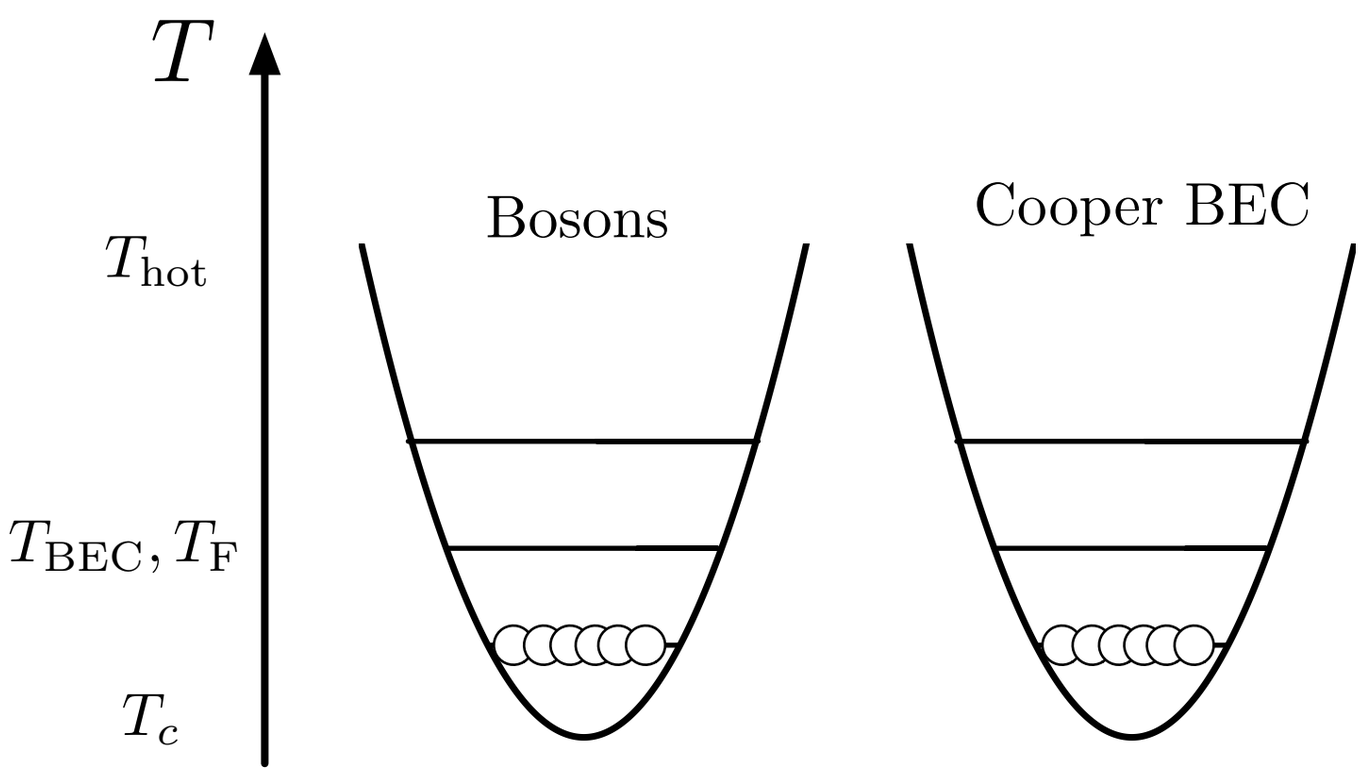
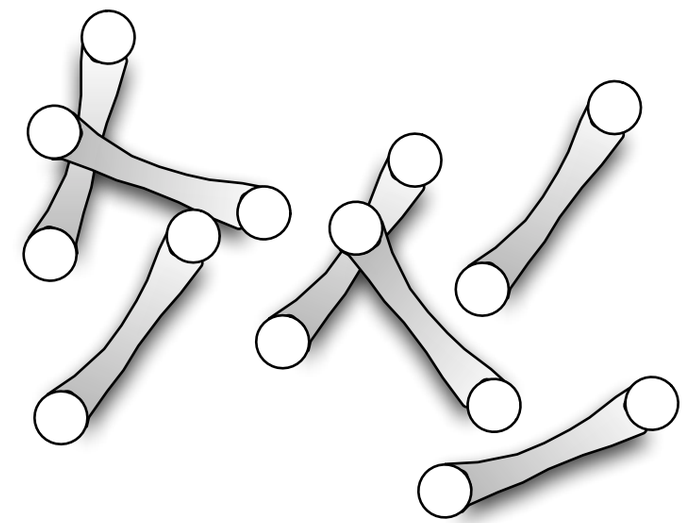
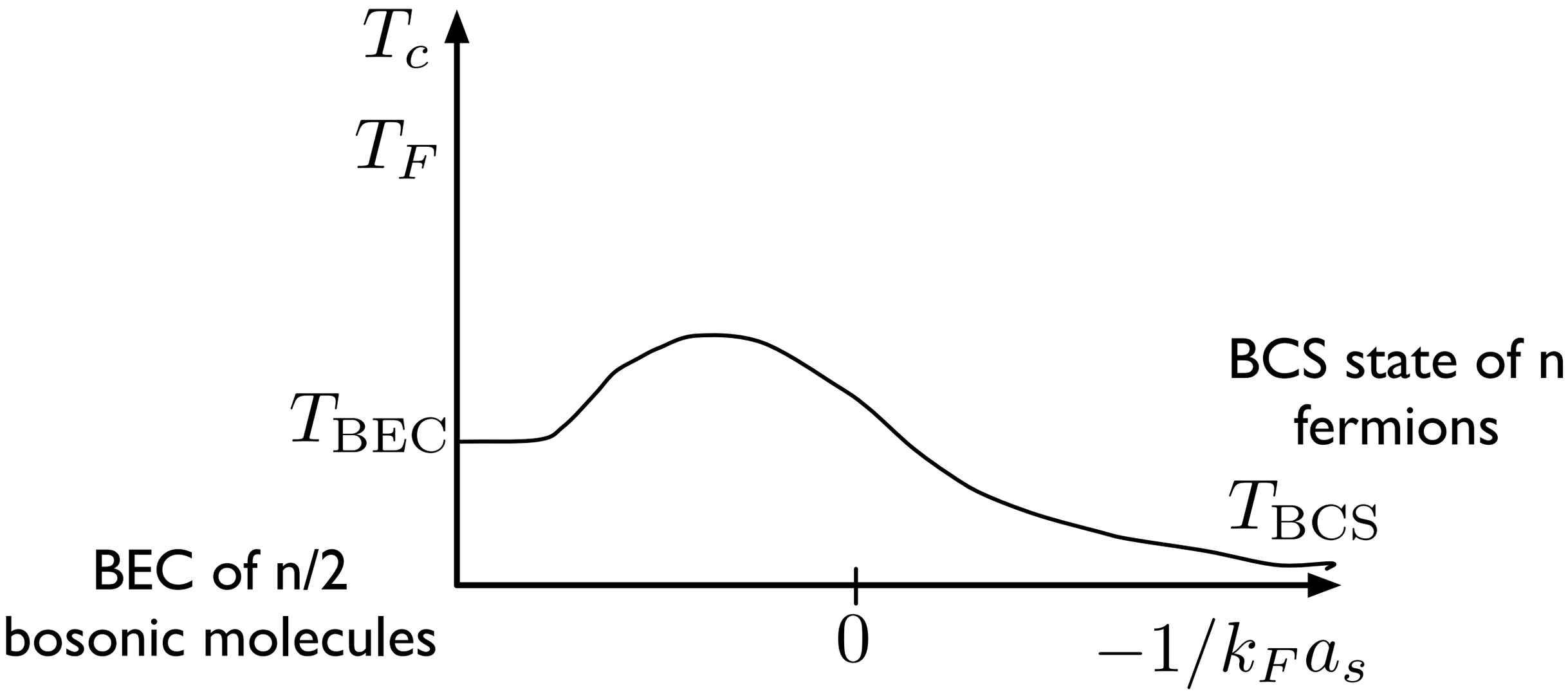




BEC-BCS crossover

Imagine "turning up" attraction



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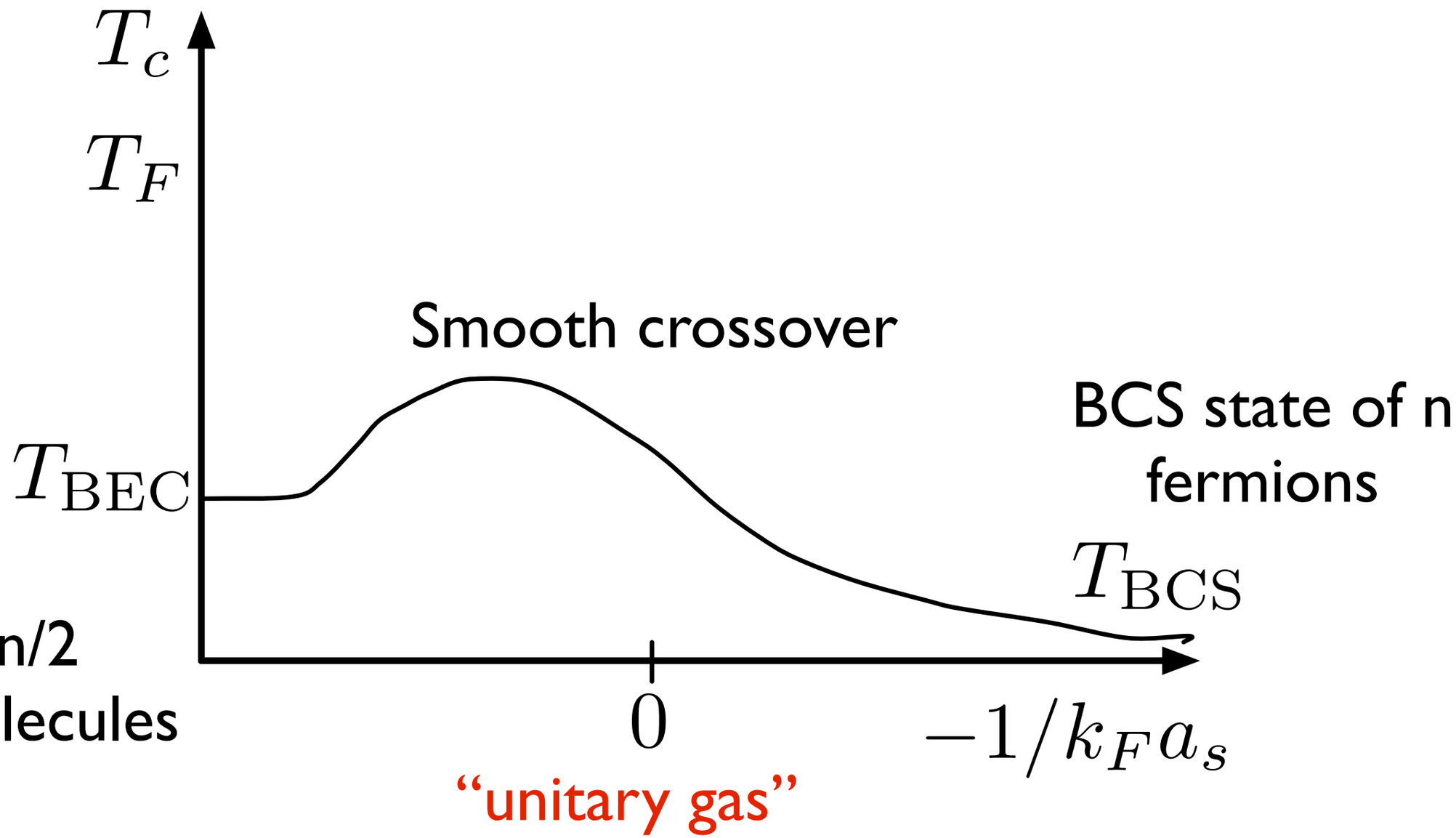

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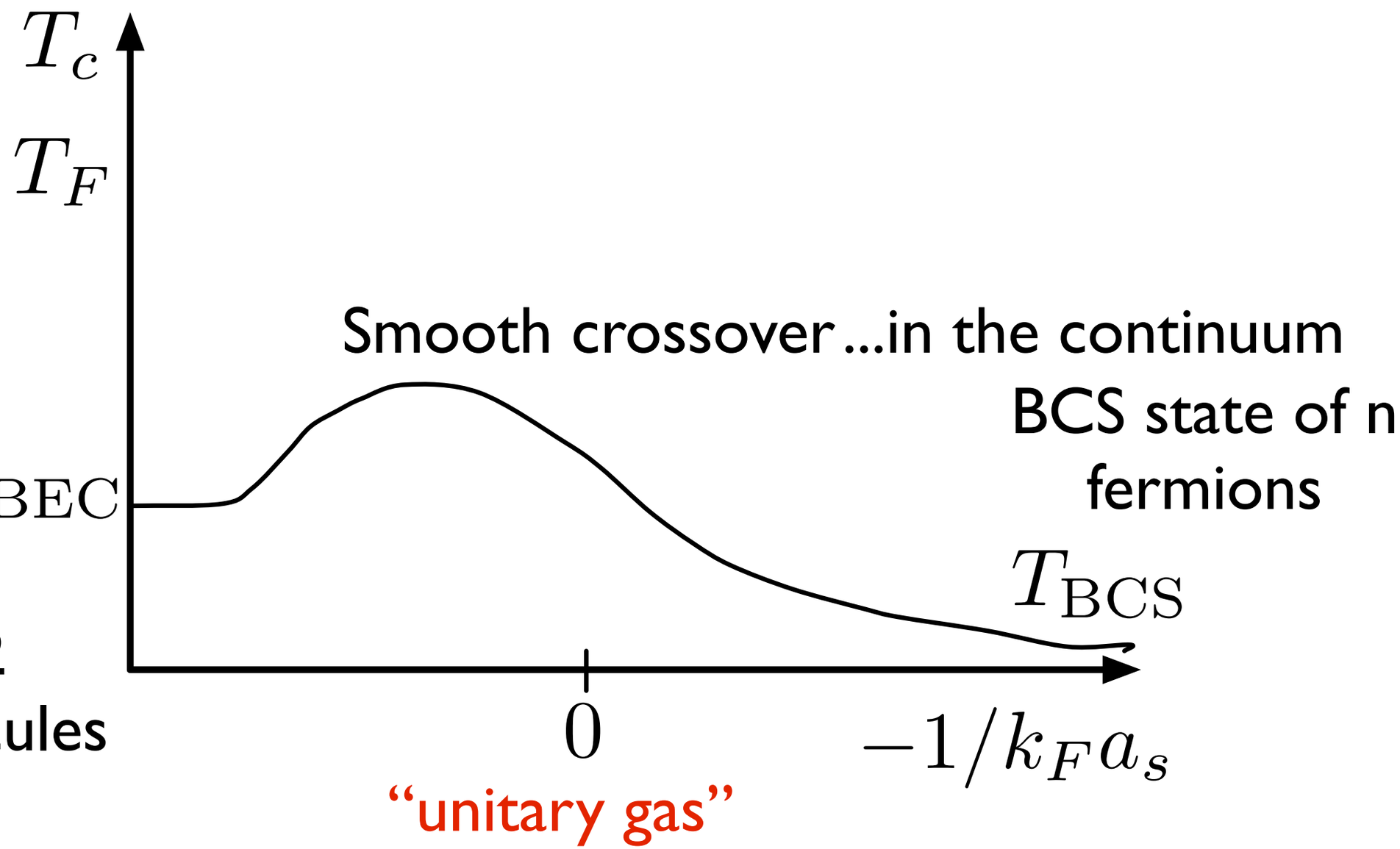
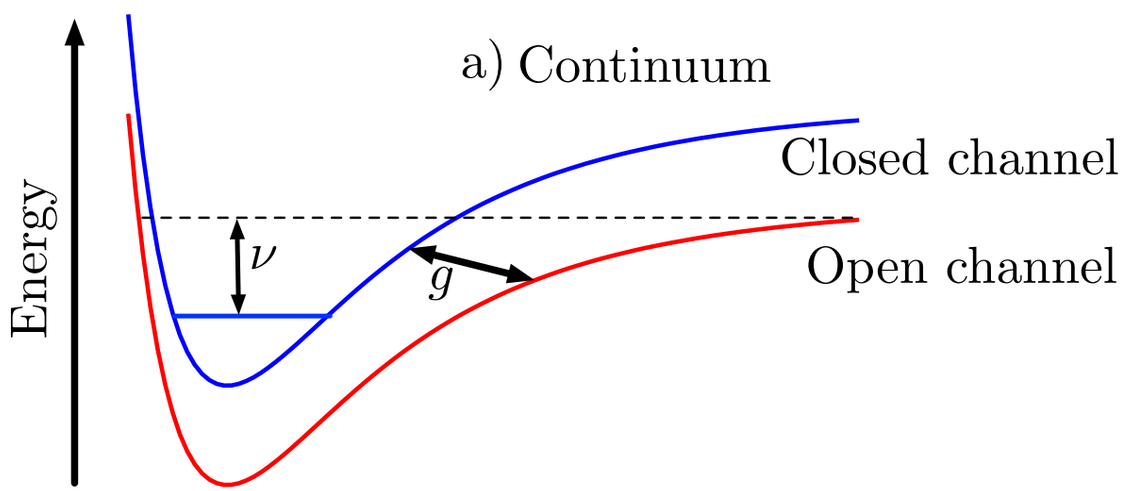
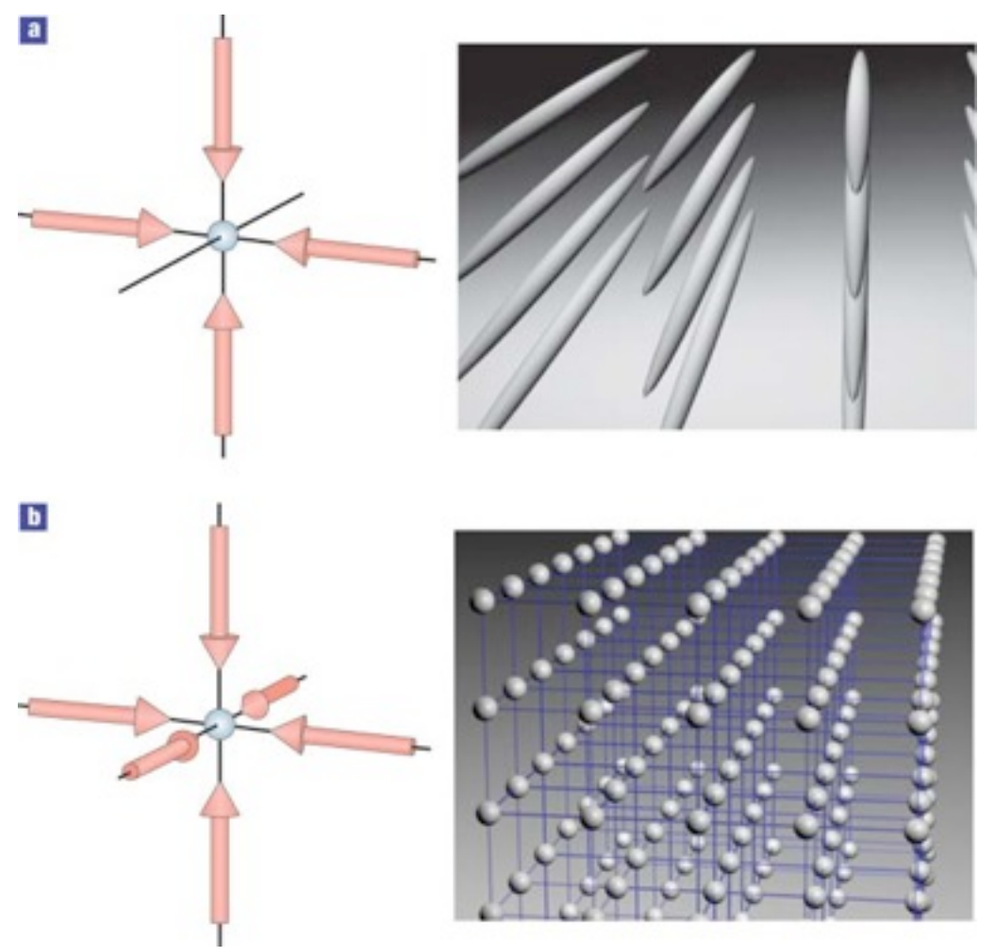


BEC of n/2 bosonic molecules

Motivation

Bring together two of the most successful tools for ultracold gases: Feshbach resonances and optical lattices





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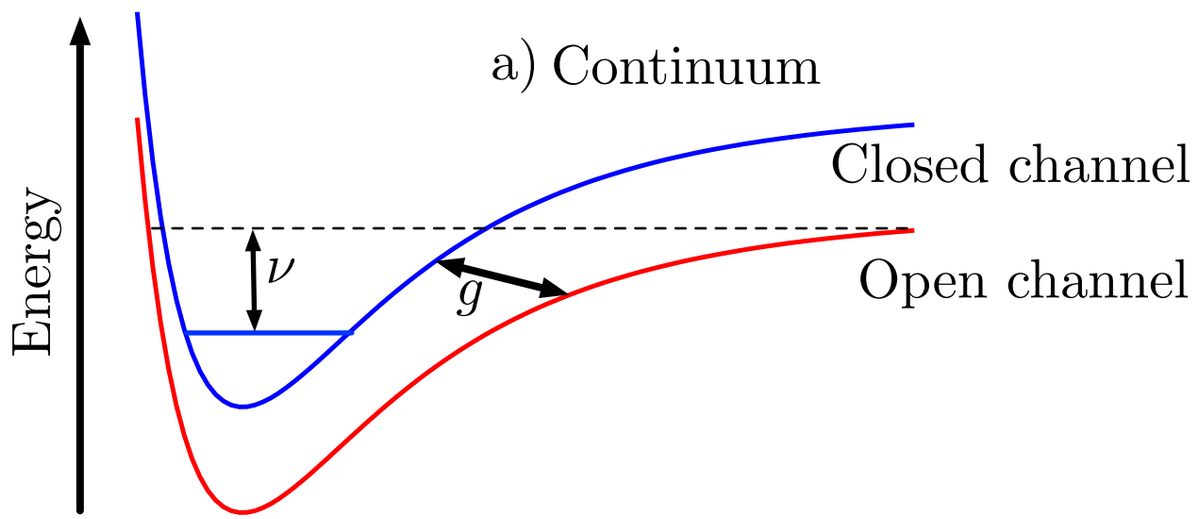
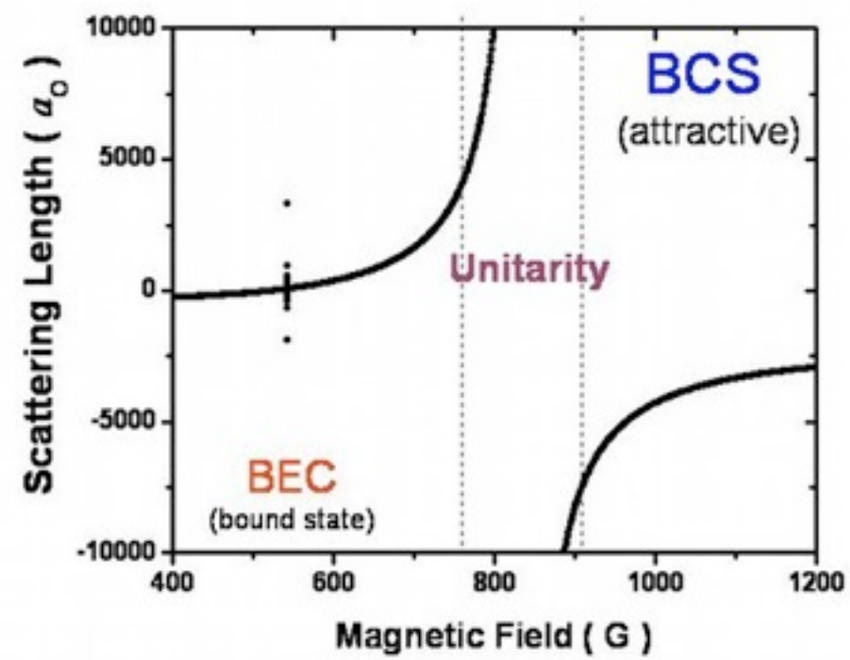
Fundamentally: how do fermions pair to form bosons?

How do we optimize making diatomic molecules from atoms?

Can we devise tractable models for many-body physics near unitarity?

Feshbach resonances





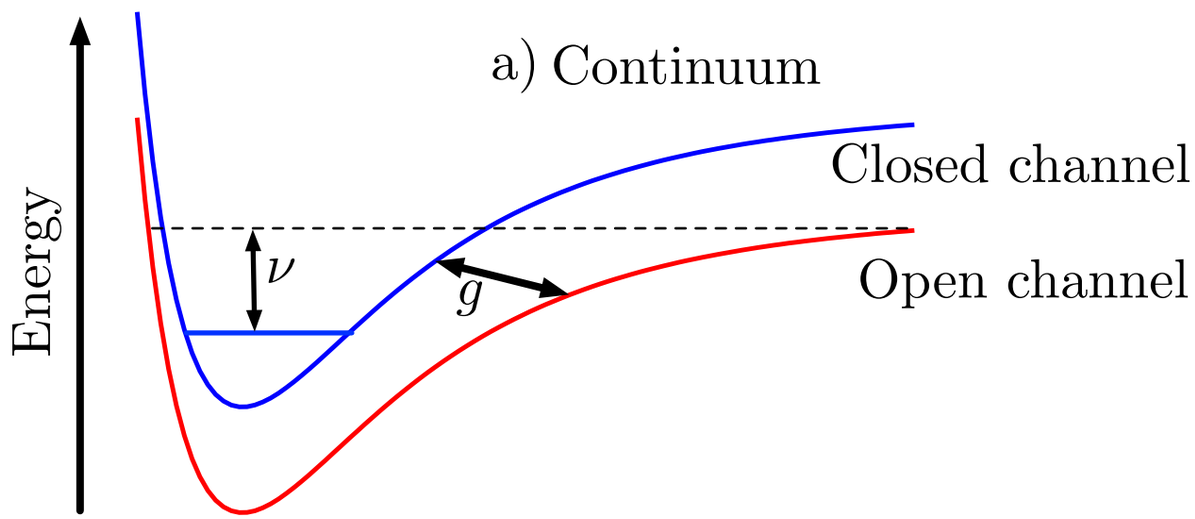
Two-channel model:

s-wave scattering length $a_s = -\frac{\mu g^2}{2\pi\hbar^2\nu}$

Effective range $r_B = \frac{\pi\hbar^4}{\mu^2 g^2}$

Scattering amplitude $f(\mathbf{k}) = -\frac{1}{1/a_s + ik + r_b k^2}$

Feshbach resonances



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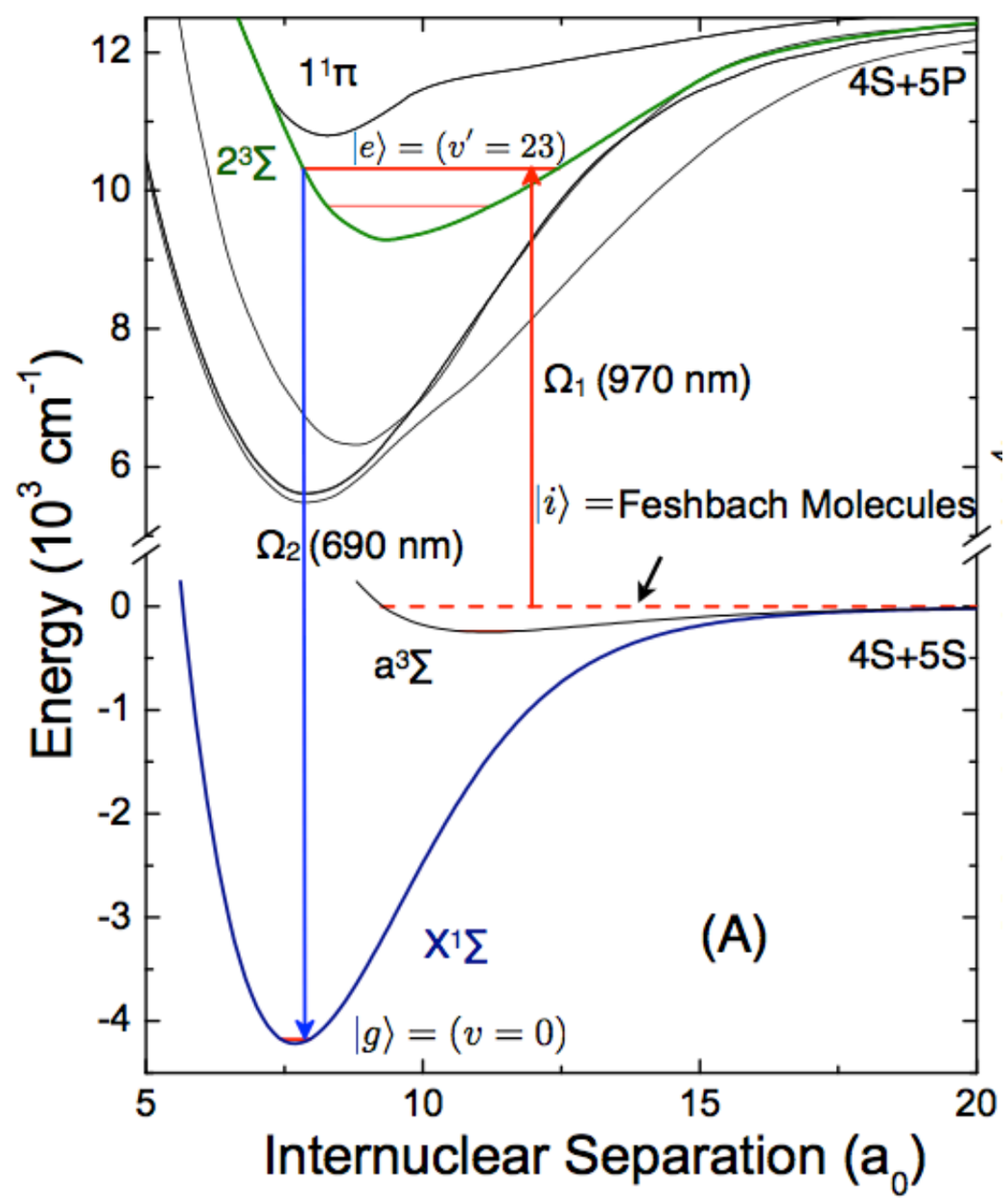


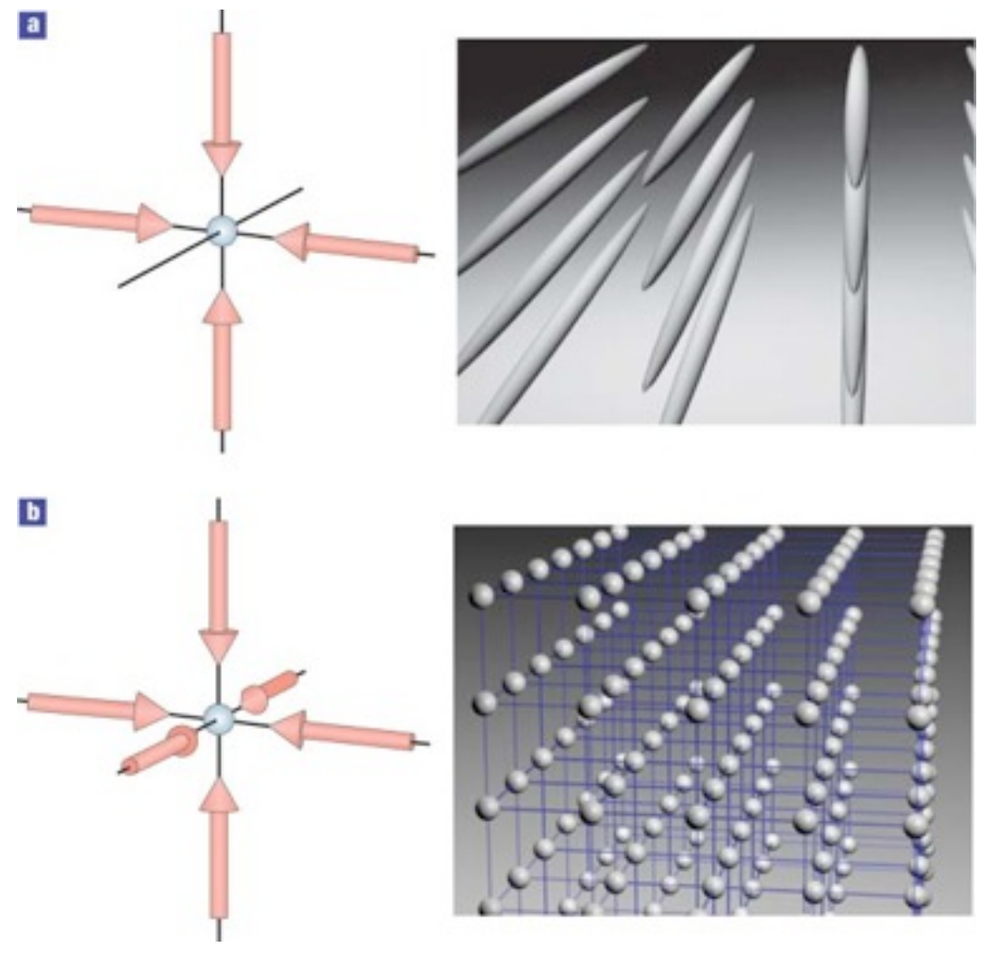
Lattice physics

$$V_{\text{latt}}(\mathbf{r}) = V \sum_{i=x,y,z} \sin^2(\pi i/a)$$

- Lattice potential is non-separable
- Couples center-of-mass and relative coordinates
- Band structure "structures continuum"
- New length and energy scales

Scale	Typical values
Lattice spacing a (nm)	550
Temperature T (nK)	$\sim$100
Recoil energy $\hbar^2\pi^2/2ma^2$ (kHz)	25 (^{40}K), 170 (^{6}Li)
Lowest band tunneling t (E_R)	$\sim$0.01
Band gap (E_R)	$\sim$ 5
Effective range r_B (nm)	$\lesssim$1
Interchannel coupling $g/E_R a^{3/2}$	$\gtrsim$16

Optical lattices



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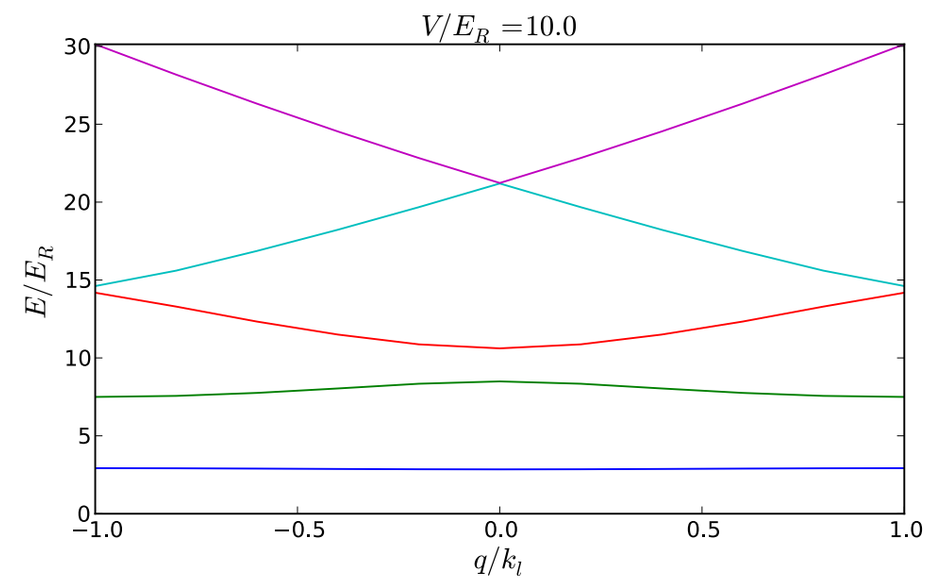

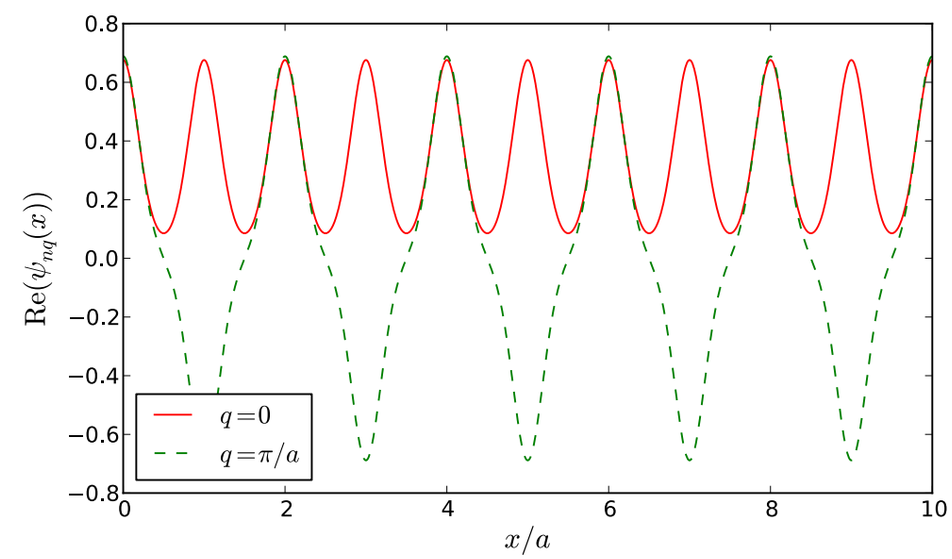

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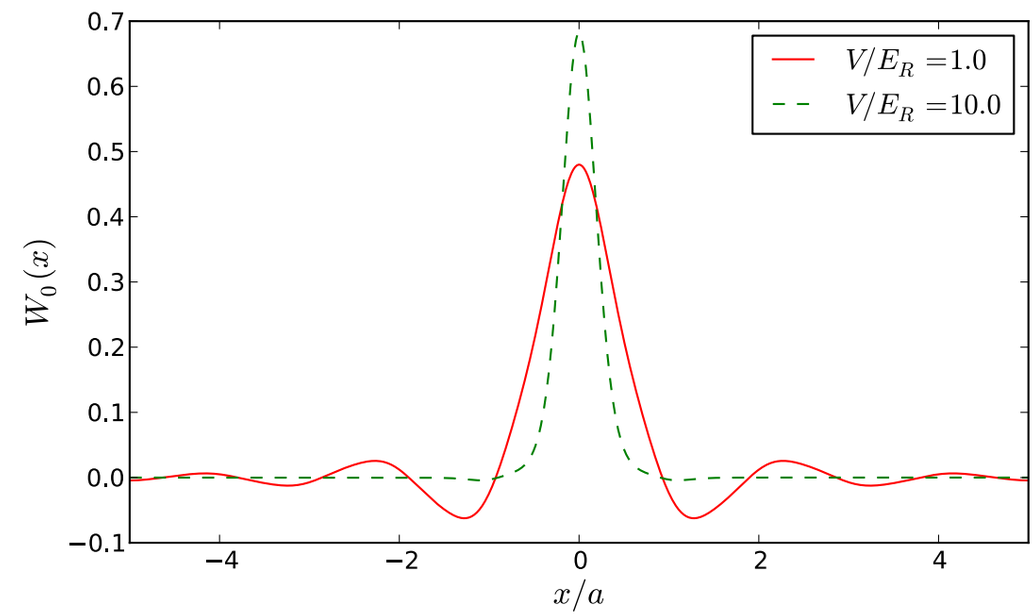

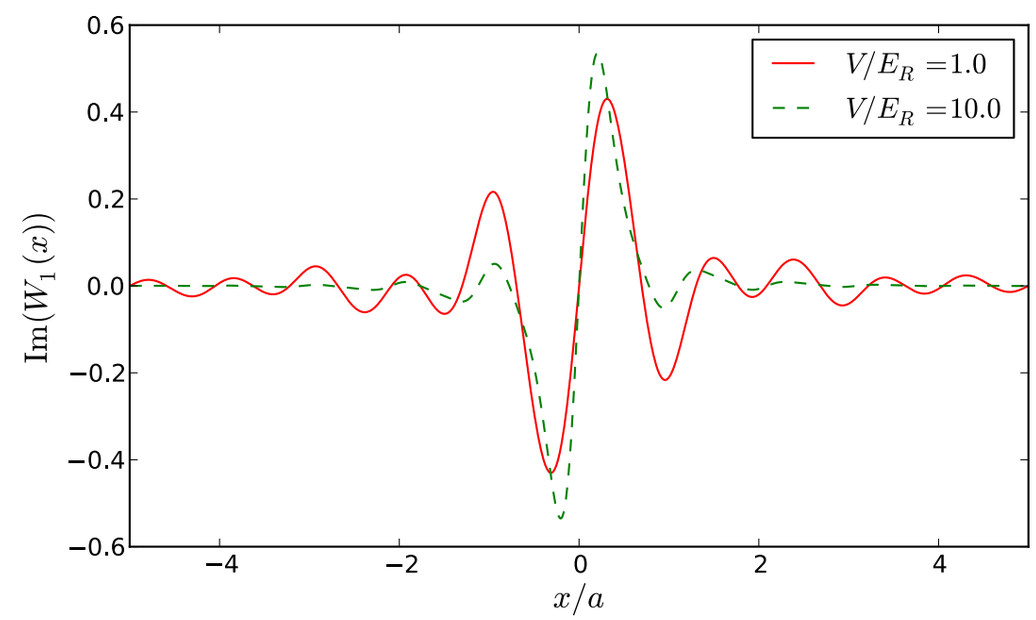

Many-body physics in a lattice

Usual assumptions

- Replace interaction with free-space pseudopotential $U(\mathbf{r}) = \frac{4\pi\hbar^2 a_s}{m}\delta(\mathbf{r})$
- Restrict to the lowest band
- Requires low energy, small s-wave scattering length

$$a_s \ll a$$

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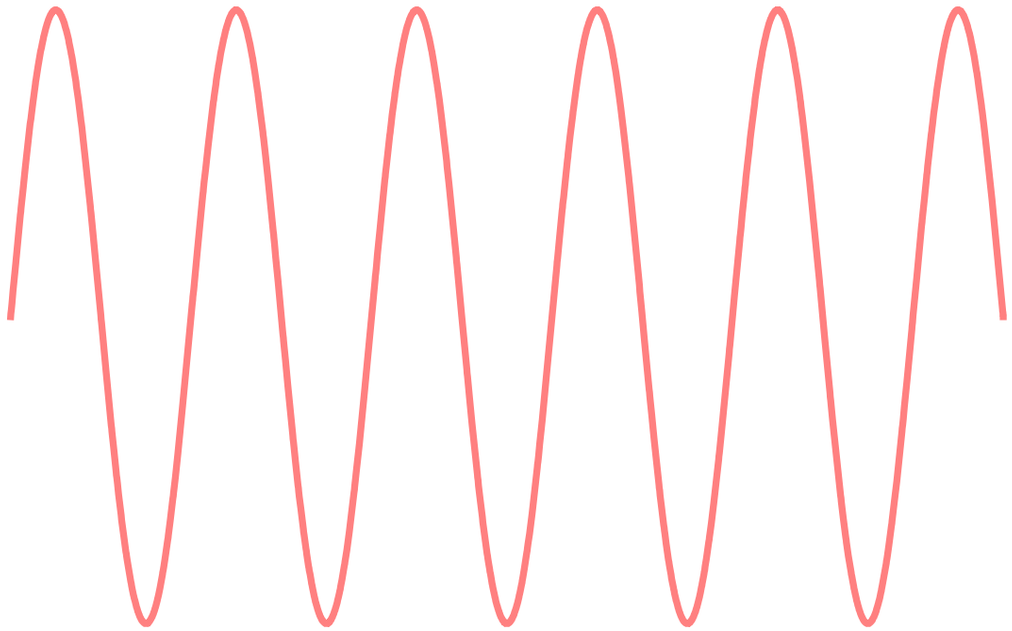

Hubbard model

- Simplest model of interacting lattice fermions

$$\hat{H} = -t \sum_{\langle i,j \rangle} \sum_{\sigma \in \{\uparrow,\downarrow\}} \left[\hat{a}^\dagger_{i,\sigma} \hat{a}_{j,\sigma} + \text{h.c.} \right] + U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow}$$

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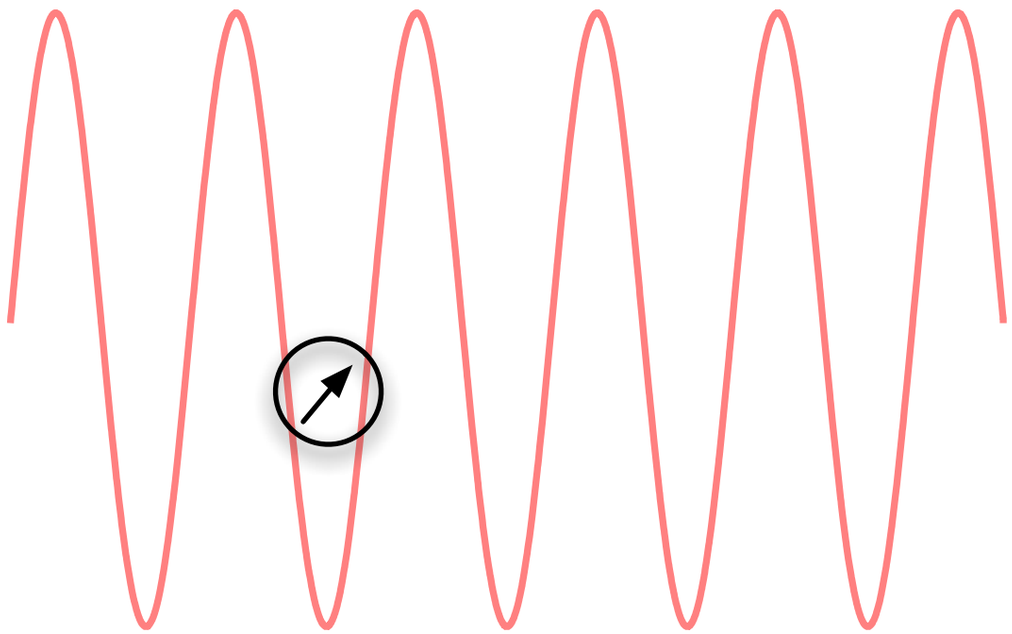

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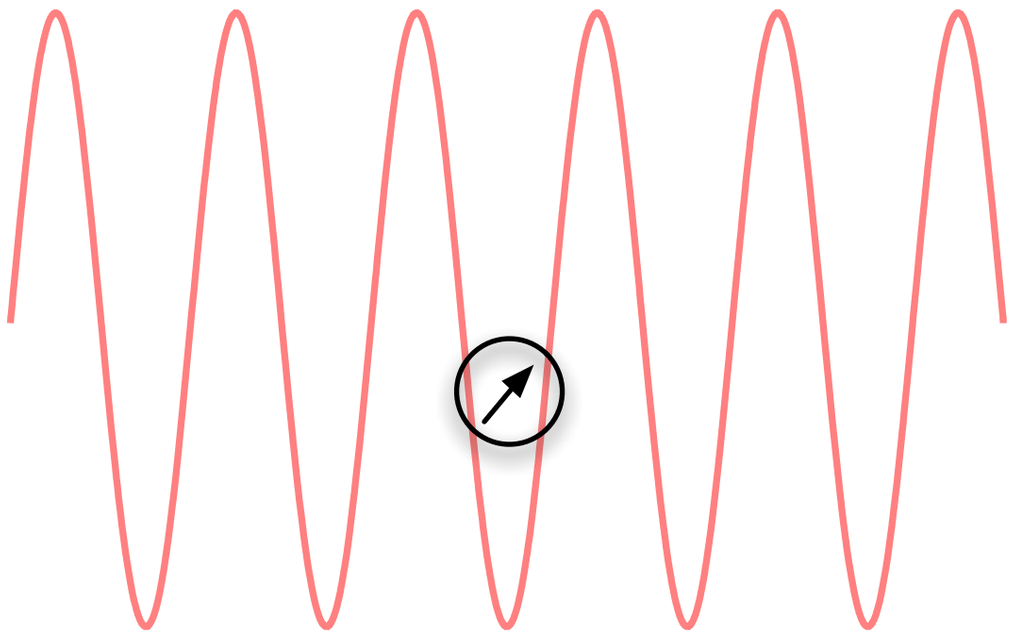

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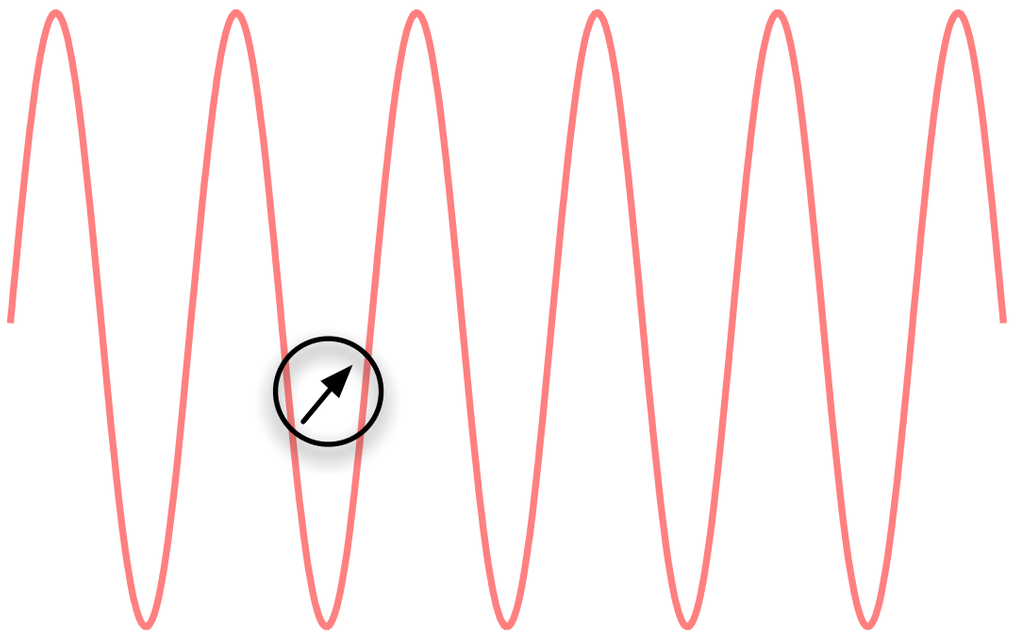

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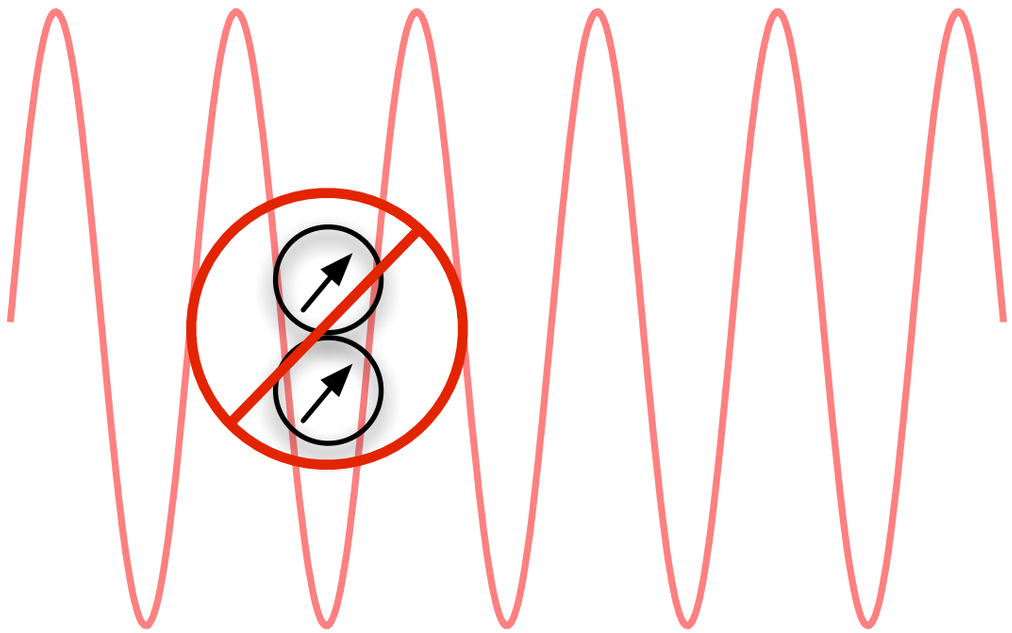

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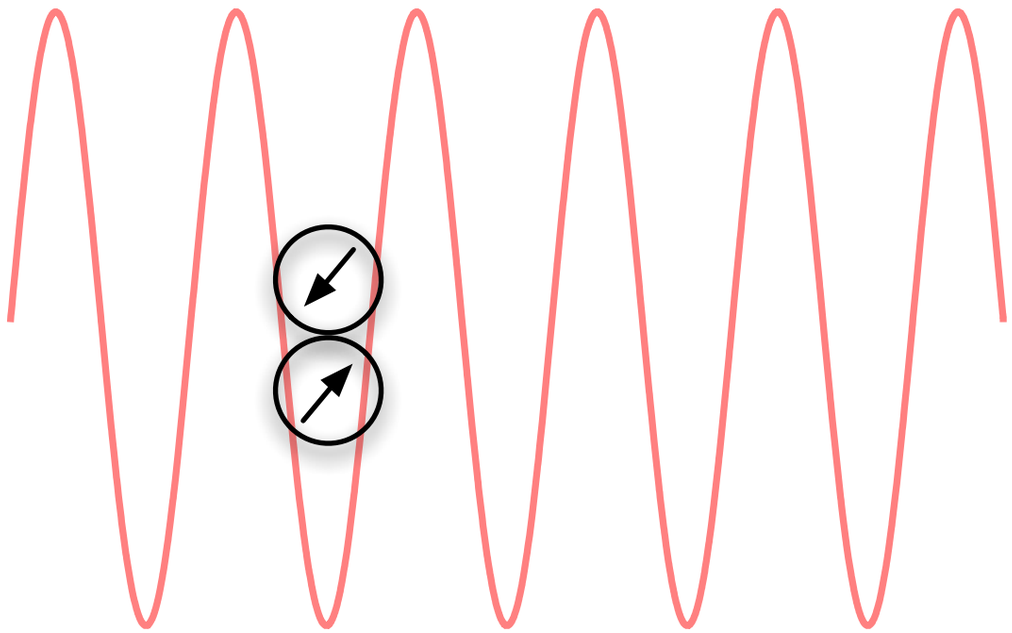

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tunneling s-wave scattering

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Hubbard parameters

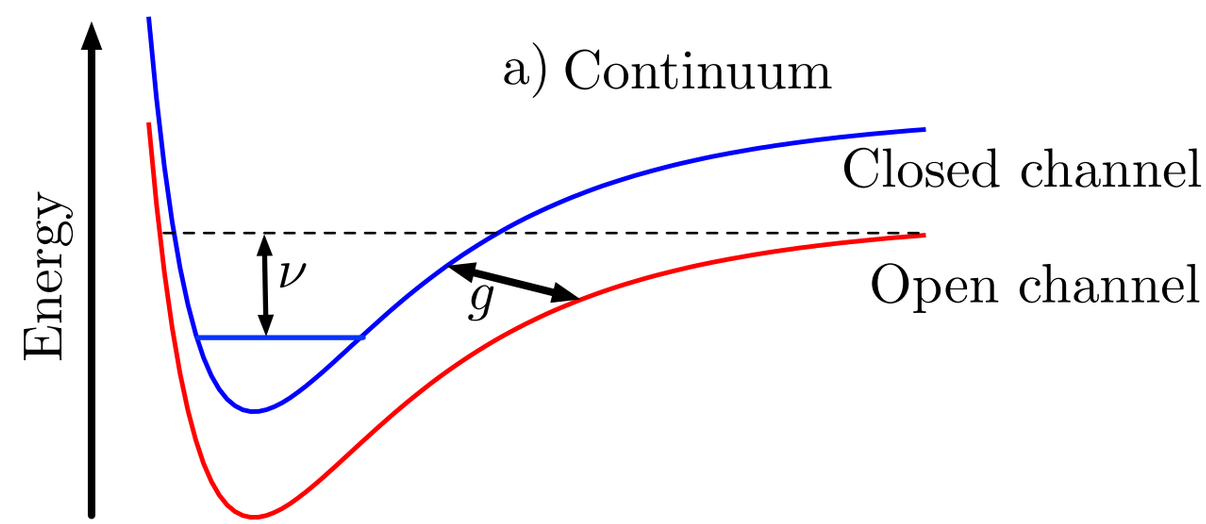
Improved ansatz

- Lattice pseudopotential set by lattice scattering properties
- Includes all bands via the lattice scattering amplitude
- Also allows for renormalization of two-particle theory

$$\lambda_{\text{Hubb.}} = \frac{U}{1 - UG(E)}$$

$$\lim_{E \to 0} \lambda_{\text{Hubb.}} = \lim_{E \to 0} \lambda_{\text{exact}}$$

$$U = \frac{1}{G(0) + 1/\lambda_{\text{exact}}}$$



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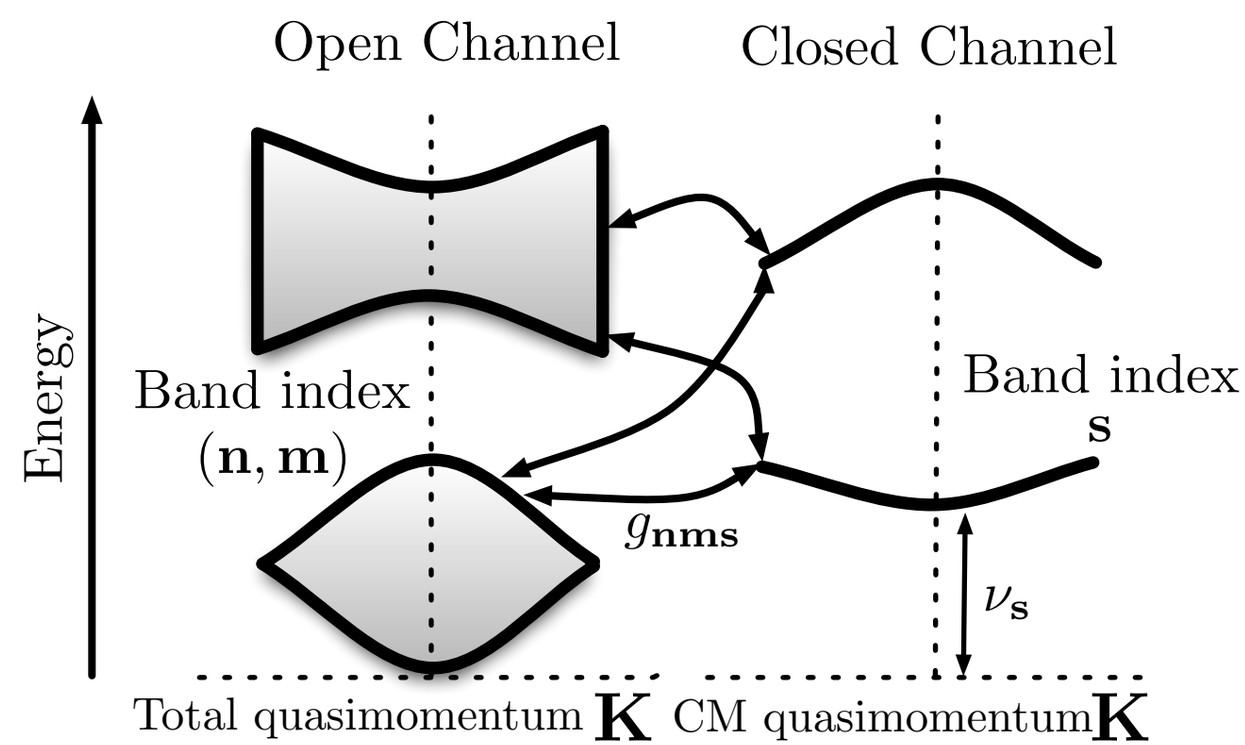

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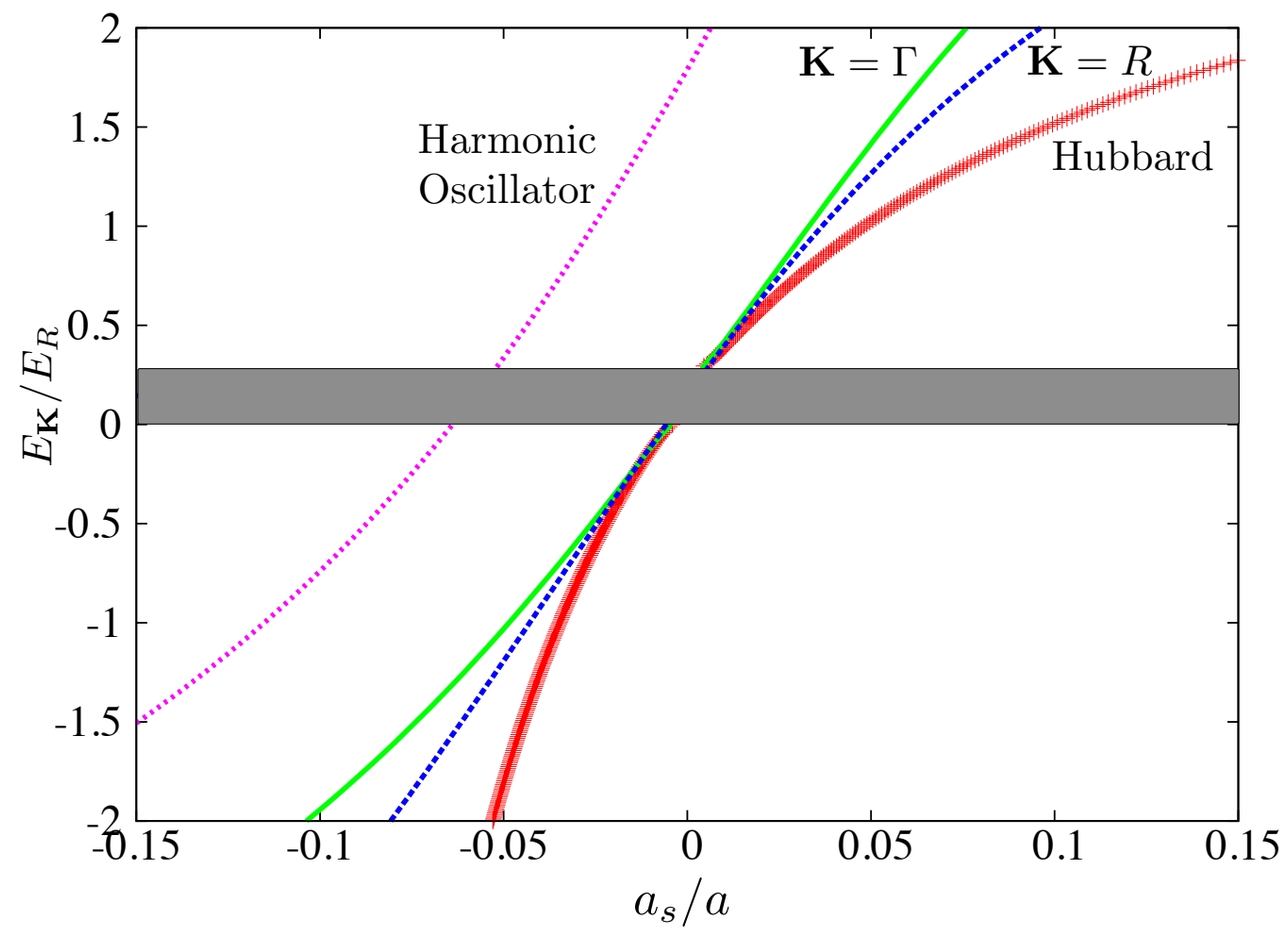


Single vs. two-channel models

Why does it fail?

- Lacks complete momentum dependence of scattering amplitude
- Actual effective potential in two-channel model:

$$U_{\mathbf{k},\mathbf{k}'} = \mathcal{P}\frac{g_{\mathbf{k}}g_{\mathbf{k}'}}{2\epsilon_{\mathbf{k}} - E}$$

$$E = \nu_0 - \mathcal{P}\sum_{\mathbf{k}}\frac{g_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}} - E}$$

- Single channel model lacks residual momentum dependence as coupling becomes pointlike

Solution: Use a two-channel many-body lattice model!

Fermi-Bose Hubbard Hamiltonian

$$\hat{H} = \hat{H}_f + \hat{H}_b + \hat{H}_{fb}$$

Multiband Fermi-Hubbard Hamiltonian

$$\hat{H}_f = -\sum_m t_m^f \sum_{\langle i,j \rangle} \sum_{\sigma \in \{\uparrow,\downarrow\}} \left(\hat{f}^\dagger_{i,\sigma;m} \hat{f}_{j,\sigma;m} + \text{h.c.} \right) - \sum_{m,m'} \left| U^f_{m,m'} \right| \sum_i \hat{n}^f_{i,\uparrow;m} \hat{n}^f_{i,\downarrow;m'} - \sum_{i,\sigma,m} \left(\mu_f - E^f_m \right) \hat{n}^f_{i,\sigma;m}$$

Multiband Bose-Hubbard Hamiltonian

$$\hat{H}_b = -\sum_m t_m^b \sum_{\langle i,j \rangle} \left(\hat{b}^\dagger_{i;m} \hat{b}_{j;m} + \text{h.c.} \right) + \sum_{m,m'} \frac{U^b_{m,m'}}{2} \sum_i \left[\hat{n}^b_{i;m} \left(\hat{n}^b_{i;m'} - \delta_{m,m'} \right) \right] - \sum_{i,\sigma,m} \left(\mu_b - E^b_m \right) \hat{n}^b_{i;m}$$

Bose-Fermi coupling

$$\hat{H}_{fb} = \sum_{i,s,m,m'} g_s^{mm'} \left(\hat{b}^\dagger_{i;s} \hat{f}_{i,\uparrow;m} \hat{f}_{i,\downarrow;m'} + \text{h.c.} \right) + \sum_{i,m,m',\sigma} V_{mm'} \hat{n}^b_{i;m'} \hat{n}^f_{i,\sigma;m}$$

Correct but awful!

- In principle infinite summation over fermionic and molecular bands
- Not amenable to treatment by modern methods

i, j site indices

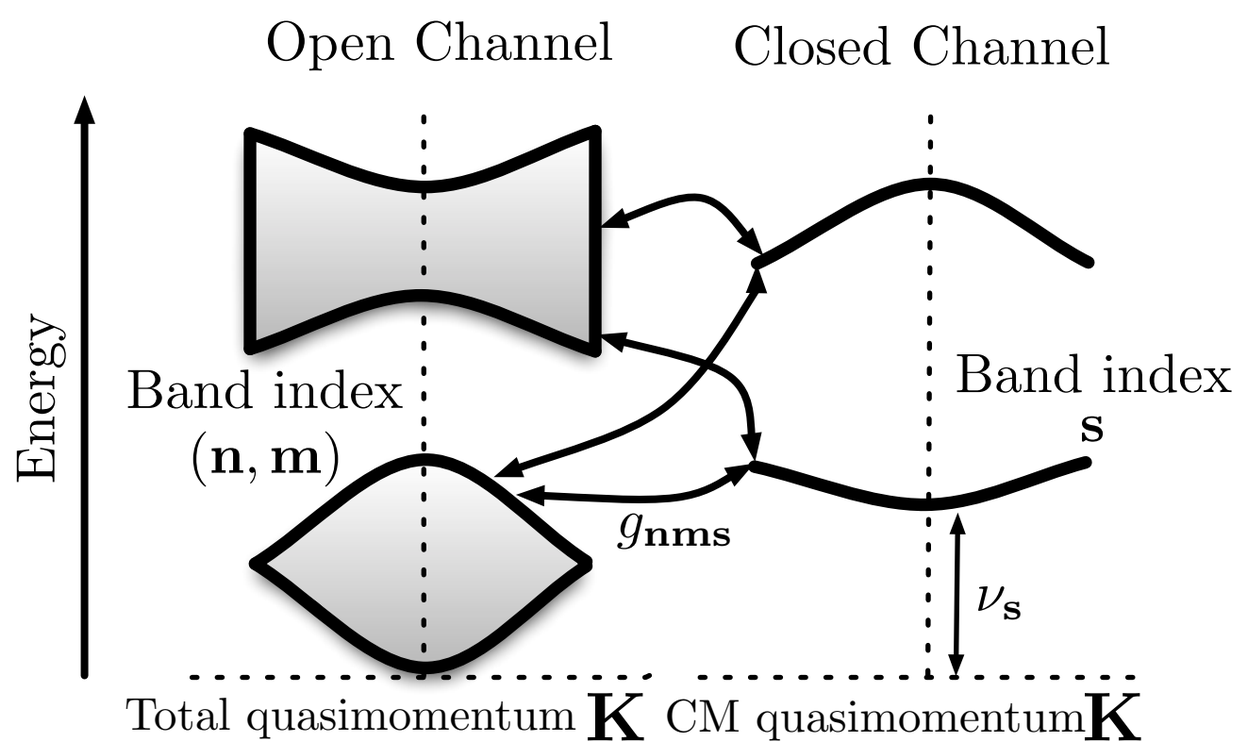
m, m' band indices

σ "spin" indices

What can we do?

Solve carefully chosen two-body problem

- Effective range is small-high energy physics is very short range
- Long wavelength physics captured by lowest band fermions
- At low density, high energy physics is two-body physics
- Recouple at many-particle level



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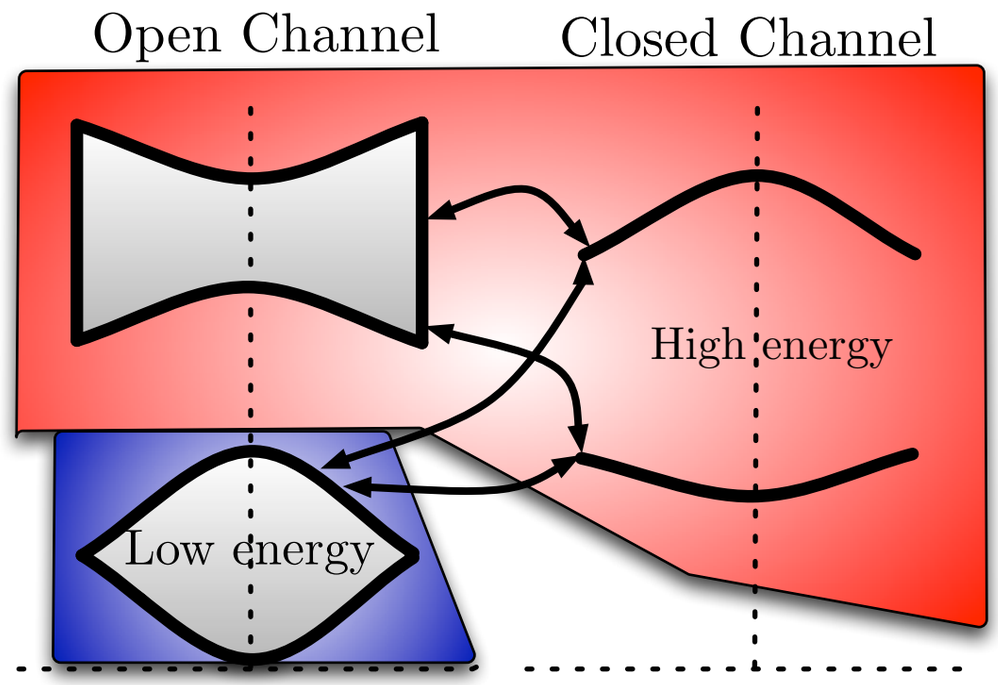
2)Partition into low and high energy spaces



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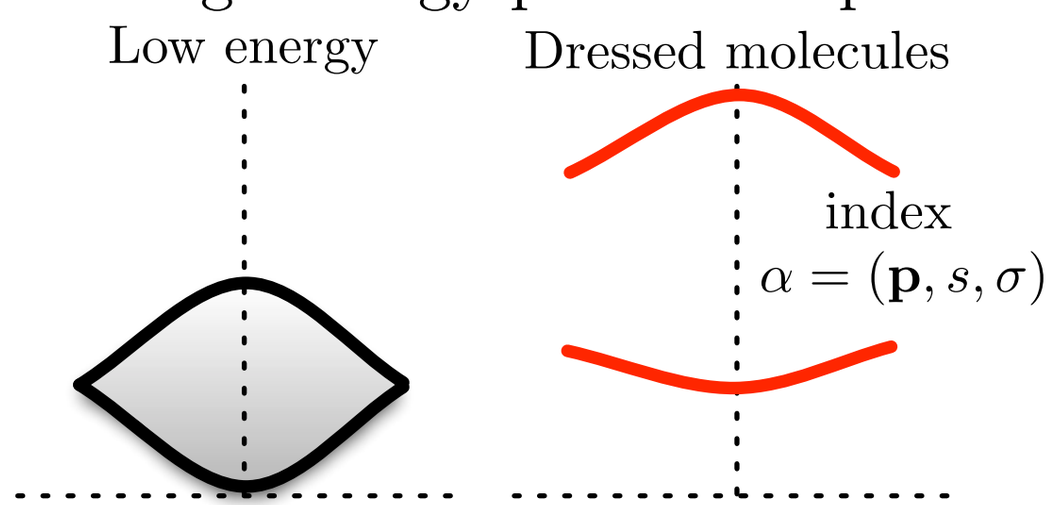
3)Solve high energy piece for 2 particles



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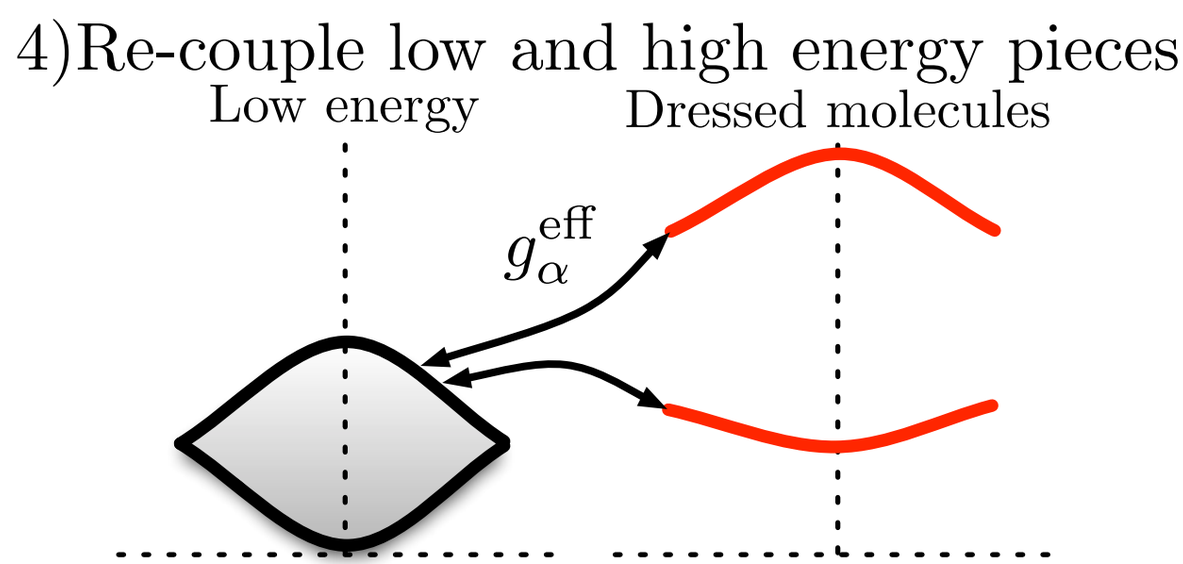
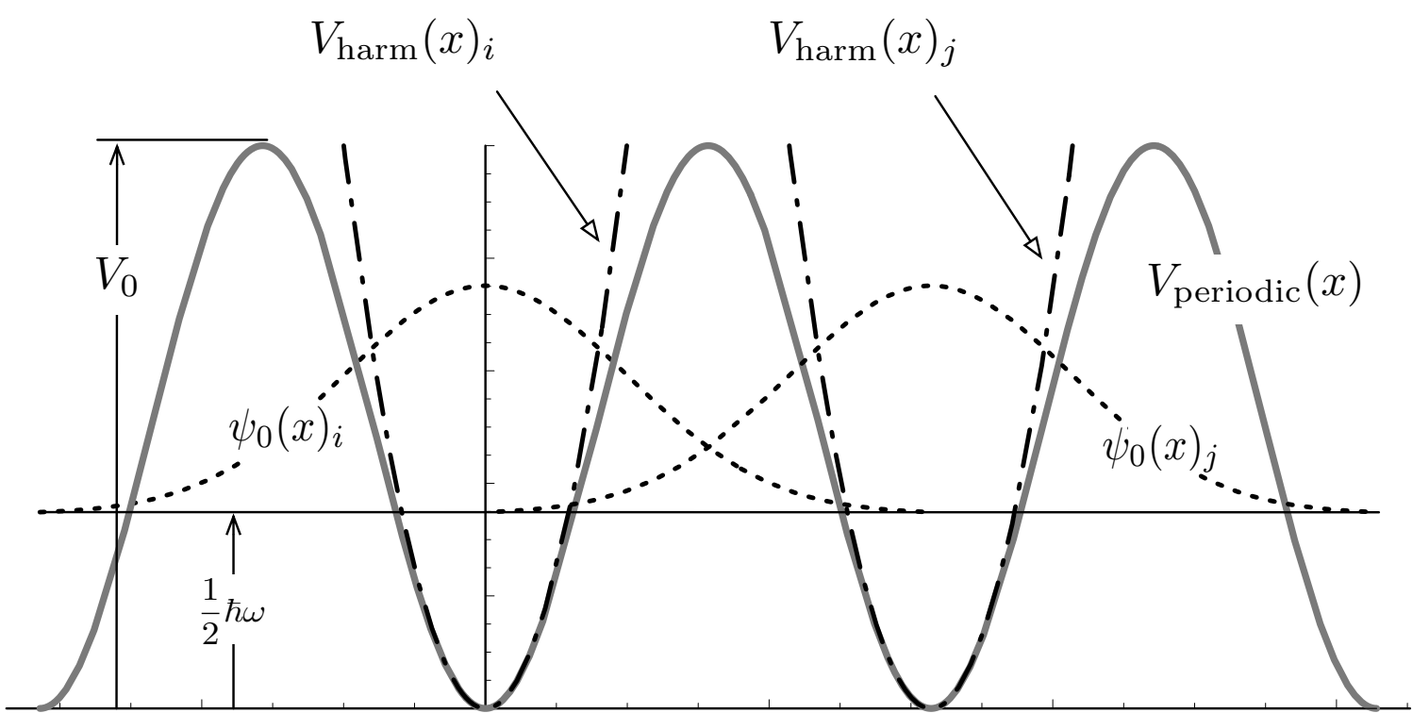
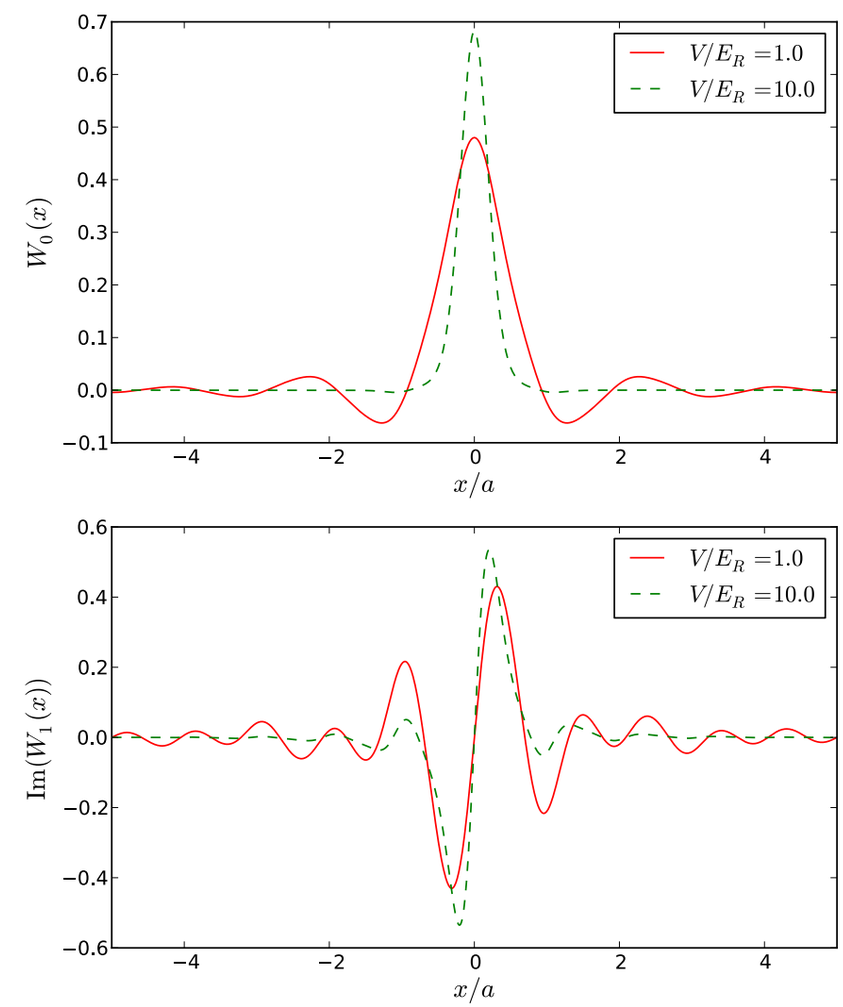


What can we do?

Solve carefully chosen two-body problem

- Effective range is small-high energy physics is very short range
- Long wavelength physics captured by lowest band fermions
- At low density, high energy physics is two-body physics
- Recouple at many-particle level





Two-particle equation

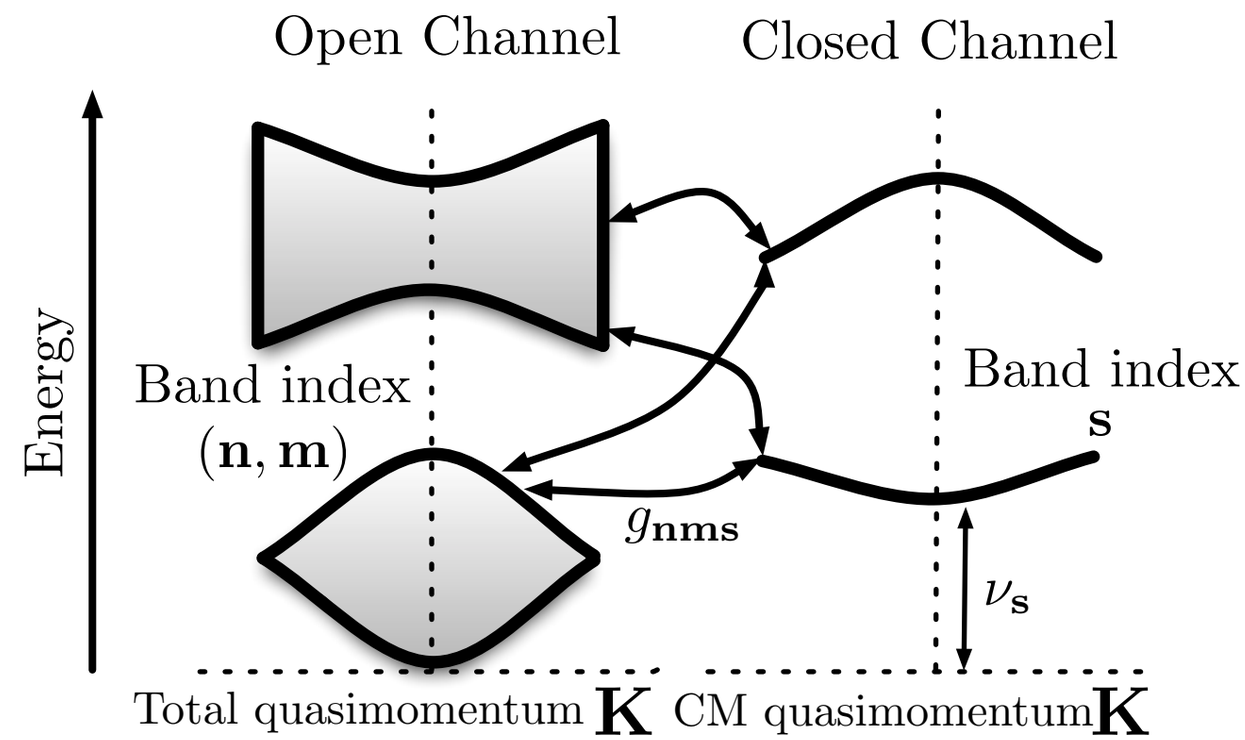
Two channel Feshbach resonance

$$\hat{H}_{\mathrm{o}}|\psi_{\mathrm{o}}\rangle + \hat{g}|\psi_{\mathrm{c}}\rangle = E|\psi_{\mathrm{o}}\rangle$$

$$\left[\hat{H}_{\mathrm{c}} + \nu\right]|\psi_{\mathrm{c}}\rangle + \hat{g}|\psi_{\mathrm{o}}\rangle = E|\psi_{\mathrm{c}}\rangle$$

$$\Rightarrow |\psi_o\rangle = \frac{1}{E - \hat{H}_0 + i\eta}\hat{g}|\psi_c\rangle$$

$$\left[E - \nu - \hat{H}_c\right]|\psi_c\rangle = \hat{g}\frac{1}{E - \hat{H}_0 + i\eta}\hat{g}|\psi_c\rangle$$



- Open channel: product of Bloch functions with total quasimomentum fixed
- Closed channel: Bloch functions with twice the mass and twice the lattice potential

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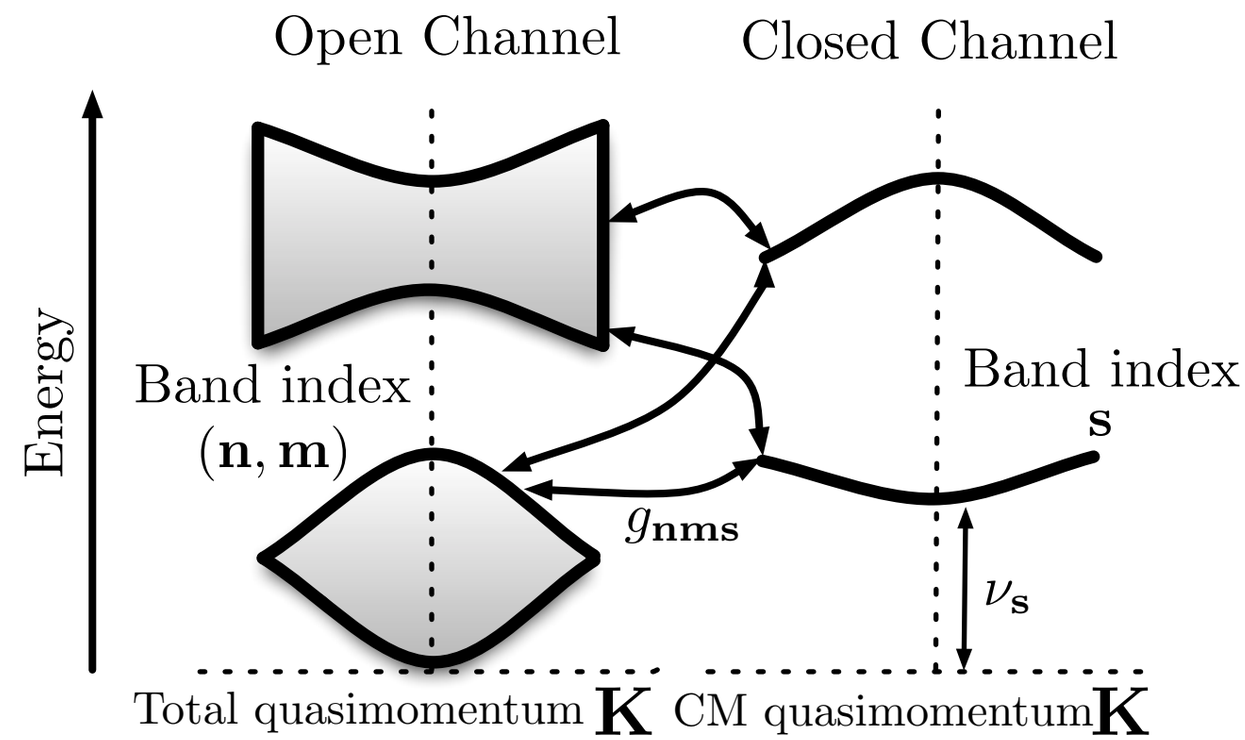

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- Closed channel: Bloch functions with twice the mass and twice the lattice potential

$$\hat{H}_o|\mathbf{K},\mathbf{q};\mathbf{n},\mathbf{m}\rangle = E^{\mathbf{K}}_{\mathbf{nm}}(\mathbf{q})\,|\mathbf{K},\mathbf{q};\mathbf{n},\mathbf{m}\rangle$$

$$\hat{H}_c|\mathbf{K};\mathbf{s}\rangle = E^{(b)}_{\mathbf{sK}}|\mathbf{K};\mathbf{s}\rangle$$

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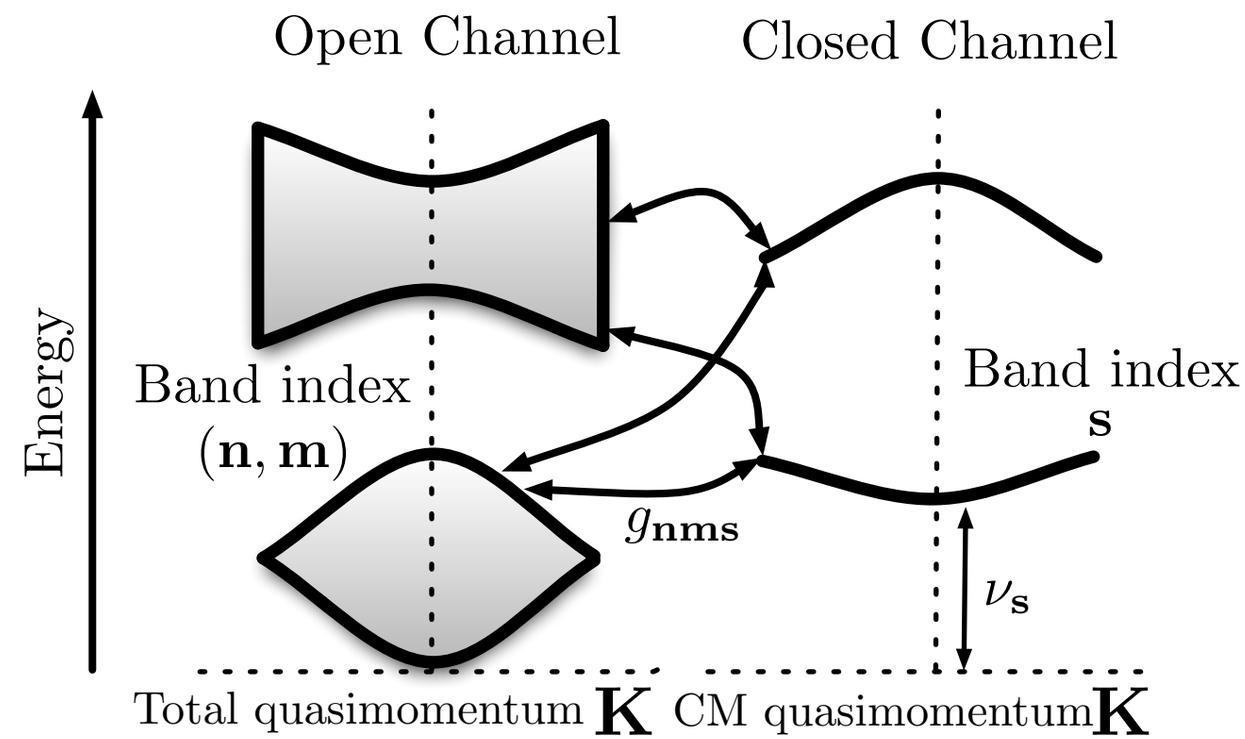

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$$\hat{g} = g\alpha_\Lambda(\mathbf{r})$$

$$\lim_{\Lambda\to\infty}\alpha_\Lambda(\mathbf{r}) = \delta(\mathbf{r})$$

$$h^{\mathbf{nm}}_{\mathbf{sK}}(\mathbf{q}) = \langle\mathbf{K};\mathbf{s}|\hat{g}|\mathbf{K},\mathbf{q};\mathbf{n},\mathbf{m}\rangle$$

Two-particle equation

Project into lattice basis

$$\left[E-\nu-\hat{H}_c\right]|\psi_c\rangle=\hat{g}\frac{1}{E-\hat{H}_0+i\eta}\hat{g}|\psi_c\rangle$$

$$\hat{H}_o|\mathbf{K},\mathbf{q};\mathbf{n},\mathbf{m}\rangle=E^{\mathbf{K}}_{\mathbf{nm}}(\mathbf{q})\,|\mathbf{K},\mathbf{q};\mathbf{n},\mathbf{m}\rangle$$

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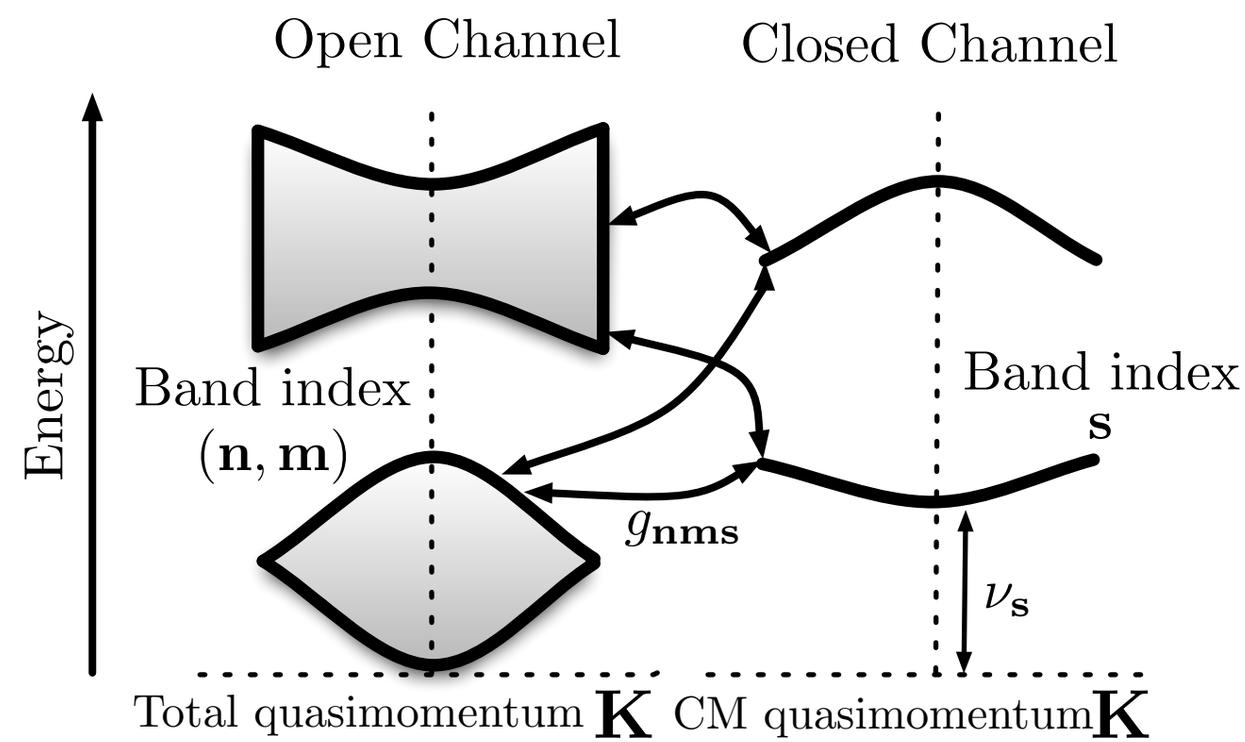

$$\left[E_{\mathbf{K}}-\nu-E^{(b)}_{\mathbf{sK}}\right]\Upsilon^{\mathbf{K}}_{\mathbf{s}}=\frac{g^2}{v}\sum_{\mathbf{t}}\chi^{\mathbf{K}}_{\mathbf{st}}(E_{\mathbf{K}})\,\Upsilon^{\mathbf{K}}_{\mathbf{t}}$$

$$\chi^{\mathbf{K}}_{\mathbf{st}}(E_{\mathbf{K}})=\sum_{\mathbf{nm}}\int\frac{d\mathbf{q}}{v_{\mathrm{BZ}}}\frac{h^{\mathbf{nm}}_{\mathbf{sK}}(\mathbf{q})\,h^{\mathbf{nm}}_{\mathbf{tK}}(\mathbf{q})}{E_{\mathbf{K}}-E^{\mathbf{K}}_{\mathbf{nm}}(\mathbf{q})+i\eta}$$

Regularization

$$\left[E_{\mathbf{K}} - \nu - E^{(b)}_{\mathbf{sK}}\right] \Upsilon^{\mathbf{K}}_{\mathbf{s}} = \frac{g^2}{v} \sum_{\mathbf{t}} \chi^{\mathbf{K}}_{\mathbf{st}} \left(E_{\mathbf{K}}\right) \Upsilon^{\mathbf{K}}_{\mathbf{t}}$$

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- Divergence of RHS as coupling becomes pointlike
- Renormalization is χ without the optical lattice
- Take limit bands->infinity, then cutoff->infinity

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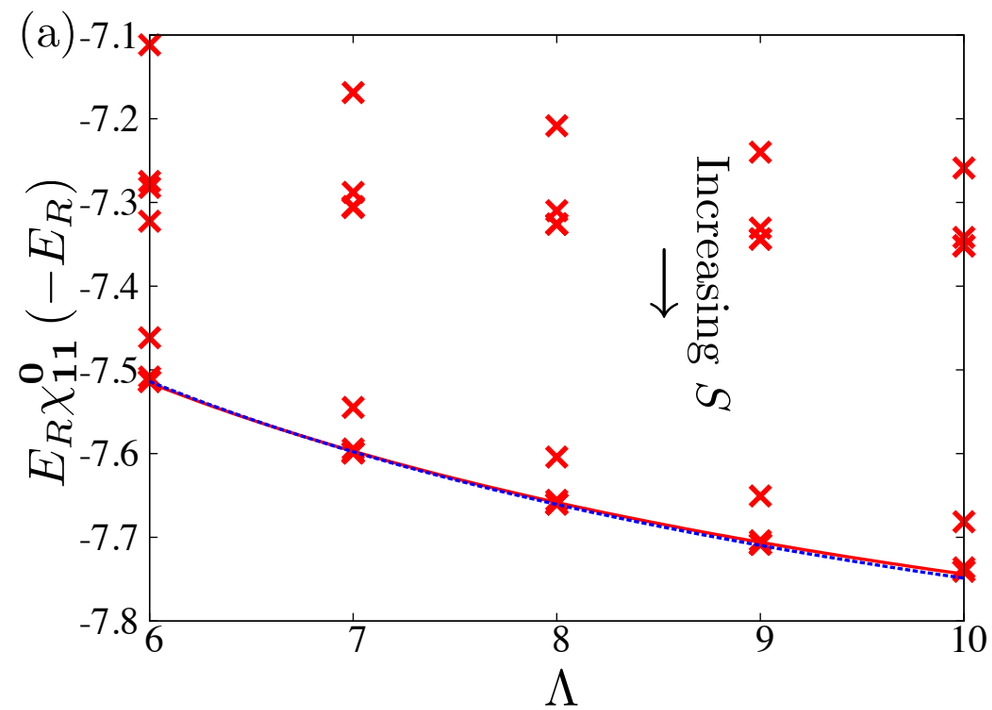

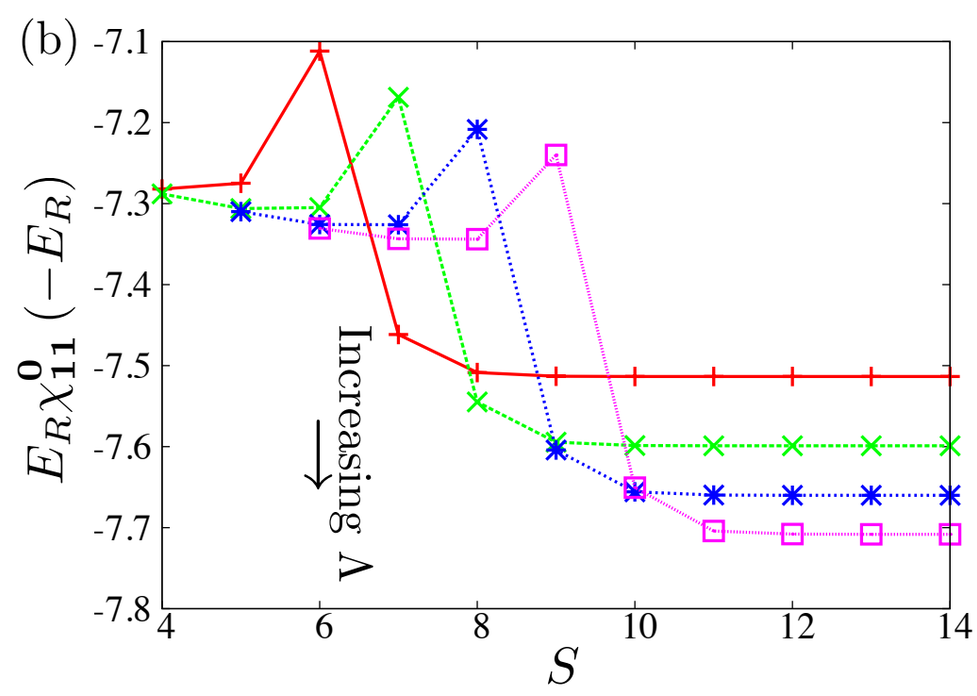

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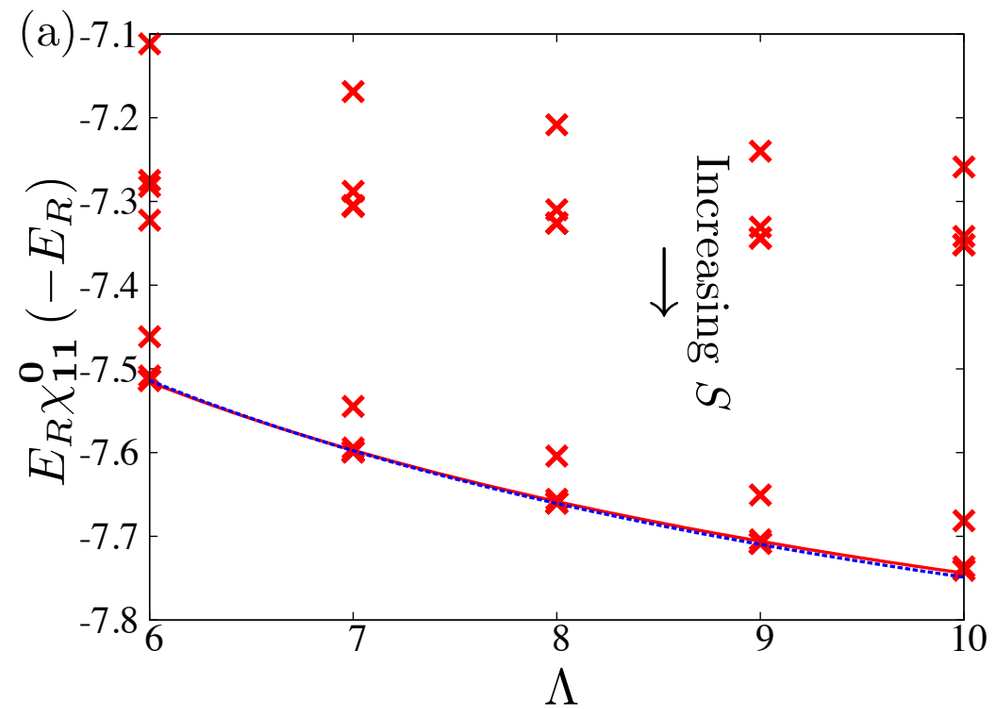

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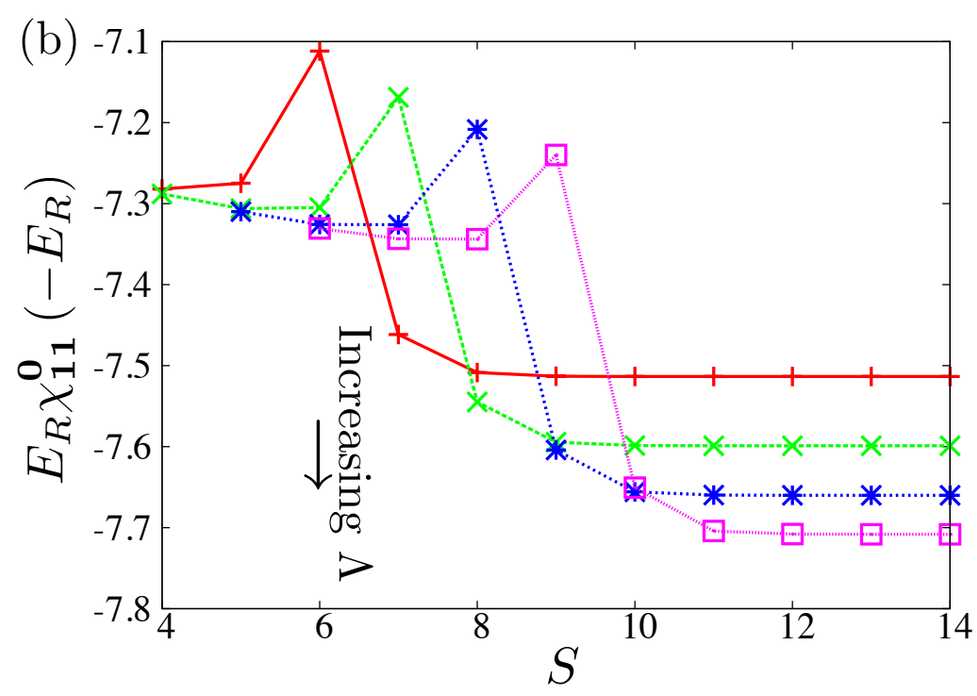
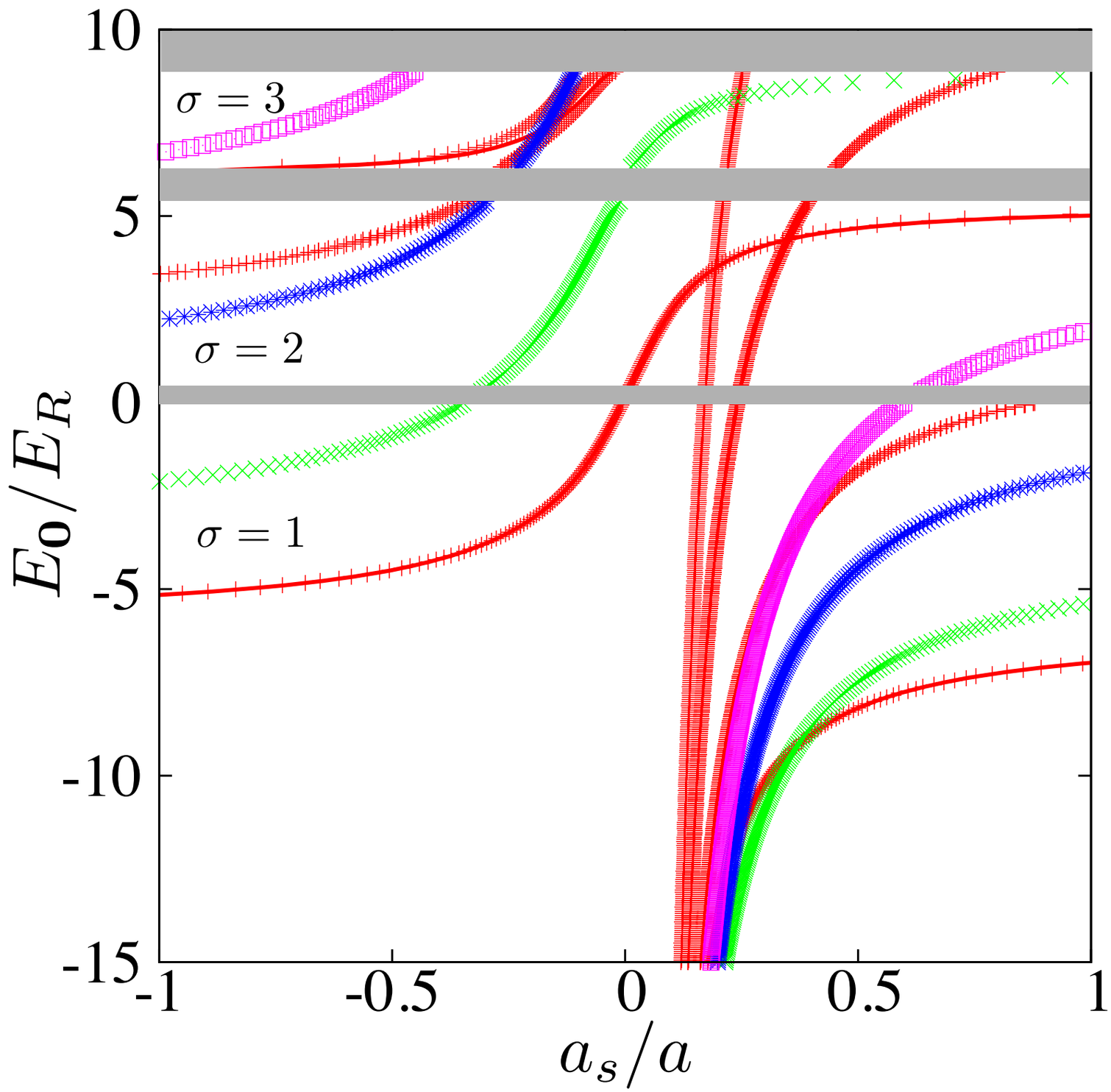
$$a_s = -mg^2/4\pi\hbar^2\nu \ , \quad E_R = \hbar^2\pi^2/2ma^2$$

$$\Rightarrow \boxed{\left(8a_s E_R/\pi a\right) \left[\chi^{\mathbf{K}}\left(E_{\mathbf{K}}\right) - \bar{\chi}^{\mathbf{K}}\right] \boldsymbol{\Upsilon}^{\mathbf{K}} - \boldsymbol{\Upsilon}^{K} = 0}$$

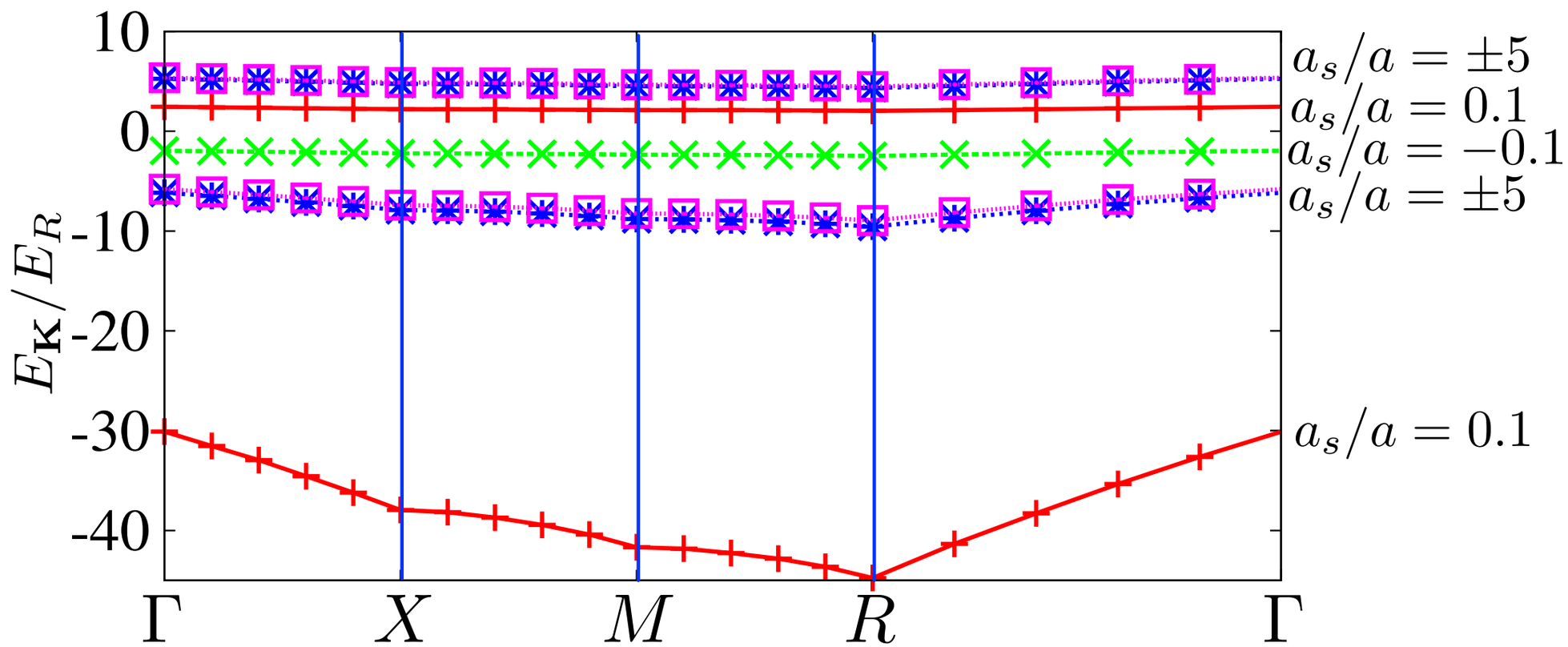


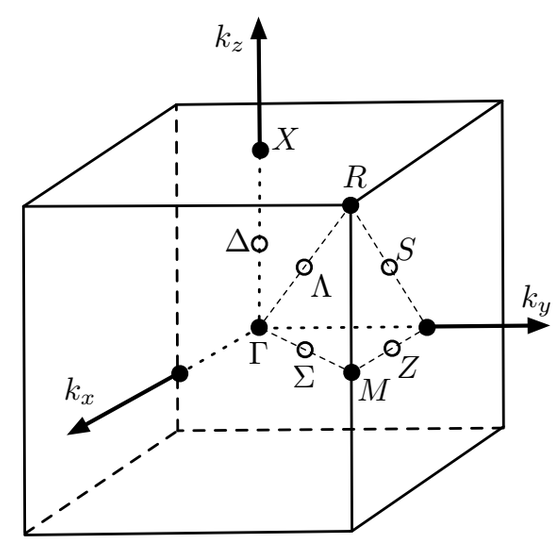
Nonlinear eigenvalue problem!

Results for K=0



“Band structures” for general quasimomentum





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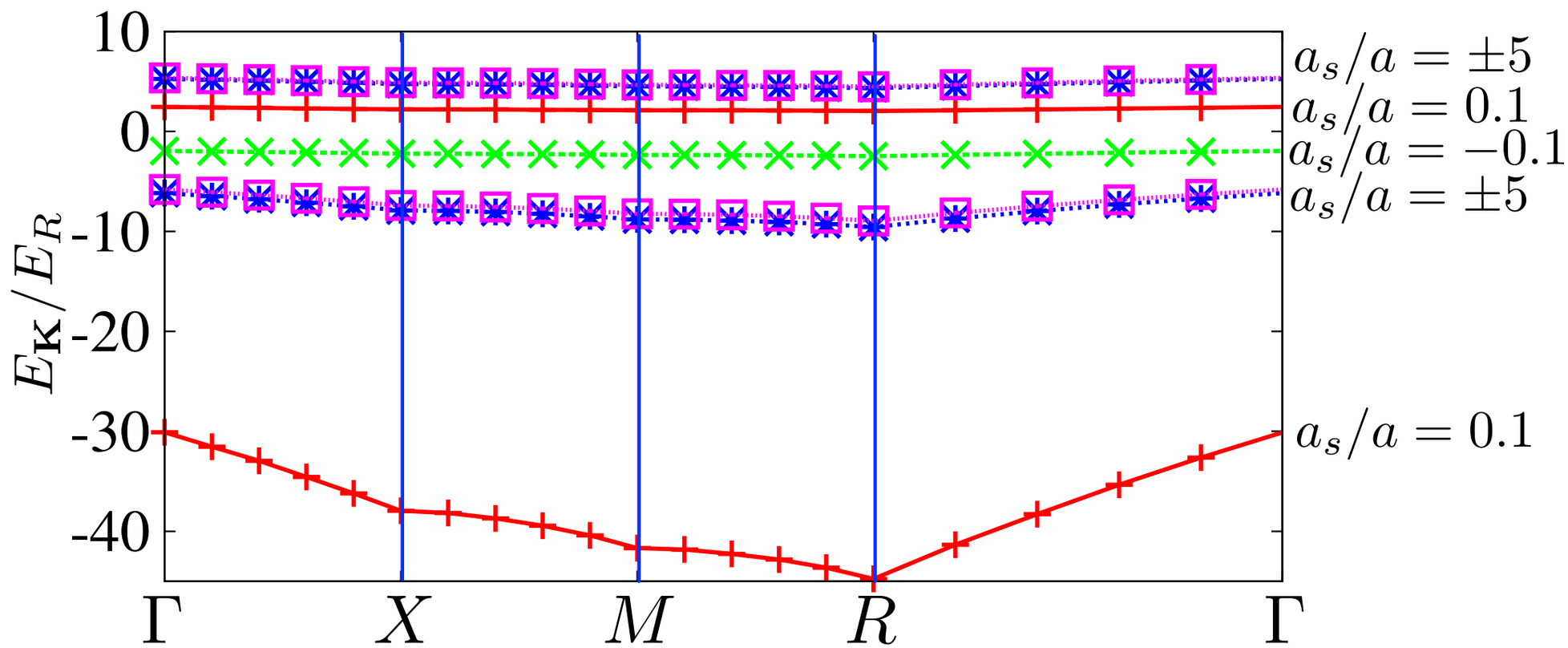
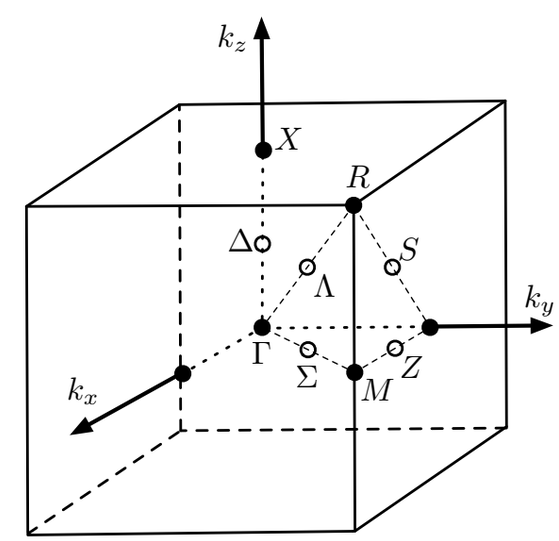




Universality

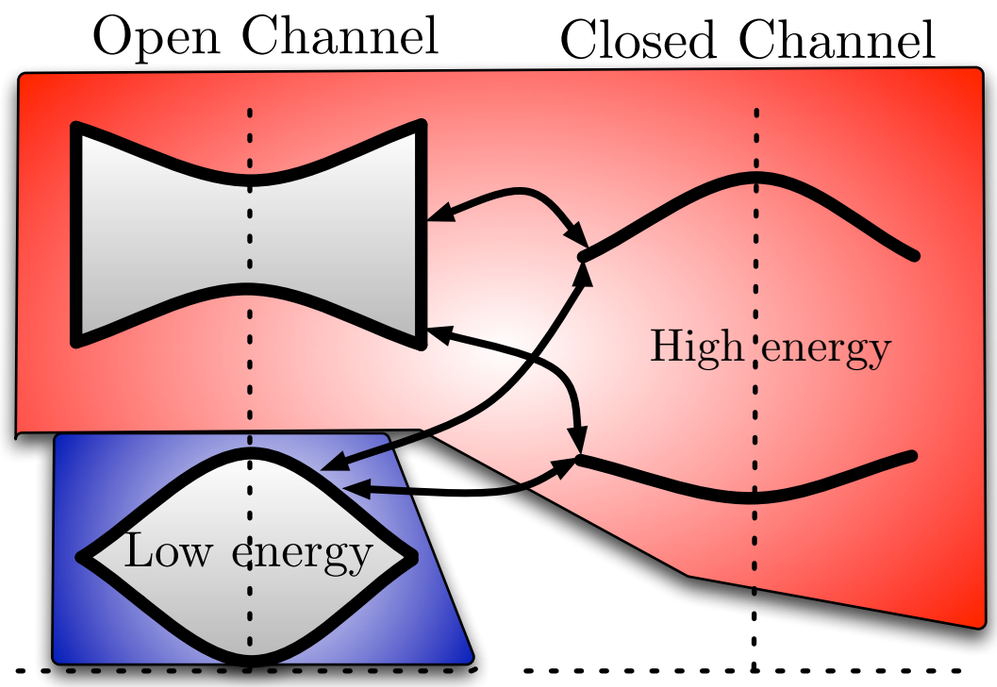
Non-separability

Projection

$$\left[E_{\mathbf{K}} - \bar{\nu} - E^{(b)}_{\mathbf{sK}}\right] \Upsilon^{\mathbf{K}}_{\mathbf{s}} = \frac{g^2}{v} \sum_{\mathbf{t}} \left[\tilde{\chi}^{\mathbf{K}}_{\mathbf{st}}\left(E_{\mathbf{K}}\right) - \bar{\chi}^{\mathbf{K}}_{\mathbf{st}}\right] \Upsilon^{\mathbf{K}}_{\mathbf{t}}$$

$$\tilde{\chi}^{\mathbf{K}}_{\mathbf{st}}\left(E_{\mathbf{K}}\right) = \sum_{\mathbf{nm}\neq(\mathbf{1},\mathbf{1})} \int \frac{d\mathbf{q}}{v_{\mathrm{BZ}}} \frac{h^{\mathbf{nm}}_{\mathbf{sK}}\left(\mathbf{q}\right) h^{\mathbf{nm}}_{\mathbf{tK}}\left(\mathbf{q}\right)}{E_{\mathbf{K}} - E^{\mathbf{K}}_{\mathbf{nm}}\left(\mathbf{q}\right) + i\eta}$$

2)Partition into low and high energy spaces



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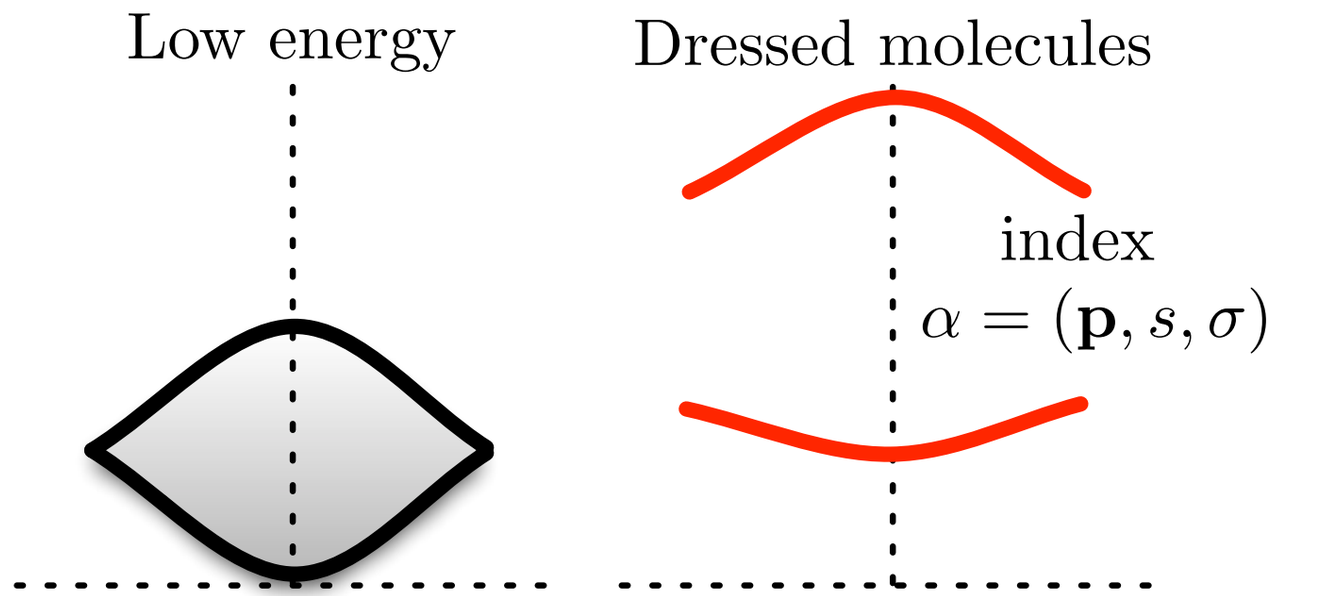
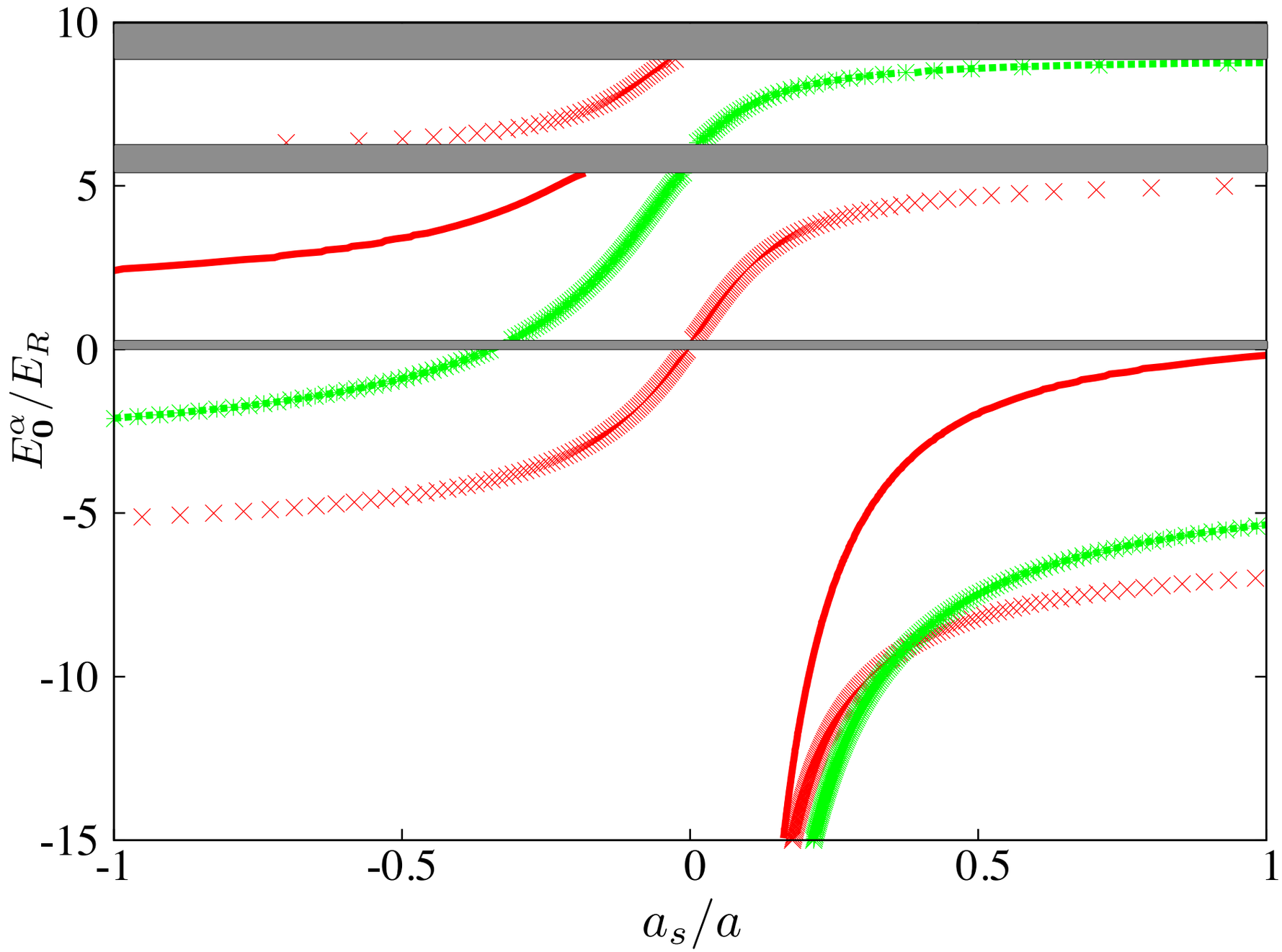
- Compute χ without contributions from lowest open channel band
- Renormalization is same as full two-body problem
- Scattering length obtained is the true two-body scattering length
- Same scaling analysis in cutoff applies
- Together with dynamical lowest band fermions, correctly reproduces all scattering states in the lowest band and nearby bound states
- If higher scattering bands are relevant, project them out and include dynamically
- Energetic range of model can be extended arbitrarily

Projection

3)Solve high energy piece for 2 particles

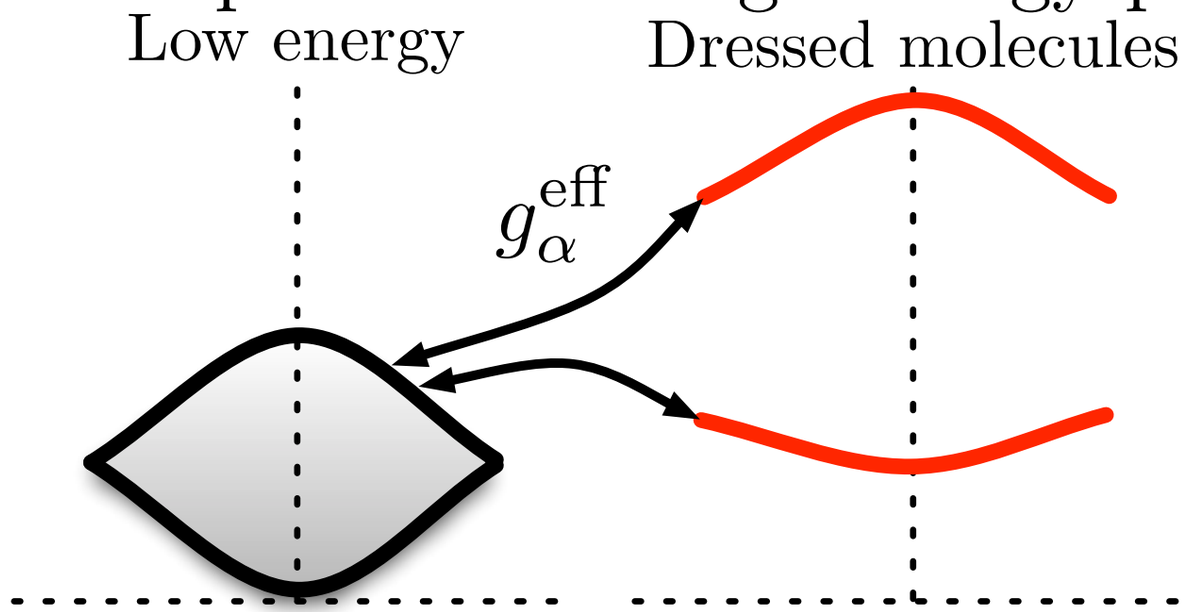


Projection



Projection

4)Re-couple low and high energy pieces



The Fermi resonance Hamiltonian

$$\hat{H}_{\text{eff}} = -t_f \sum_{\sigma\in\{\uparrow,\downarrow\}} \sum_{\langle i,j\rangle} \hat{a}^\dagger_{i\sigma}\hat{a}_{j\sigma} + E_0 \sum_{\sigma\in\{\uparrow,\downarrow\}} \sum_i \hat{n}^{(f)}_{i\sigma} - \sum_{\alpha\in\mathcal{M}} \sum_{i,j} t^\alpha_{i,j} \hat{d}^\dagger_{i,\alpha}\hat{d}_{j,\alpha} + \sum_{\alpha\in\mathcal{M}} \bar{\nu}_\alpha \sum_i \hat{n}^{(b)}_{i\alpha}$$

$$+ \sum_{\alpha\in\mathcal{M}} \sum_{ijk} g^\alpha_{i-j,i-k} \left[\hat{d}^\dagger_{i,\alpha}\hat{a}_{j,\uparrow}\hat{a}_{k,\downarrow} + \text{h.c.}\right]$$

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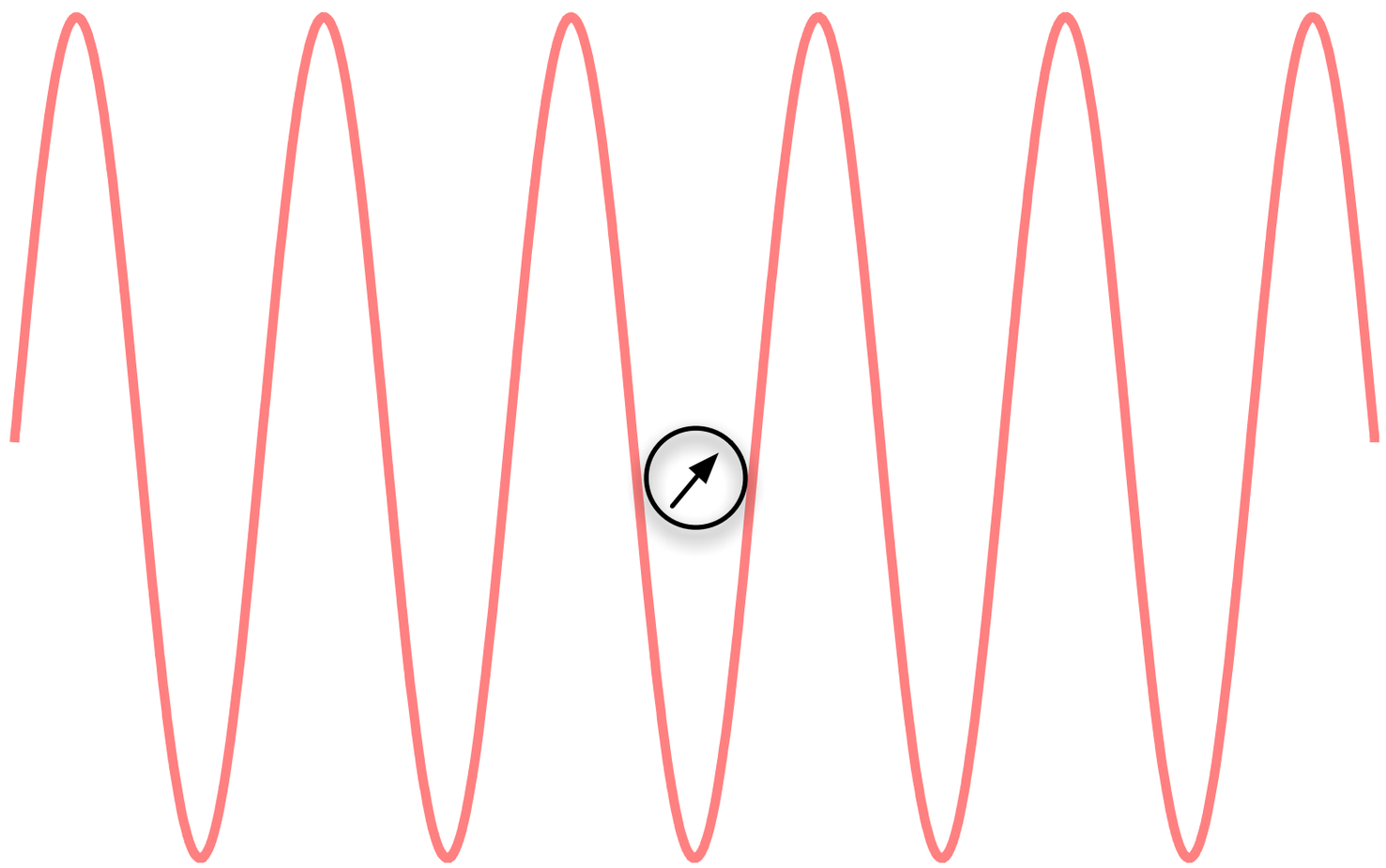

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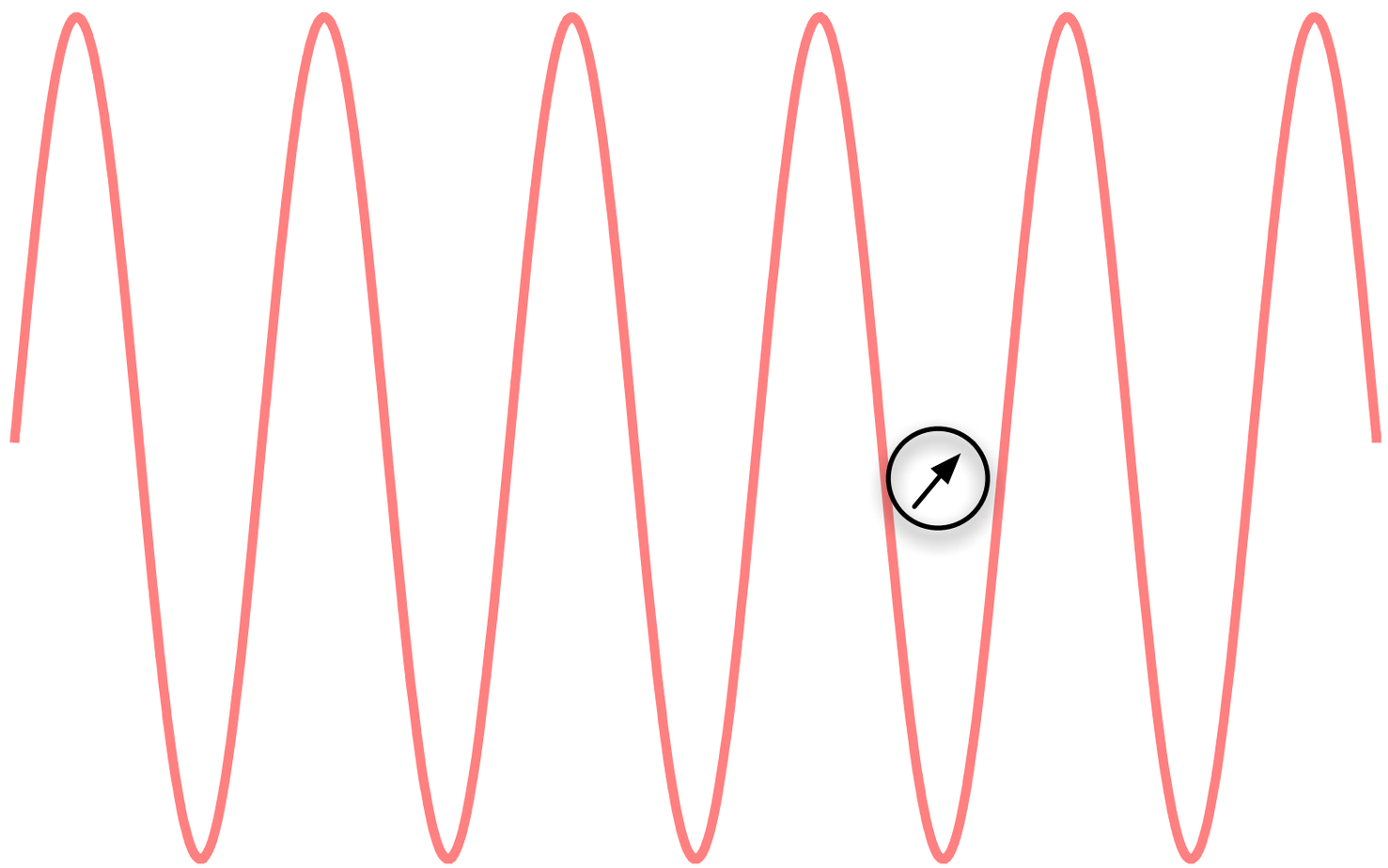

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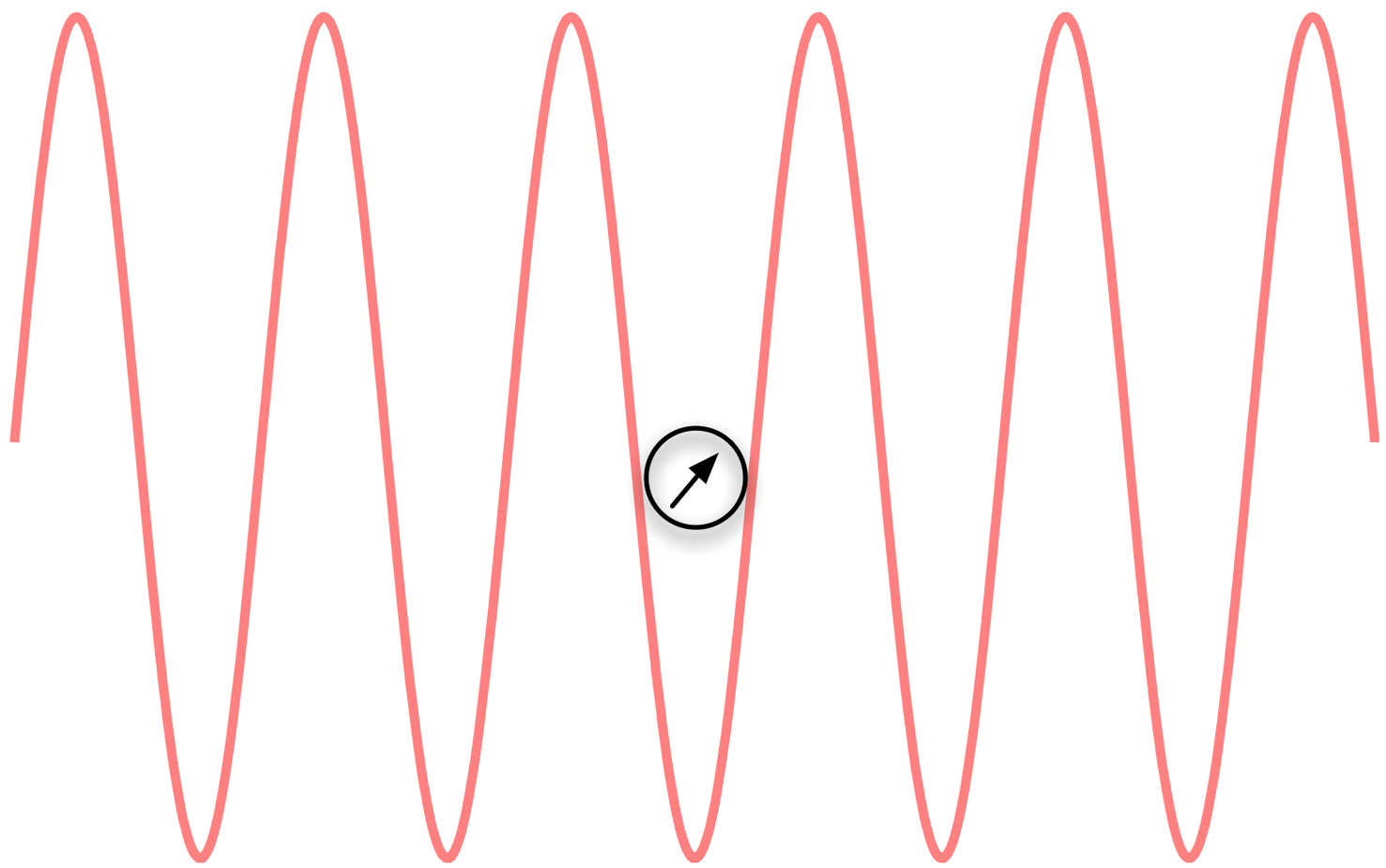

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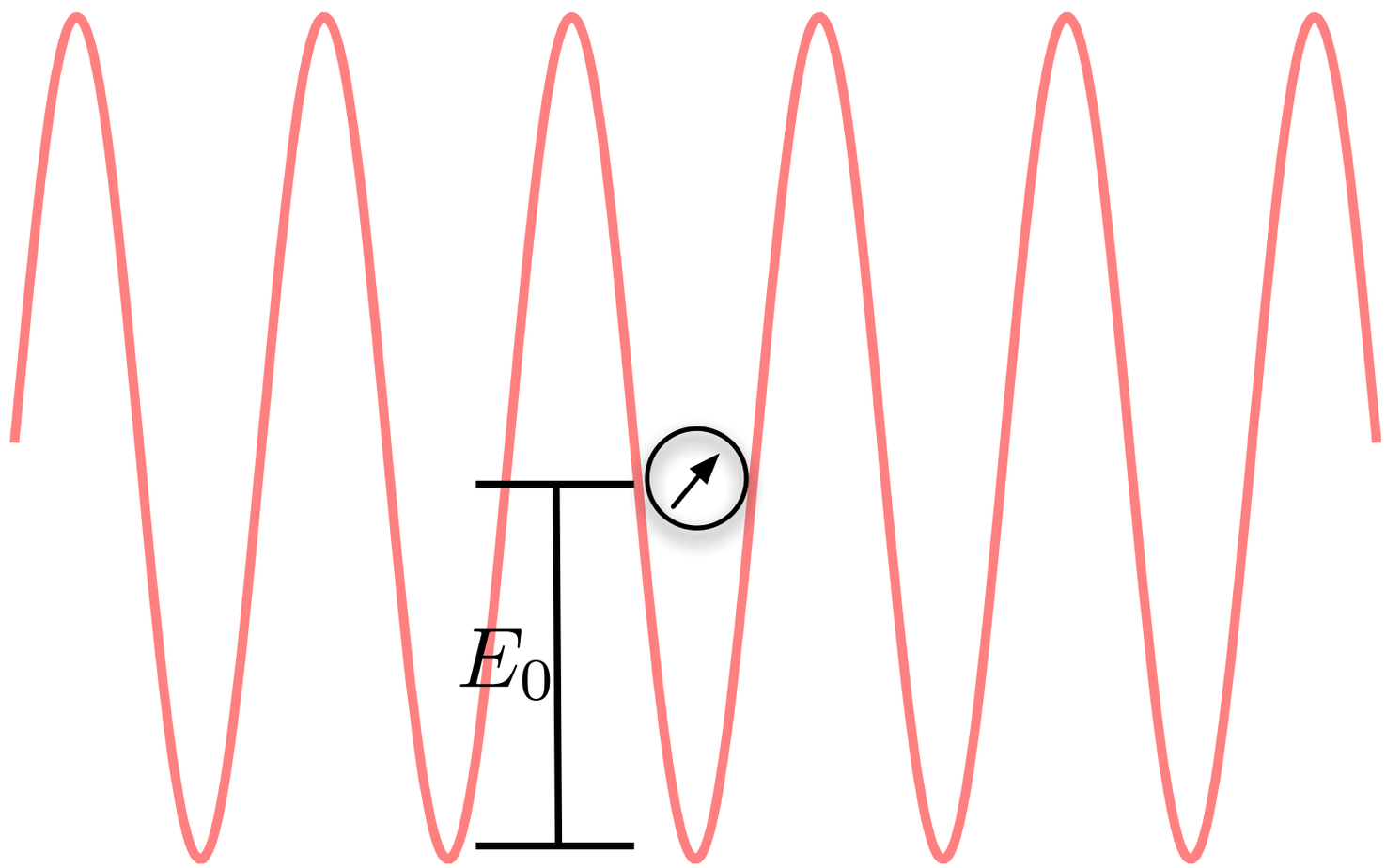

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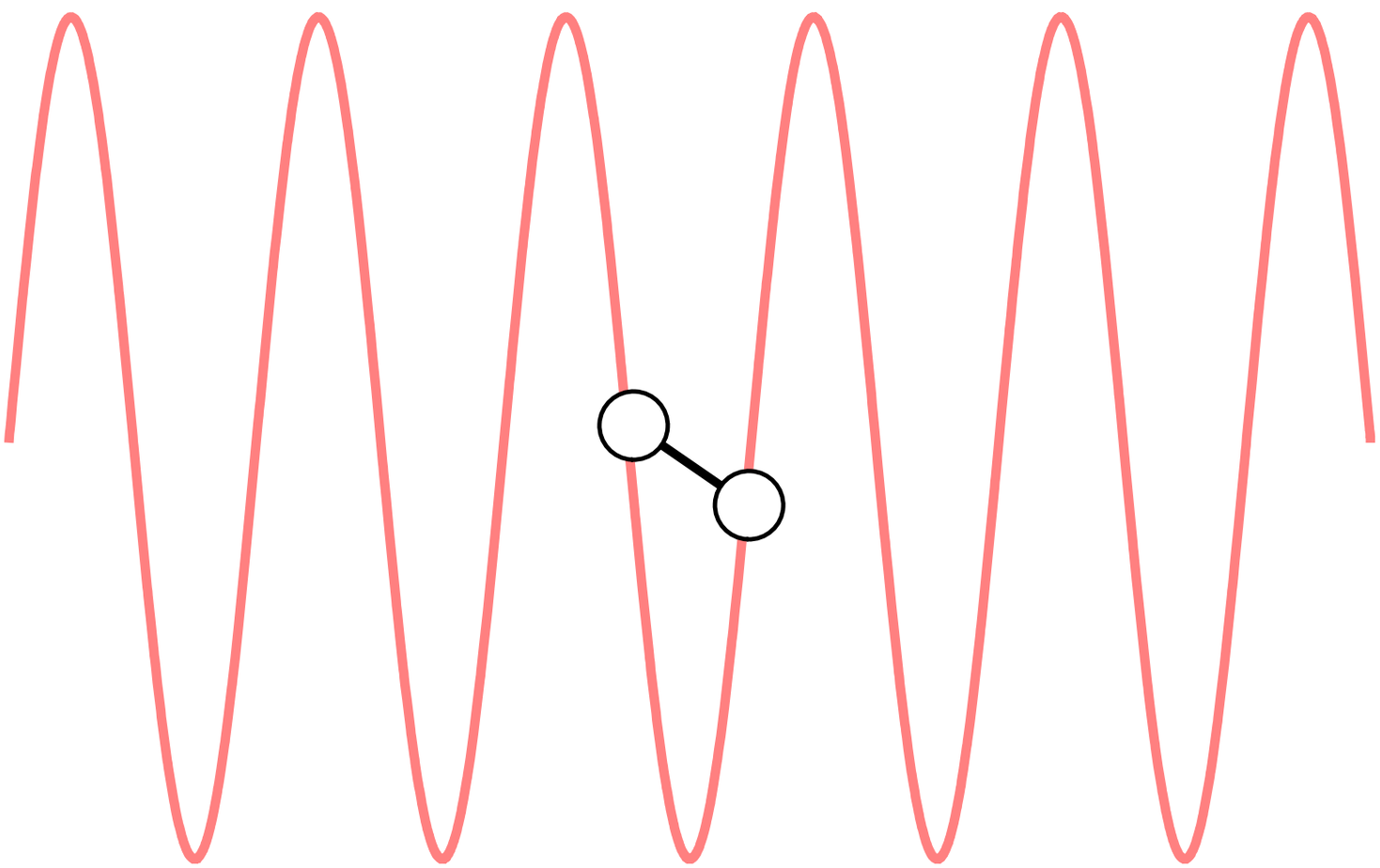

The Fermi resonance Hamiltonian

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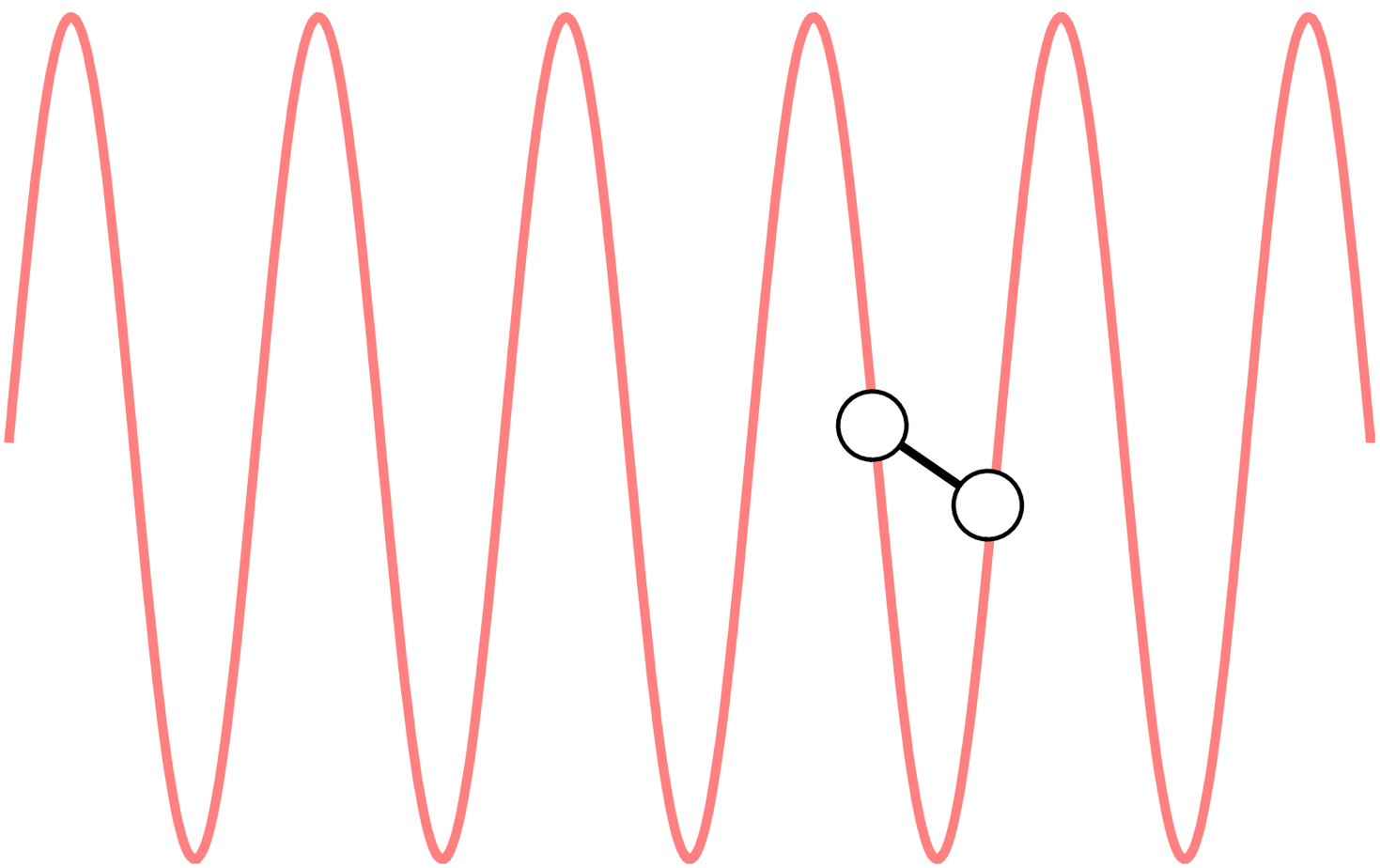

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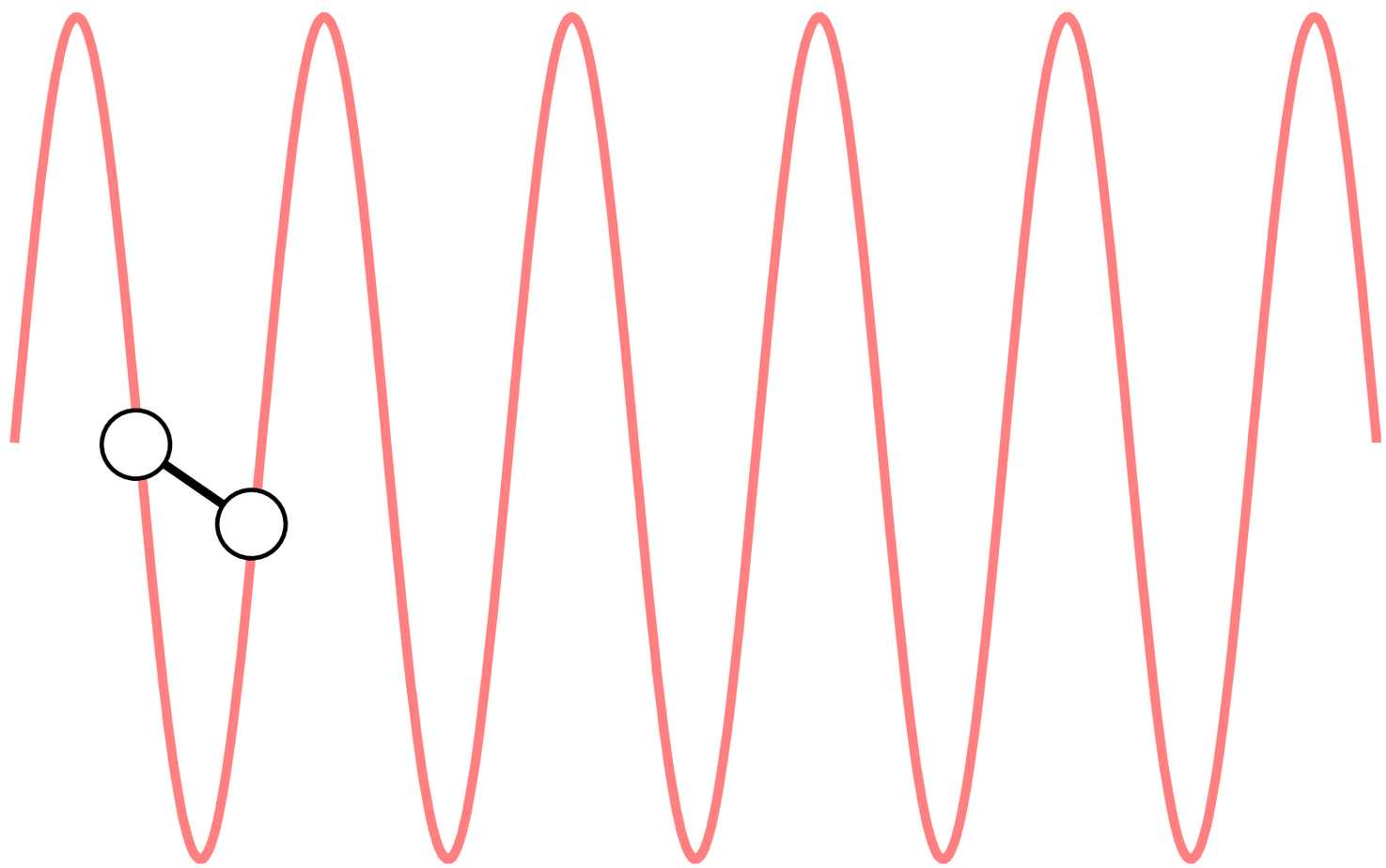

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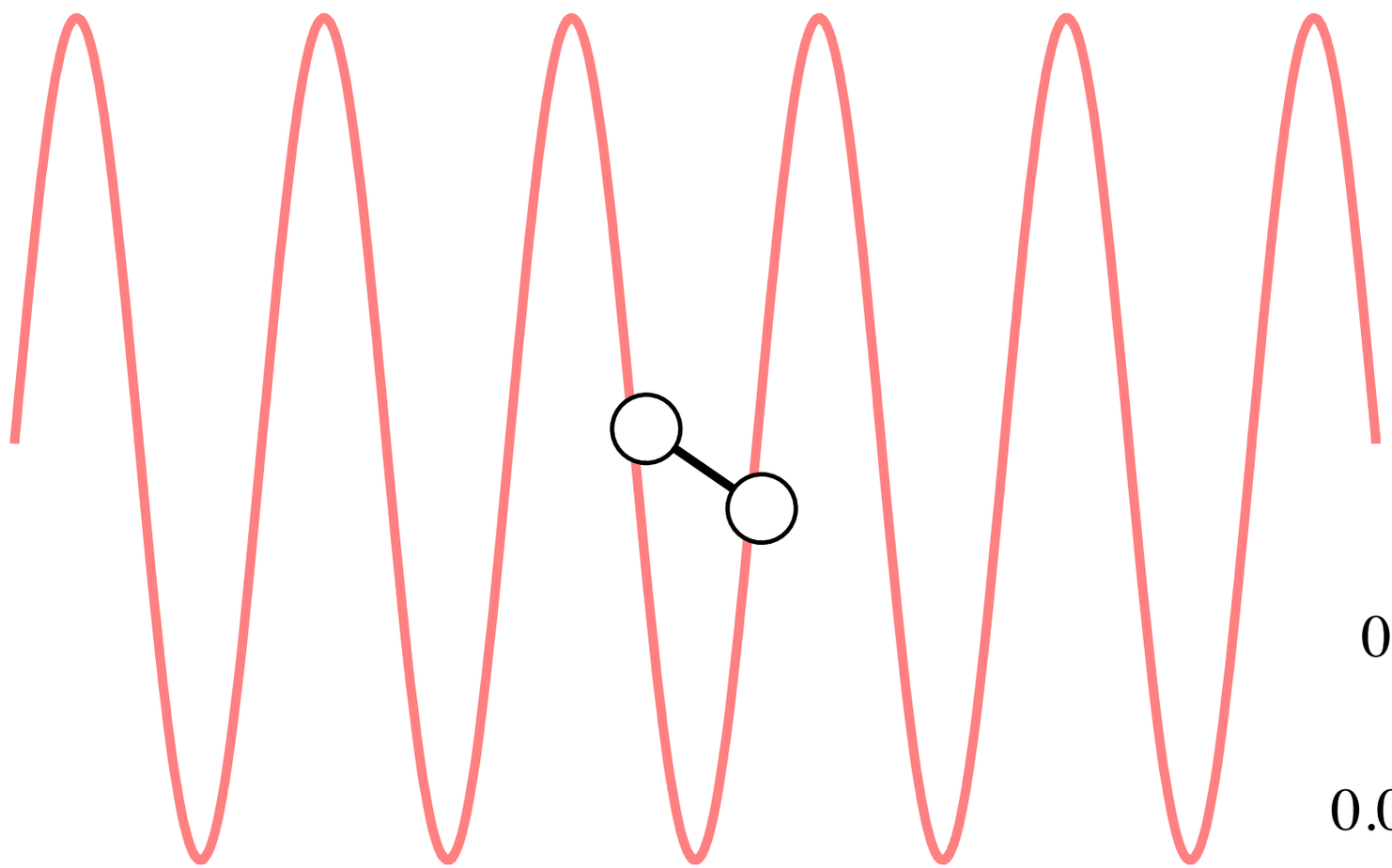

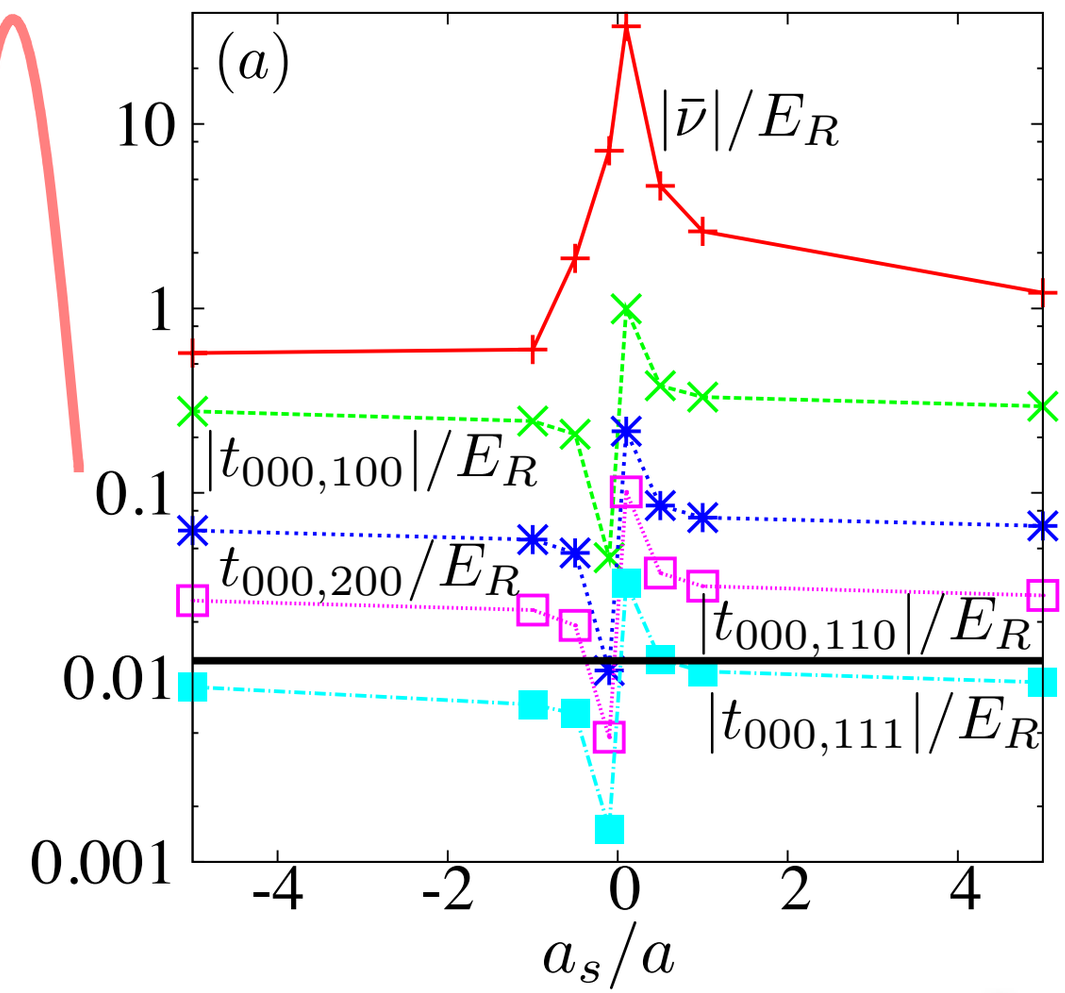

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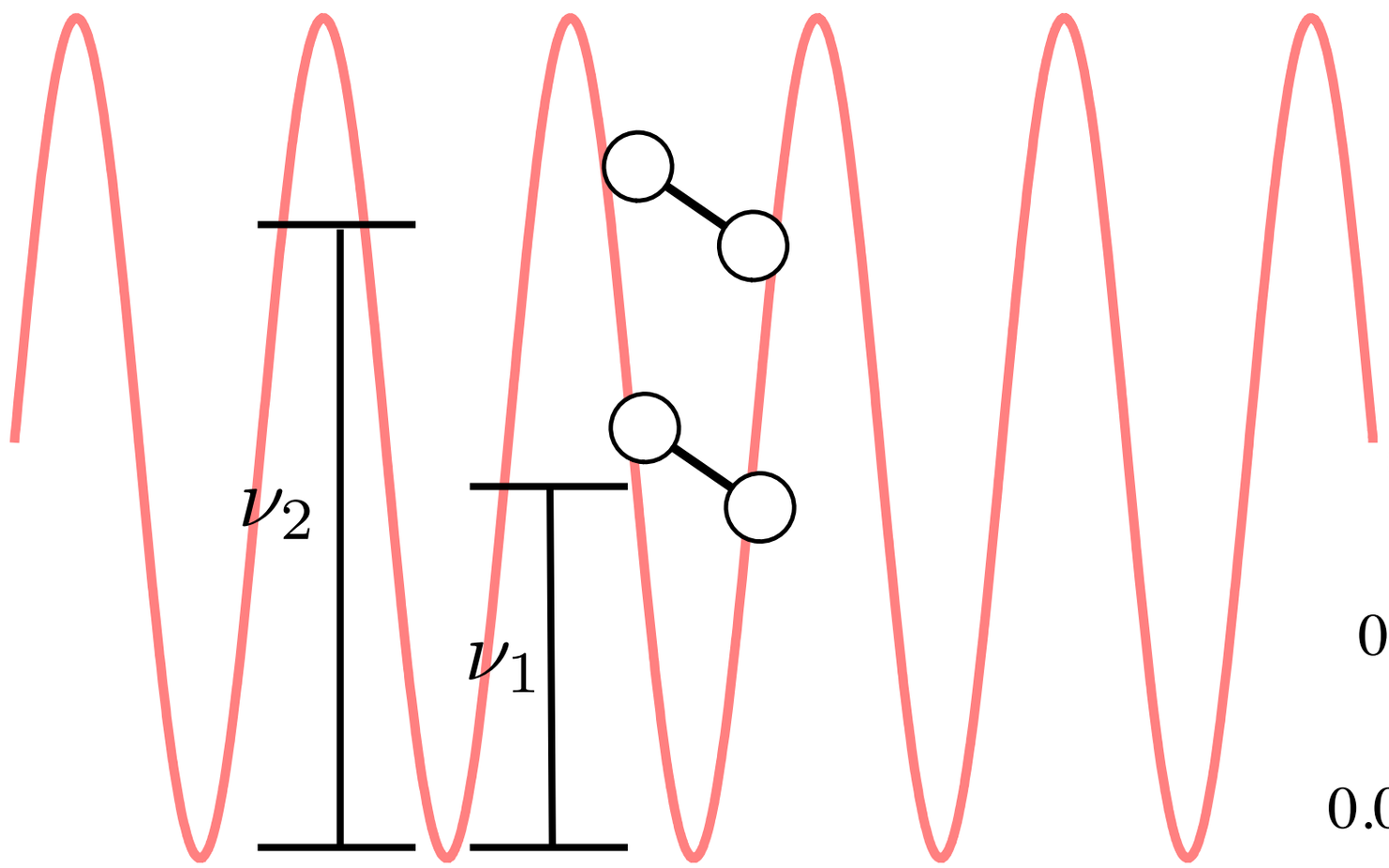

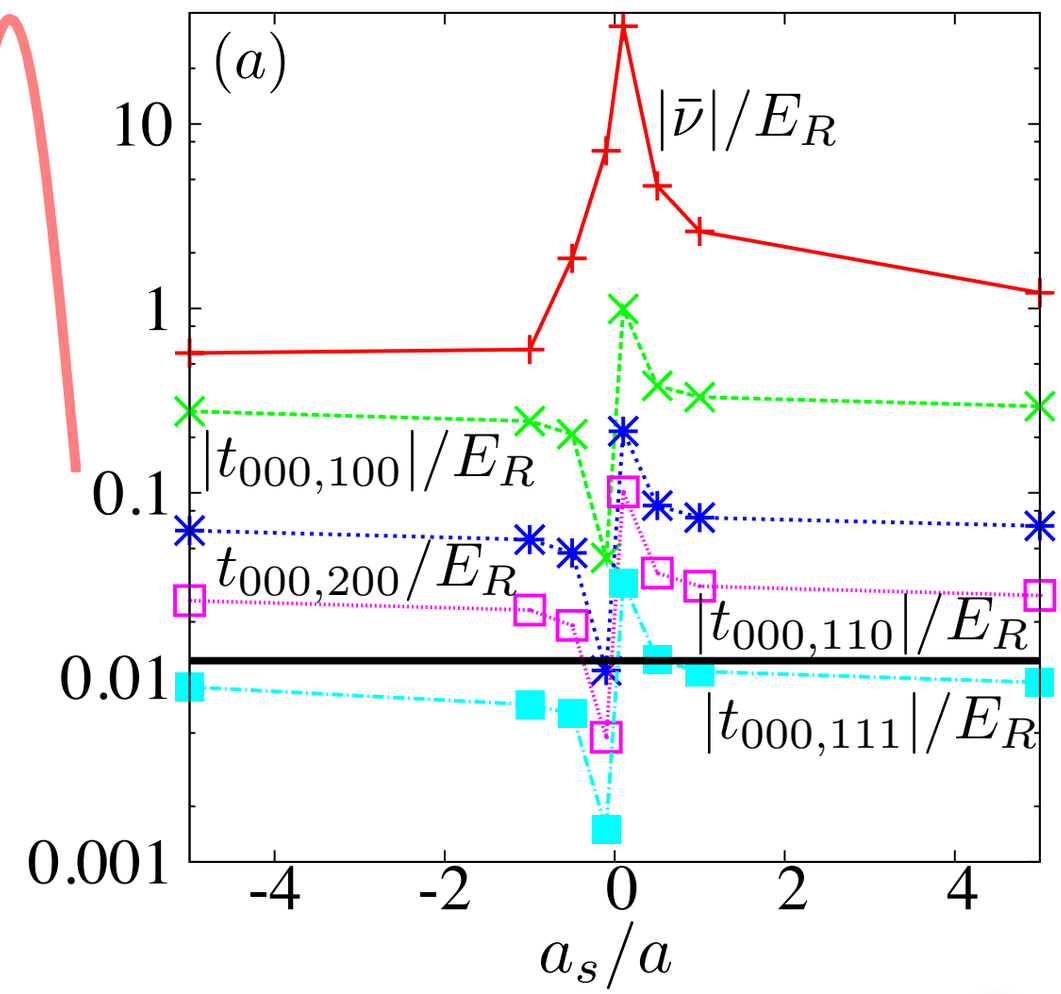

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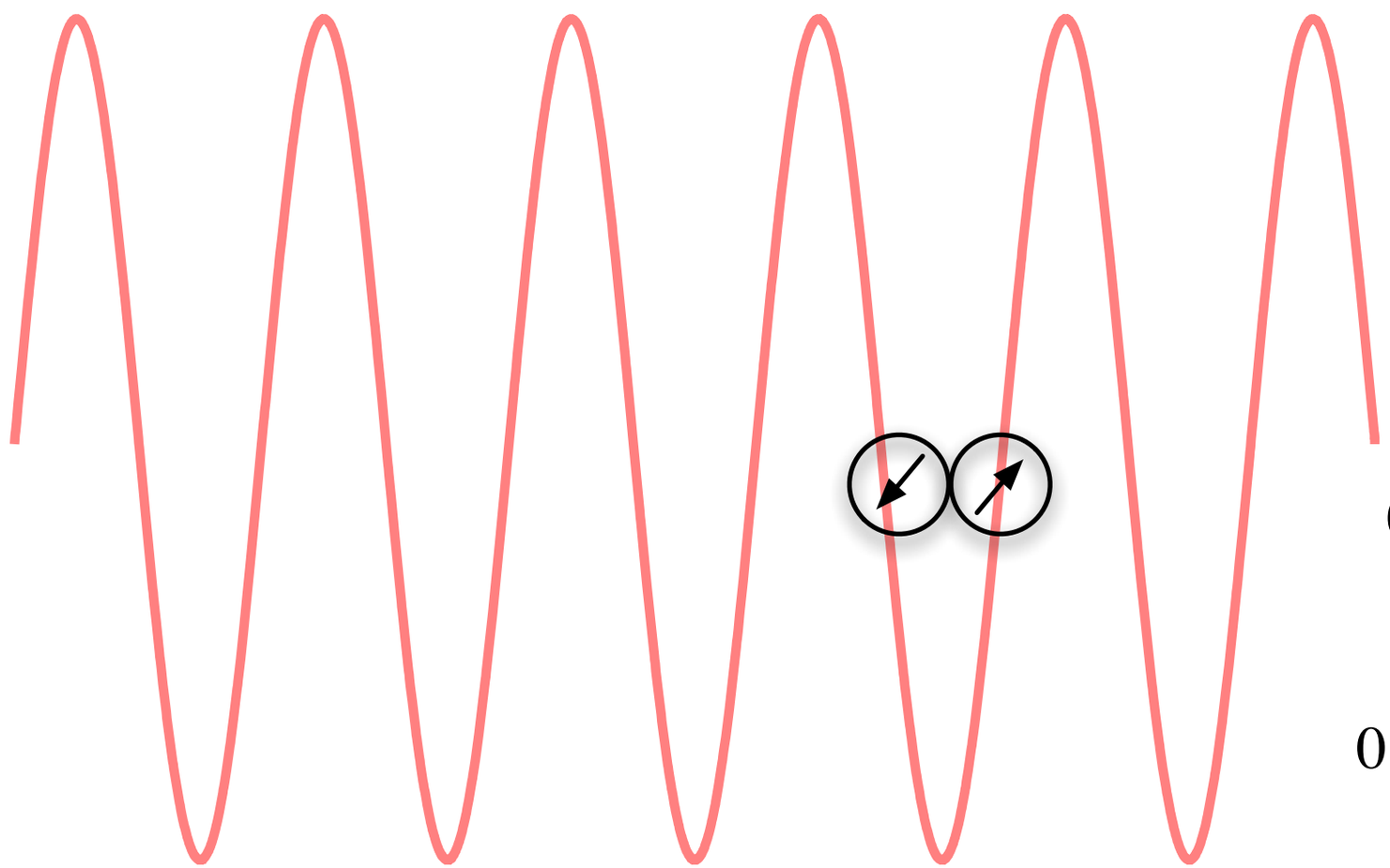

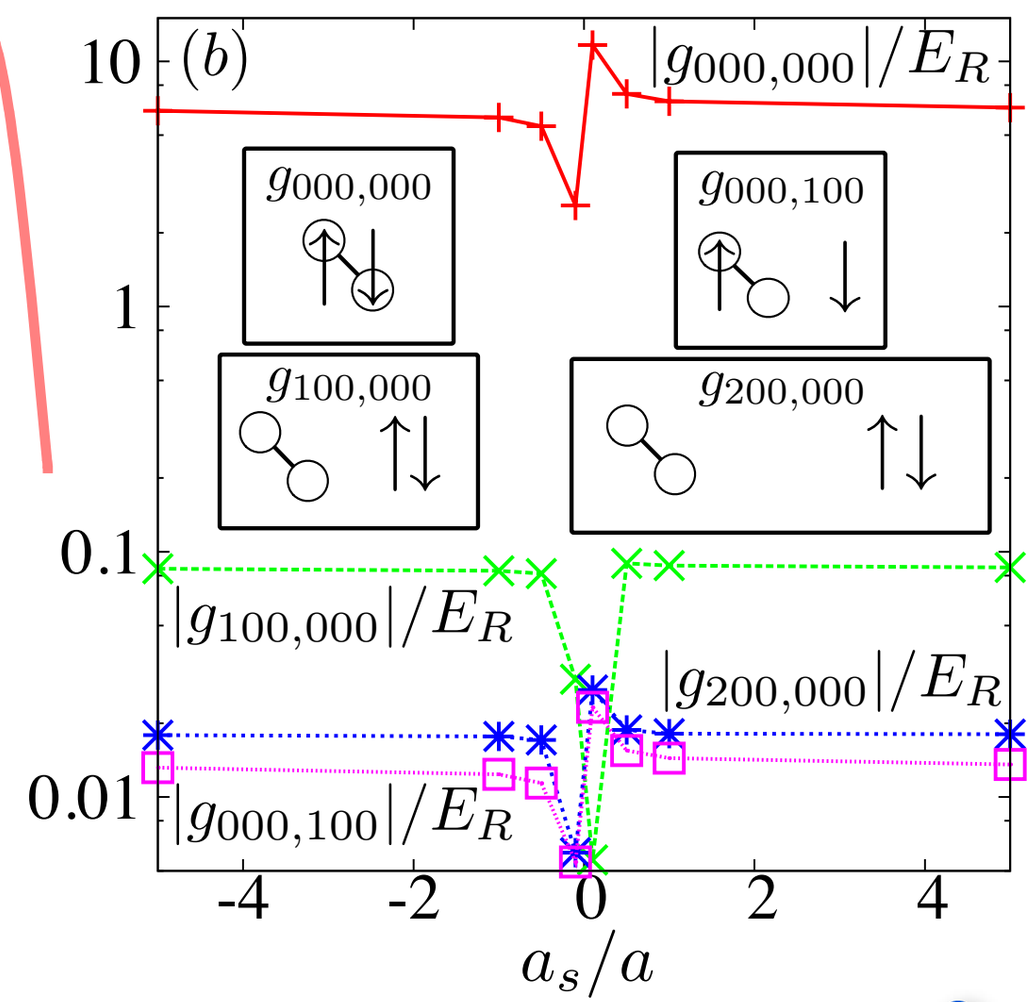

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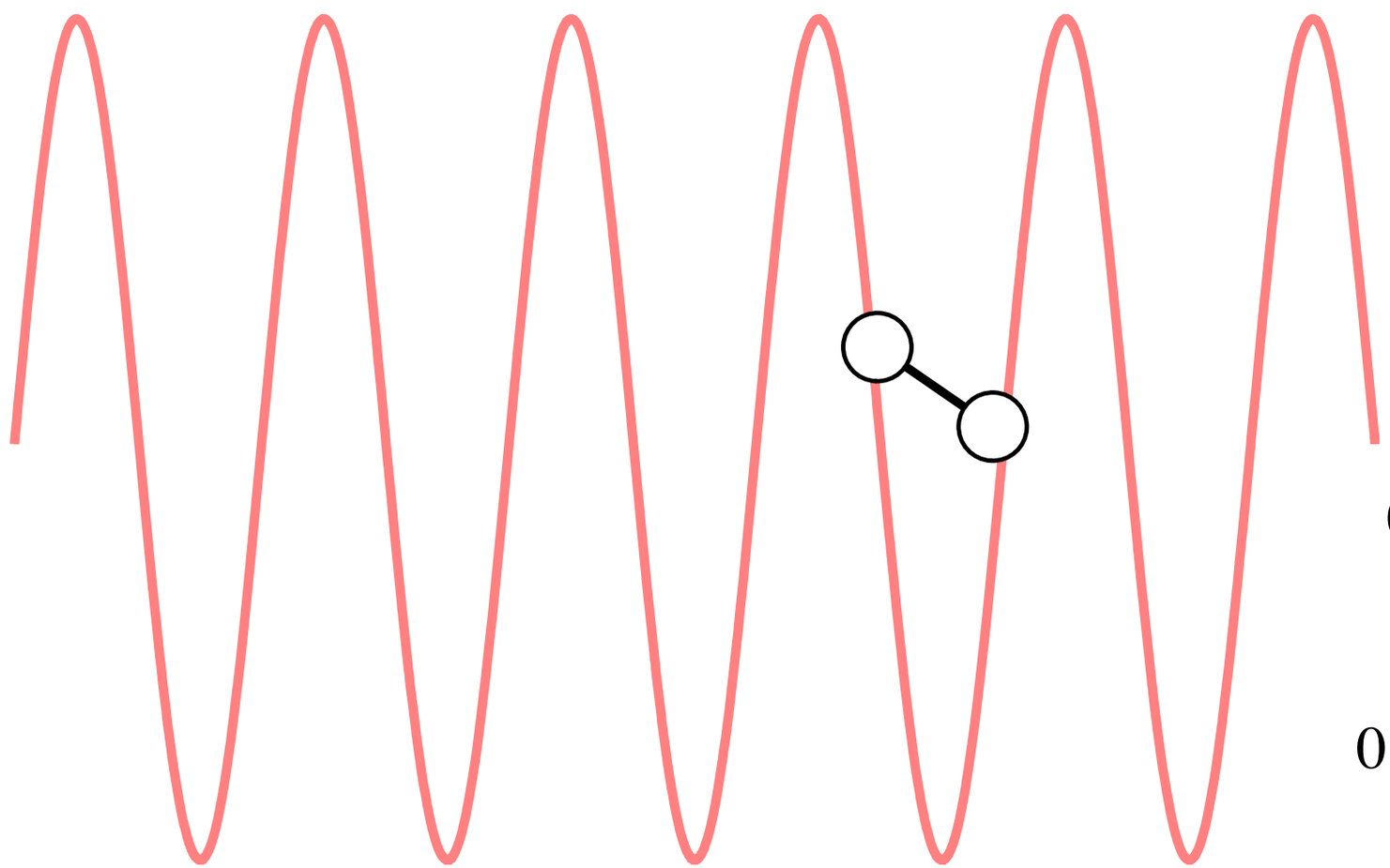

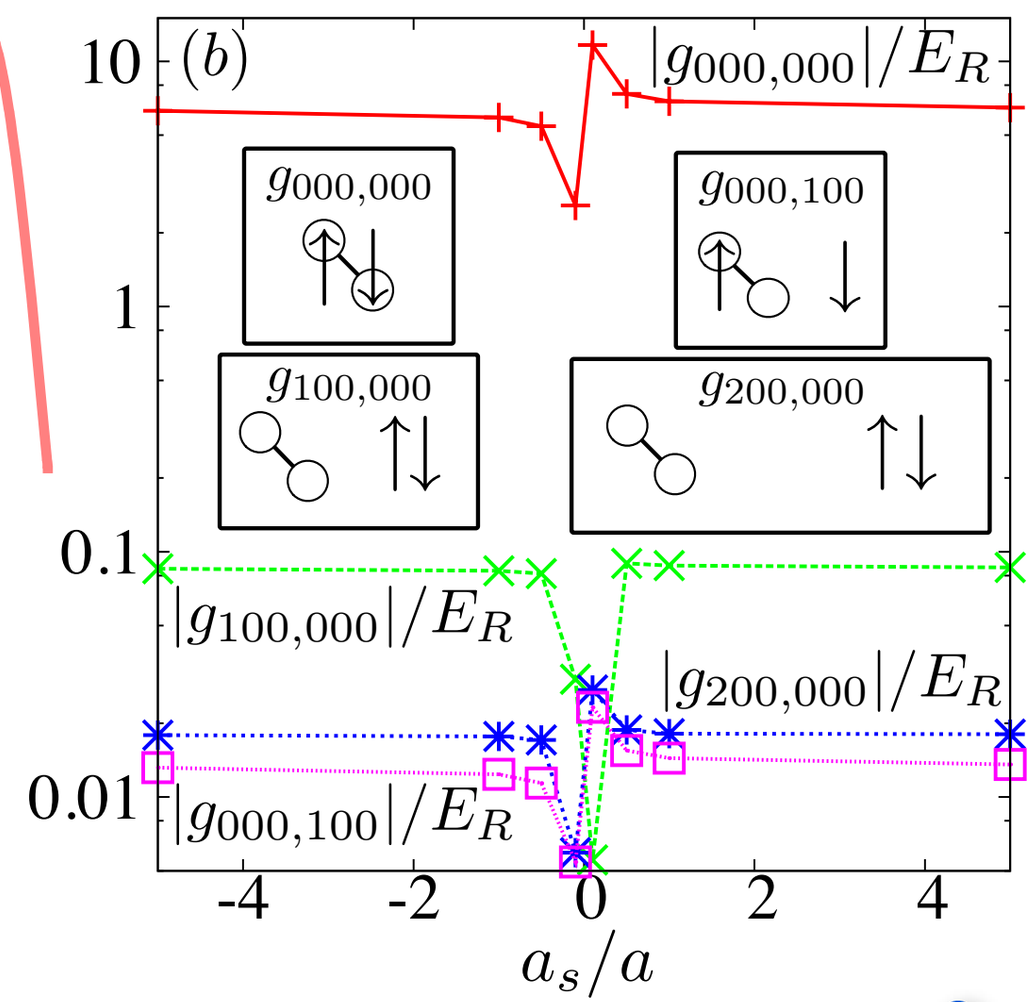

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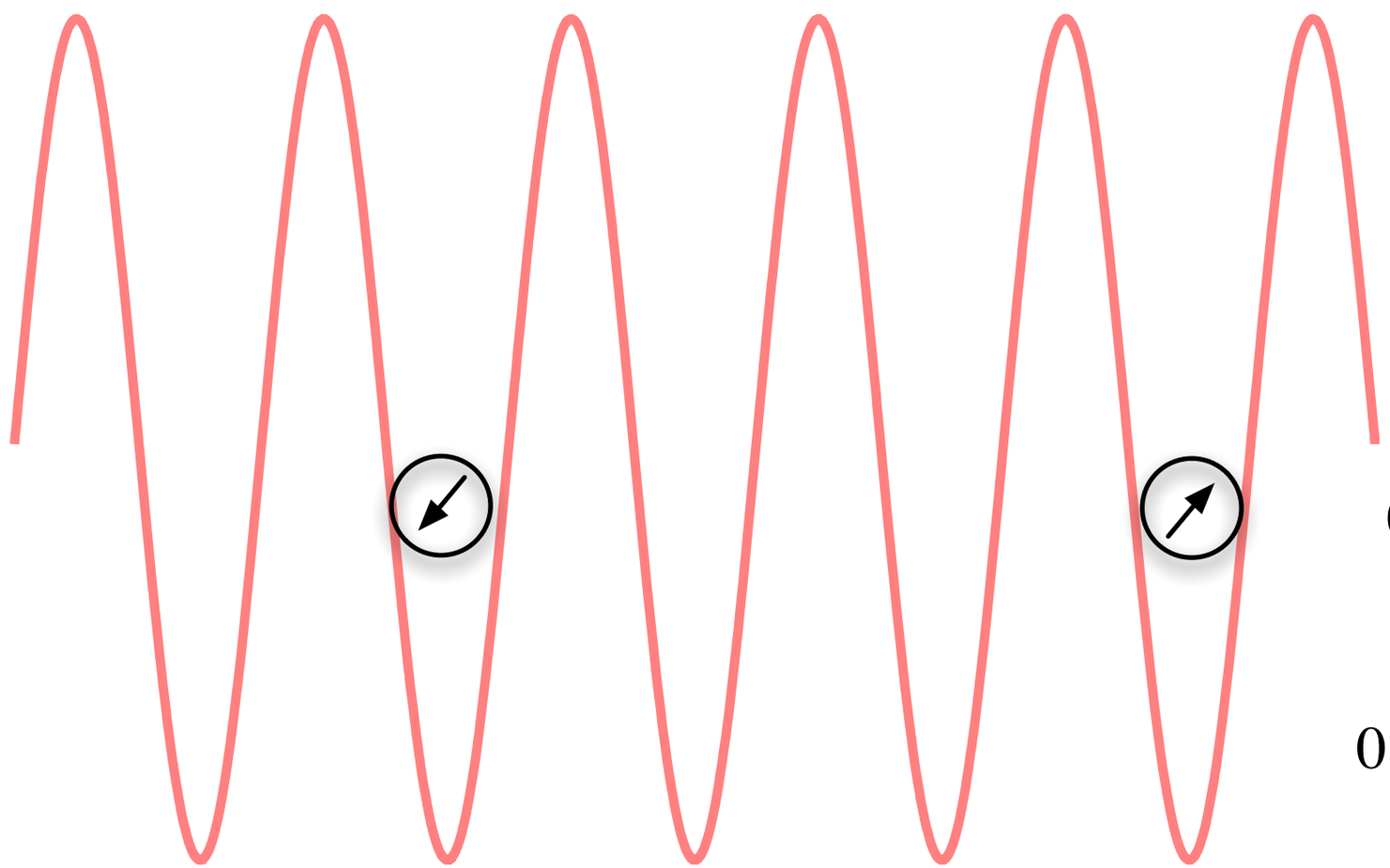

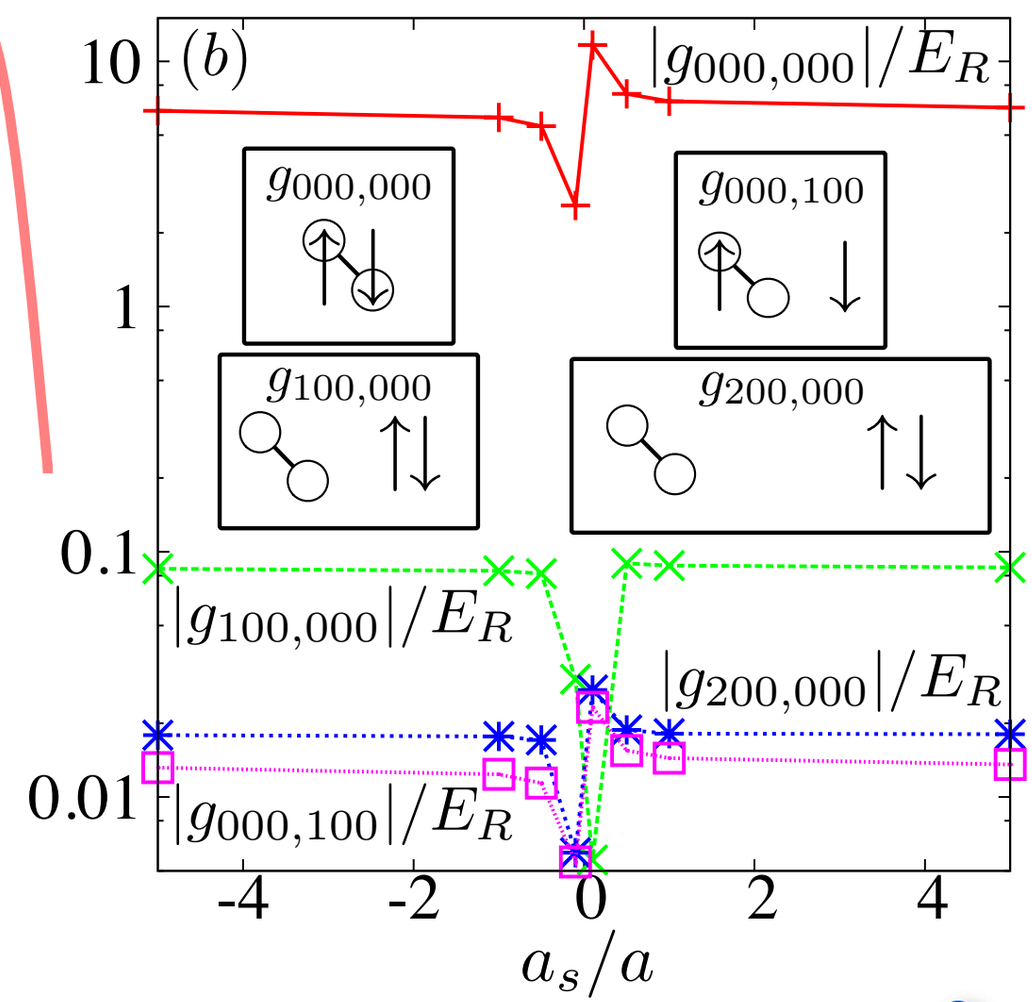

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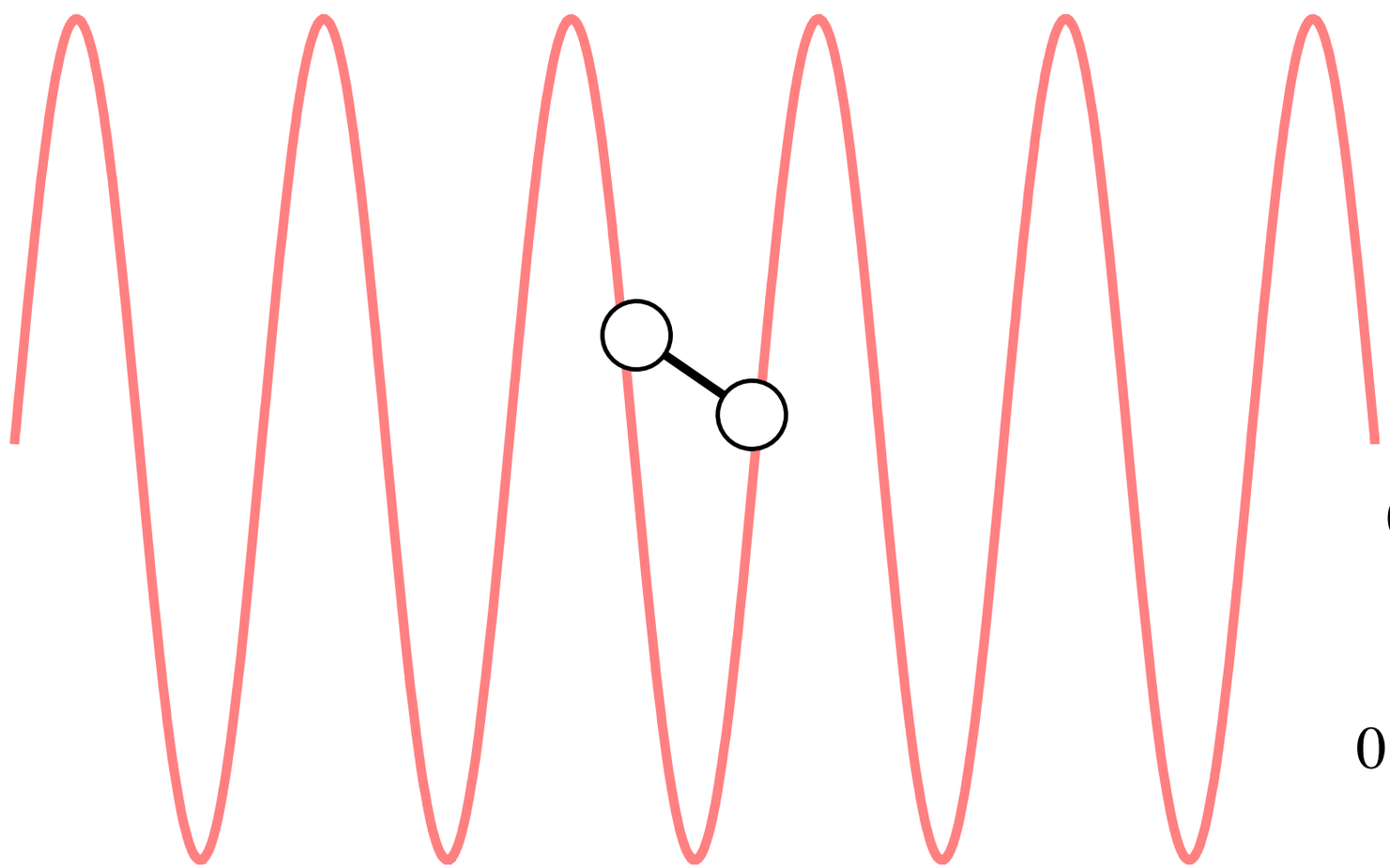

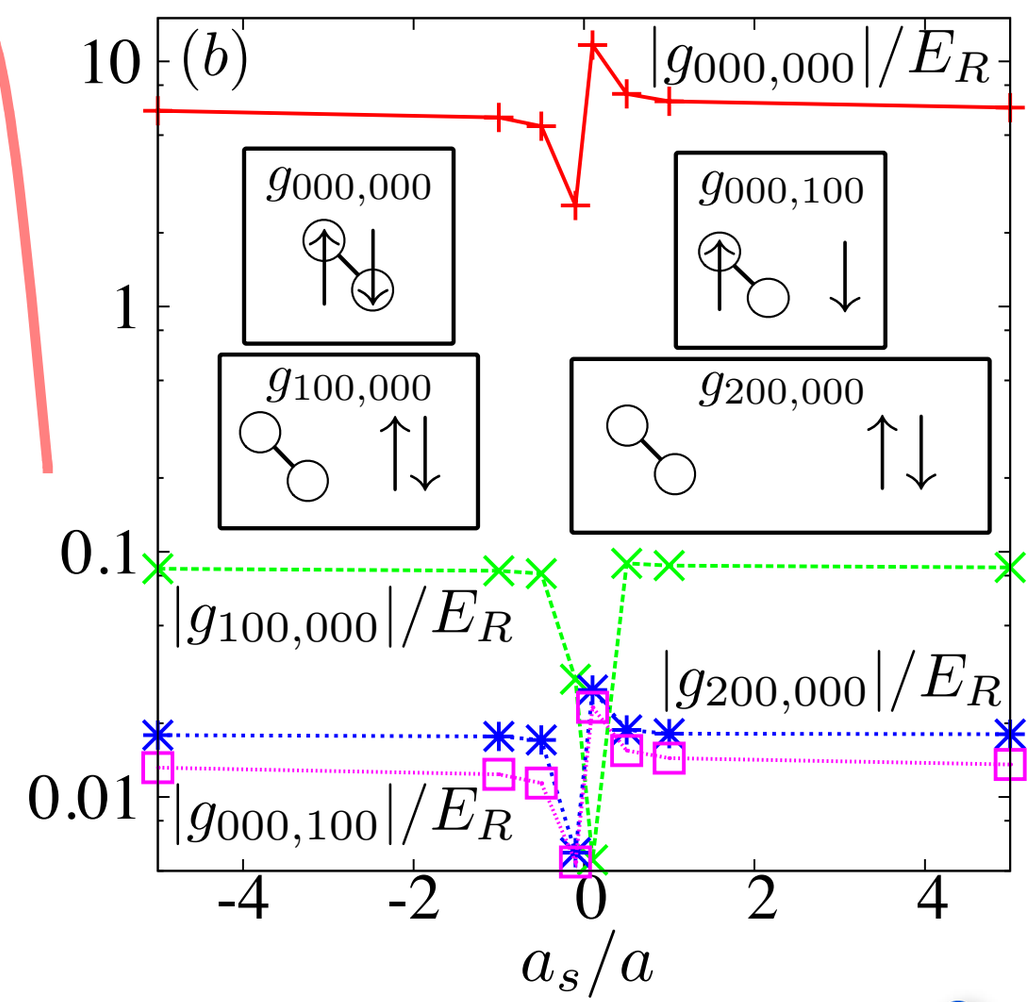

Extensions and Conclusions

Extensions

- Multichannel resonances-basis expanded to include a channel index
- Higher-l pairing/ more general two-body potential
- Arbitrary single particle potentials

Conclusions

- Investigated BEC-BCS crossover in a lattice at the two-body level
- Derived effective few-band model for low-energy, low filling limit
- Significant qualitative and quantitative differences with models derived using separable potential approximations
- All parameters microscopically attainable

Thank You!

Narrow resonances